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NOTE ON FUZZY WEAKLY COMPLETELY PRIME - IDEALS IN TERNARY SEMIGROUPS

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ABSTRACT

Ideal theory in ternary semigroups is introduced by F.M. Sioson. Samit Kumar Majumder introduced and studied on fuzzy completely prime ideals in gamma semigroups. In this paper we proved some properties of prime -ideals and fuzzy weakly completely prime -ideals in ternary semigroups. It is prove that If μ be a non-empty fuzzy subset of a ternary semigroup T, then $1-\mu$ is a fuzzy ternary sub semigroup of if and only if μ is a fuzzy weakly completely prime - ideal in T.

Key words: *Ternary semigroup, fuzzy set, Fuzzy left (lateral, right) ideals, Fuzzy weakly completely prime ideals.*

INTRODUCTION

The theory of ternary algebraic system was introduced by D.H. Lehmer in 1932 but earlier such structures were studied by Kasner in1904, who gave the idea of n-ary algebras. Ternary semigroups are universal algebras with one associative ternary operation. The Concept of quasi-ideals in semigroups was introduced in 1956 by O. Steinfeld. Dixit and Dewan studied quasi-ideals and biideals in ternary semigroups. The introduction of fuzzy sets by L.A. Zadeh. After N. Kuroki introduced and studied the notion of fuzzy semigroups. He also studied the concept of fuzzy bi-ideals (1976) and fuzzy quasi-ideals(1982) of semigroups. Shabir, Jun and Bano introduced and studied the prime, strongly prime, semiprime and irreducible fuzzy bi-ideals of semigroups. They characterized those semigroups for which each fuzzy bi-ideal is semiprime and also characterized those semigroups for which each fuzzy bi-ideal is strongly prime. Several researchers conducted the researches on the generalizations of the notions of fuzzy sets with huge applications in computer, logics and many branches of pure and applied mathematics. Chinram and Saelee in 2010 studied the concept of fuzzy ideals and fuzzy filters of ordered ternary semigroups.

1. BASIC DEFINITIONS AND PRELIMINARIES

1.1 Definition: A non-empty set T is said to be ternary semigroup if there exists a ternary operation $\cdot : T\times T\times T \to T$ written as $(a,b,c) \to a.b.c$ satisfies the following identity

$$(a.b.c).d.e = a.(b.c.d).e = a.b.(c.d.e) \text{ for any } a,b,c,d,e \in T.$$

1.2 Definition: A non-empty subset A of a ternary semigroup T is called a ternary subsemigroup of T if $AAA \subseteq A$.

1.3 Definition: A non-empty subset A of a ternary semigroup T is called a left (right, lateral) ideal in T if

$$TTA \subseteq A\left(ATT \subseteq A, TAT \subseteq A\right).$$

1.4 Definition: A non-empty subset A of a ternary semigroup T is called a two sided ideal of T if it is both left and right ideal in T.

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1.5 Definition: A non-empty subset A of a ternary semigroup T is called a ideal in T if it is left, right and lateral ideal in T.

1.6 Definition: Let T be a non-empty set. A fuzzy subset of a ternary semigroup T is a function $\mu : T \to [0,1]$.

1.7 Definition: Let μ be a fuzzy subset of a non-empty set T for any $t \in [0,1]$, the subset $\mu_t = \{x \in T : \mu(x) \geq t\}$ of T is called a level set of μ.

1.8 Definition: For any two fuzzy subsets μ_1 and μ_2 of a non-empty set T, the union and the intersection of μ_1 and μ_2 denoted by $\mu_1 \cup \mu_2$ and $\mu_1 \cap \mu_2$ are fuzzy subsets of T and defined as

$$(\mu_1 \cup \mu_2)(x) = \max\{\mu_1(x), \mu_2(x)\} = \mu_1(x) \vee \mu_2(x) \text{ and } S \text{ for all } x \in T.$$

where $\vee$ denotes maximum or supremum and $\wedge$ denotes minimum or infimum.

