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f -DERIVATIONS IN ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

In this paper, we introduce the concept of f - derivation in an Almost Distributive Lattice (ADL) and derive some important properties of f -derivations in ADLs.

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1. INTRODUCTION

The notion of derivation, introduced from the analytic theory, is helpful for the research of structure and property in an algebraic system. Several authors ([5], [2]) have studied derivations in rings and near rings after Posner[9] has given the definition of the derivation in ring theory. The concept of a derivation in lattices was introduced by G.Szasz in 1974[15]. X. L. Xin *et al.* [16] applied the notion of derivation in the ring theory to lattices and investigated some properties. Later, several authors ([1], [3], [4], [6], [7], [8] and [18]) have worked on this concept. The concept of an f - derivation on lattices was introduced by Yilmaz Ceven [3] in 2008.

In 1980, the concept of an Almost Distributive Lattice (ADL) was introduced by U.M.Swamy and G.C Rao [14]. This class of ADLs include most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other.

In this paper, we introduce the concept of an f -derivation in an ADL and invistigate some important properties. Also, we introduce the concept of an isotone f -derivation in ADLs and we establish a set of conditions which are sufficient for a derivation on an ADL with a maximal element to become an isotone derivation. We define the fixed set F_d of an $f-$ derivation d in an ADL L and prove that it is an ideal of L if f is a constant function. Also, we give some equivalent conditions under which an f -derivation on an ADL becomes an isotone f -derivation. Finally, we prove that if f -is a join-homomorphism, then an f -derivation on an ADL is a meet homomorphism if and only if it is a join homomorphism.

2. PRELIMINARIES

In this section, we recollect basic concepts and important results on Almost Distributive Lattices.

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Definition 2.1 [10]: *An algebra* $(L,\vee,\wedge)$ *of type* $(2,2)$ *is called an Almost Distributive Lattice, if it satisfies the following axioms:*

$L_1:\ (a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)\ \ (RD\wedge)$

$L_2:\ a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)\ \ (LD\wedge)$

$L_3:\ (a\vee b)\wedge b=b$

$L_4:\ (a\vee b)\wedge a=a$

$L_5:\ a\vee(a\wedge b)=a$

Definition 2.2 [10]: *Let* X *be any non-empty set. Define, for any* $x,y\in L,\ x\vee y=x$ *and* $x\wedge y=y.$ *Then* $(X,\vee,\wedge)$ *is an ADL and such an ADL, we call discrete ADL.*

Through out this paper L stands for an ADL $(L,\vee,\wedge)$ unless otherwise specified.

Lemma 2.3 [10]: *For any* $a,b\in L,$ *we have*

$(i)\ \ a\wedge a=a$

$(ii)\ \ a\vee a=a.$

$(iii)\ (a\wedge b)\vee b=b$

$(iv)\ \ a\wedge(a\vee b)=a$

$(v)\ \ a\vee(b\wedge a)=a.$

(vi) $a\vee b=a$ if and only if $a\wedge b=b$

(vii) $a\vee b=b$ if and only if $a\wedge b=a.$

Definition 2.4 [10]: *For any* $a,b\in L,$ *we say that* a *is less than or equal to* b *and write* $a\le b,$ *if* $a\wedge b=a$ *or, equivalently,* $a\vee b=b.$

Theorem 2.5 [10]: *For any* $a,b,c\in L$ *, we have the following*

(i) The relation $\le$ is a partial ordering on $L.$

(ii) $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c).\ \ (LD\vee)$

(iii) $(a\vee b)\vee a=a\vee b=a\vee(b\vee a).$

(iv) $(a\vee b)\wedge c=(b\vee a)\wedge c.$

(v) The operation $\wedge$ is associative in $L.$

(vi) $a\wedge b\wedge c=b\wedge a\wedge c.$

Theorem 2.6 [10]: *For any* $a,b\in L,$ *the following are equivalent.*

$(i)\ \ (a\wedge b)\vee a=a$

$(ii)\ \ a\wedge(b\vee a)=a$

$(iii)\ \ (b\wedge a)\vee b=b$

$(iv)\ \ b\wedge(a\vee b)=b$

$(v)\ \ a\wedge b=b\wedge a$

$(vi)\ \ a\vee b=b\vee a$

(vii) The supremum of a and b exists in L and equals to $a\vee b$

$(viii)$ there exists $x\in L$ such that $a\le x$ and $b\le x$

(ix) the infimum of a and b exists in L and equals to $a\wedge b.$

Definition 2.7 [10]: L *is said to be associative, if the operation* $\vee$ *in* L *is associative.*

Theorem 2.8 [10]: *The following are equivalent.*

(i) L is a distributive lattice.