1.9 Definition: Let μ_1, μ_2 and μ_3 are any three fuzzy sets of a ternary semigroup T. Then their fuzzy product $\mu_1 \circ \mu_2 \circ \mu_3$ is defined by $\mu \lhd$

1.10 Definition: A fuzzy set μ of a ternary semigroup T is called a fuzzy ternary subsemigroup of T if $\mu(xyz) \geq \{\mu(x) \wedge \mu(y) \wedge \mu(z)\}$ for all $x, y, z \in T$.

1.11 Definition: A fuzzy set μ of a ternary semigroup T is called a fuzzy left (right, lateral) ideal in T if $\mu(xyz) \geq \mu(z)$, ($\mu(xyz) \geq \mu(x)$, $\mu(xyz) \geq \mu(y)$) for all $x, y, z \in T$..

1.12 Definition: A fuzzy set μ of a ternary semigroup T is a fuzzy ideal in T if it is fuzzy left, right and lateral ideal in T.

1.13 Definition: Let A be a non-subset of a ternary semigroup T. Then the characteristic function of A is defined by

$$C_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}$$

We denote the characteristic function $C_T \ of \ T$ i.e., $T = C_T$ thus $T(x) = 1$ for all $x \in T$.

1.14 Definition: A subset A of a ternary semigroup T is said to be a prime ideal in T if $xyz \in A$ implies $x \in A$ or $y \in A$ or $z \in A$.

1.15 Definition: A fuzzy ideal μ of a ternary semigroup T is called a fuzzy weakly completely prime ideal in T if $\mu(x) \geq \mu(xyz)$ or $\mu(y) \geq \mu(xyz)$ or $\mu(z) \geq \mu(xyz)$ for all $x, y, z \in T$.

1.16 Definition: A fuzzy ideal μ of a ternary semigroup T is called a fuzzy prime ideal in T if

$$\inf \mu(xyz) \geq \max\{\mu(x), \mu(y), \mu(z)\} \text{ for all } x, y, z \in T.$$

1.17 Proposition: If μ_1 μ_2 μ_3 μ_4 and μ_5 are fuzzy subsets of a non-empty set T then

(i) $\mu_1 \cap (\mu_2 \cup \mu_3) \cap \mu_4 = (\mu_1 \cap \mu_2 \cap \mu_4) \cup (\mu_1 \cap \mu_3 \cap \mu_4)$.

(ii) $(C_A \circ C_T \circ C_T) \cap (C_T \circ C_A \circ C_T) \cap (C_T \circ C_T \circ C_A)$

(iii) $\mu_1 \circ (\mu_2 \cup \mu_3) \circ \mu_4 = (\mu_1 \circ \mu_2 \circ \mu_4) \cup (\mu_1 \circ \mu_3 \circ \mu_4)$.

(iv) $\mu_1 \circ \mu_2 \circ (\mu_3 \cup \mu_4) = (\mu_1 \circ \mu_2 \circ \mu_3) \cup (\mu_1 \circ \mu_2 \circ \mu_4)$.

(v) $(\mu_1 \circ \mu_2 \circ \mu_3) \circ \mu_4 \circ \mu_5 = \mu_1 \circ (\mu_2 \circ \mu_3 \circ \mu_4) \circ \mu_5 = \mu_1 \circ \mu_2 \circ (\mu_3 \circ \mu_4 \circ \mu_5)$.

1.18 Proposition: If μ_1 and μ_2 are two fuzzy subsets of a non-empty set T then

(i) $((\mu_1 \cap \mu_2) \circ T \circ T) \subseteq (\mu_1 \circ T \circ T) \cap (\mu_2 \circ T \circ T)$

(ii) $(T \circ (\mu_1 \cap \mu_2) \circ T) \subseteq (T \circ \mu_1 \circ T) \cap (T \circ \mu_2 \circ T)$

(iii) $(T \circ T \circ (\mu_1 \cap \mu_2)) \subseteq (T \circ T \circ \mu_1) \cap (T \circ T \circ \mu_2)$.

Proof:(i) Let $x \in T$. If $x \neq pqr$ for any $p, q, r \in T$ then

$$((\mu_1 \circ T \circ T) \cap (\mu_2 \circ T \circ T))(x) = 0 = ((\mu_1 \cap \mu_2) \circ T \circ T)(x).$$

If $x = pqr$ for any $p, q, r \in T$ then

$$\begin{aligned}((\mu_1 \cap \mu_2) \circ T \circ T)(x) &= \vee_{x=pqr} \{(\mu_1 \cap \mu_2)(p) \wedge T(q) \wedge T(r)\} \\ &= \vee_{x=pqr} \{\mu_1(p) \wedge \mu_2(p) \wedge 1 \wedge 1\} \\ &= \vee_{x=pqr} \{\mu_1(p) \wedge \mu_2(p)\} \\ &\leq \{\vee_{x=pqr} \{\mu_1(p)\}\} \subseteq \\ &= \{\vee_{x=pqr} \{\mu_1(p) \wedge 1 \wedge 1\}\} \wedge \{\vee_{x=pqr} \{\mu_2(p) \wedge 1 \wedge 1\}\} \\ &= \{\vee_{x=pqr} \{\mu_1(p) \wedge T(q) \wedge T(r)\}\}\end{aligned}$$

$$\begin{aligned}\wedge \{\vee_{x=pqr} \{\mu_2(p) \wedge T(q) \wedge T(r)\}\} &= (\mu_1 \circ T \circ T)(x) \wedge (\mu_2 \circ T \circ T)(x) \\ &= ((\mu_1 \circ T \circ T) \cap (\mu_2 \circ T \circ T))(x)\end{aligned}$$

$$((\mu_1 \cap \mu_2) \circ T \circ T)(x) \leq ((\mu_1 \circ T \circ T) \cap (\mu_2 \circ T \circ T))(x)$$

Therefore $((\mu_1 \cap \mu_2) \circ T \circ T) \subseteq ((\mu_1 \circ T \circ T)\ T$.

Similarly we can prove that

(ii) $(T \circ (\mu_1 \cap \mu_2) \circ T) \subseteq (T \circ \mu_1 \circ T) \cap (T \circ \mu_2 \circ T)$

and (iii) $(T \circ T \circ (\mu_1 \cap \mu_2)) \subseteq (T \circ T \circ \mu_1) \cap (T \circ T \circ \mu_2)$.

RESULTS

2.1 Theorem: Let μ be a non-empty fuzzy subset of a ternary semigroup T . Then $1-\mu$ is a fuzzy ternary subsemigroup of T if and only if μ is a fuzzy weakly completely prime ideal in T .

Proof: Let μ be a non-empty fuzzy subset of a ternary semigroup T .

Assume that $\mu^{/} = 1-\mu$ is a fuzzy ternary subsemigroup of T and let $x, y, z \in T$. Then

$$\begin{aligned}1-\mu(xyz) &= 1-\min\{\mu(x), \mu(y), \mu(z)\} \\ &\geq \min\{1-\mu(x), 1-\mu(y), 1-\mu(z)\}\end{aligned}$$

$$1-\mu(xyz) \geq 1-\max\{\mu(x), \mu(y), \mu(z)\}$$

$$-\mu(xyz) \geq -\max\{\mu(x), \mu(y), \mu(z)\}$$

$$\mu(xyz) \leq \max\{\mu(x), \mu(y), \mu(z)\}$$

i.e., $\max\{\mu(x), \mu(y), \mu(z)\} \geq \mu(xyz)$

Therefore $\mu(x) \geq \mu(xyz)$ or $\mu(y) \geq \mu(xyz)$ or $\mu(z) \geq \mu(xyz)$

Hence μ is a fuzzy weakly completely prime ideal in T .

Conversely, assume that μ is a fuzzy weakly completely prime ideal in T. Then we have

$$\mu(x) \geq \mu(xyz) \text{ or } \mu(y) \geq \mu(xyz) \text{ or } \mu(z) \geq \mu(xyz).$$

$$\max\{\mu(x), \mu(y), \mu(z)\} \geq \mu(xyz)$$

$$1 - \max\{\mu(x), \mu(y), \mu(z)\} \leq 1 - \mu(xyz)$$

$$\min\{1-\mu(x), 1-\mu(y), 1-\mu(z)\} \leq 1 - \mu(xyz)$$

$$\mu^{/}(xyz) \geq \min\{\mu^{/}(x), \mu^{/}(y), \mu^{/}(z)\}$$

Therefore $\mu^{/} = 1 - \mu$ is a fuzzy ternary subsemigroup of T.

2.2 Theorem: Let $\{\mu_i : i \in I\}$ be a family of fuzzy weakly completely prime ideals in a ternary semigroup T. Then $\bigcap_{i\in I}\mu_i$ is also a fuzzy weakly completely prime ideal in T.

Proof: Let $\{\mu_i : i \in I\}$ be a family of fuzzy weakly completely prime ideals in a ternary semigroup T. Then we have $\mu_i(x) \geq \mu_i(xyz)$ or $\mu_i(y) \geq \mu_i(xyz)$ or $\mu_i(z) \geq \mu_i(xyz)$ for all $x, y, z \in T$, $i \in I$.

Then $\bigcap_{i\in I}\mu_i(xyz) = \inf\{\mu_i(xyz) : i \in I\}$

$$\therefore \bigcap_{i\in I}\mu_i(xyz) \leq \inf\{\mu_i(x) : i \in I\}$$

or $\bigcap_{i\in I}\mu_i(xyz) \leq \inf\{\mu_i(y) : i \in I\}$

or $\bigcap_{i\in I}\mu_i(xyz) \leq \inf\{\mu_i(z) : i \in I\}$

Hence $\bigcap_{i\in I}\mu_i$ is fuzzy weakly ternary completely prime ideal in T.

2.3 Theorem: Let T be a ternary semigroup and μ be a non-empty fuzzy subset of T. Then the following are equivalent.

(1) μ is a fuzzy weakly completely prime ideal in T.

(2) For any $t \in [0,1]$, μ_t (if it is non-empty) is a prime ideal in T.

Proof: Let μ be a fuzzy weakly completely prime ideal in a ternary semigroup T. Then we have $\mu(x) \geq \mu(xyz)$ or $\mu(y) \geq \mu(xyz)$ or $\mu(z) \geq \mu(xyz)$ for all $x, y, z \in T$. Let $t \in [0,1]$ be such that μ_t is non-empty. Let $x, y, z \in T$, $xyz \in \mu_i$. Then $\mu(xyz) \geq t$. Since μ is a fuzzy weakly completely prime ideal in a ternary semigroup T, so, we have $\mu(x) \geq \mu(xyz)$ or $\mu(y) \geq \mu(xyz)$ or $\mu(z) \geq \mu(xyz)$ for all $x, y, z \in T$. Then $\mu(x) \geq t$ or $\mu(y) \geq t$ or $\mu(z) \geq t$ which implies that $x \in \mu_t$ or $y \in \mu_t$ or $z \in \mu_t$. Hence μ_t is a prime -ideal in T.

Conversely, let us suppose that μ_t is a prime ideal in a ternary semigroup T. Let μ_t is non-empty. Then $xyz \in \mu_t$ $\Rightarrow \mu_t(xyz) \geq t.$ Since μ_t is a prime ideal in T, we have $x \in \mu_t$ or $y \in \mu_t$ or $z \in \mu_t$. Then $\mu(x) \geq t$ or $\mu(y) \geq t$ or $\mu(z) \geq t$ which implies that $\mu(x) \geq \mu(xyz)$ or $\mu(y) \geq \mu(xyz)$ or $\mu(z) \geq \mu(xyz)$. Hence μ is a fuzzy weakly completely prime ideal in a ternary semigroup T.

2.4 Theorem: Let A be a non-empty subset of a ternary semigroup T and C_A be the characteristic function of A. Then A is a left ideal of T if and only if C_A is a fuzzy left ideal of T.

Proof: Let A be non-empty subset of a ternary semigroup T and C_A be a characteristic function of A.

We assume that A is left ideal of T then $TTA \subseteq A$.

We have to prove that C_A is fuzzy left ideal of T. i.e., $C_A(xyz) \geq C_A(z)$

Let $a = xyz$ for all $x, y, z \in T$.

Consider
$$\begin{aligned} C_A(x\,y\,z) &\geq C_{TTA}(xyz) \\ &= (C_T \circ C_T \circ C_A)(xyz) \\ &= \vee_{xyz=pqr} (C_T(p) \wedge C_T(q) \wedge C_A(r)) \\ &= (C_T(x) \wedge C_T(y) \wedge C_A(z)) \\ &= (1 \wedge 1 \wedge C_A(z)) \\ &= C_A(z) \end{aligned}$$

$$\therefore \quad C_A(xyz) \geq C_A(z)$$

Therefore C_A is a fuzzy left ideal of T.

Conversely, suppose C_A is a fuzzy left ideal in T. Then $C_A(xyz) \geq C_A(z)$ for all $x, y, z \in T$.

To prove that A is a left ideal in T. i.e., $TTA \subseteq A$.

Let $x, y, z \in T$ then $C_T(x) = 1,\ C_T(y) = 1,\ C_T(z) = 1$

Consider
$$\begin{aligned} C_A(xyz) &\geq C_A(z) \\ &= (1 \wedge 1 \wedge C_A(z)) \\ &= (C_T(x) \wedge C_T(y) \wedge C_A(z)) \\ &= (C_T \circ C_T \circ C_A)(xyz) \end{aligned}$$

$$C_A(xyz) = (C_T \circ C_T \circ C_A)(xyz)$$
$$\Rightarrow C_A \supseteq C_T \circ C_T \circ C_A$$
$$\Rightarrow C_A \supseteq C_{TTA} \Rightarrow ATT \subseteq A.$$

Therefore A is a left ideal in a ternary semigroup T.

2.5 Theorem: Let A be a non-empty subset of a ternary semigroup T and C_A be the characteristic function of A. Then A is a right ideal (lateral ideal, ideal) in T if and only if C_A is a fuzzy right ideal (fuzzy lateral ideal, fuzzy ideal) in T.

Proof: Similar to the proof of Theorem 3.3.4.

2.6 Theorem: Let T be a ternary semigroup and A be a non-empty subset of T. Then the following are equivalent

(1) A is prime ideal in a ternary semigroup T

(2) The characteristic function C_A of A is a fuzzy weakly completely prime ideal in T.

Proof: Let A be a prime ideal in a ternary semigroup T and C_A be the characteristic function of A Since $A \neq \phi$, so, C_A is non-empty. Let $x, y, z \in T$. Suppose $xyz \in A$ Then $C_A(x\,y\,z) = 1$. Since A is a prime ideal in T,

$x \in A$ or $y \in A$ or $z \in A$ which implies that $C_A(x) = 1$ or $C_A(y) = 1$ or $C_A(z) = 1$. Hence $C_A(x) \geq C_A(xyz)$ or $C_A(y) \geq C_A(xyz)$ or $C_A(z) \geq C_A(xyz)$. Suppose $xyz \in A$. Then $C_A(xyz) = 0$. Since A be a prime ideal in T, $x \notin A$ or $y \notin A$ or $z \notin A$ which implies that $C_A(x) = 0$ or $\mu_A(y) = 0$ or $C_A(z) = 0$. Hence $C_A(y) \geq C_A(xyz)$ or $C_A(z) \geq C_A(xyz)$. Consequently C_A is a fuzzy weakly completely prime ideal in T.

Conversely, let the characteristic function C_A of A is a fuzzy weakly completely prime ideal in T. Then C_A is a fuzzy ideal in T. By the theorem 2.1.4, A is an ideal in T. Let $x, y, z \in T$ be such that $xyz \in A$. Then $C_A(xyz) = 1$. Let if possible $x \notin S$ and $y \notin T$ and $z \notin T$. Then $C_A(x) = C_A(y) = C_A(z) = 0$ which implies $C_A(x) < C_A(xyz)$ and $C_A(y) < C_A(xyz)$ and $C_A(z) < C_A(xyz)$. This contradicts our assumption that C_A is a fuzzy weakly completely prime ideal of T. Hence A is prime ideal in T.

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