(ii) the poset $(L,\leq)$ is directed above.

(iii) $a\wedge(b\vee a)=a,$ for all $a,b\in L.$

(iv) the operation $\vee$ is commutative in $L.$

(v) the operation $\wedge$ is commutative in $L.$

(vi) the relation $\theta:=\{(a,b)\in L\times L\mid a\wedge b=b\}$ is anti-symmetric.

(vii) the relation θ defined in (vi) is a partial order on $L.$

Lemma 2.9 [10]: *For any* $a,b,c,d\in L$*,we have the following*

(i) $a\wedge b\leq b$ and $a\leq a\vee b$

(ii) $a\wedge b=b\wedge a$ whenever $a\leq b$.

(iii) $[a\vee(b\vee c)]\wedge d=[(a\vee b)\vee c]\wedge d.$

(iv) $a\leq b$ implies $a\wedge c\leq b\wedge c,$ $c\wedge a\leq c\wedge b$ and $c\vee a\leq c\vee b.$

Definition 2.10 [10]: *An element* $0\in L$ *is called zero element of* $L,$ *if* $0\wedge a=0$ *for all* $a\in L.$

Lemma 2.11 [10]: *If* L *has* 0*, then for any* $a,b\in L$ *, we have the following*

(i) $a\vee 0=a,$ (ii) $0\vee a=a$ and (iii) $a\wedge 0=0.$

(iv) $a\wedge b=0$ if and only if $b\wedge a=0.$

An element $x\in L$ is called maximal if, for any $y\in L$, $x\leq y$ implies $x=y$.

We immediately have the following.

Lemma 2.12 [10]: *For any* $m\in L$*, the following are equivalent:*

(1) m is maximal

(2) $m\vee x=m$ for all $x\in L$

(3) $m\wedge x=x$ for all $x\in L$.

Definition 2.13 [10]: *A nonempty subset* I *of* L *is said to be an ideal if and only if it satisfies the following:*

(1) $a,b\in I\Rightarrow a\vee b\in I$

(2) $a\in I, x\in L\Rightarrow a\wedge x\in I$.

Definition 2.14 [10]: *A function* $f:L\to L$ *is said to be an ADL homomorphism if it satisfies the following:*

(1) $f(x\wedge y)=fx\wedge fy$,

(2) $f(x\vee y)=fx\vee fy$ for all $x,y\in L$.

3. f - DERIVATIONS IN ADLs

We begin this section with the following definition of a derivation in an ADL.

Definition 3.1 [13]: *A function* $d:L\to L$ *is called a derivation on* L*, if* $d(x\wedge y)=(dx\wedge y)\vee(x\wedge dy)$ for all $x,y\in L.$

The following definition introduces the notion of an f-derivation on ADLs.

Definition 3.2: *A function* $d:L\to L$ *is called an* f*-derivation on* L *if there exists a function* $f:L\to L$ *such that* $d(x\wedge y)=(dx\wedge fy)\vee(fx\wedge dy)$ *for all* $x,y\in L$.

Definition 3.3: *An* f*-derivation* d *on* L *is called an isotone* f*-derivation if* $da\leq db$ *for all* $a,b\in L$ *with* $a\leq b$.

Example 3.4: *Let d be a derivation on L. If we choose f as the identity function on L, then we get that d is an f-derivation on L. Hence every derivation on L is an f-derivation.*

Example 3.5: *Every constant function on L is an f-derivation , but not a derivation.*

Example 3.6: *Define $d : L \to L$ by $dx = a \wedge fx$ for all $x \in L$ and for some $a \in L$ where $f : L \to L$ is a function satisfies $f(x \wedge y) = fx \wedge fy$ for all $x, y \in L$.*
Then d is an f-derivation on L. In addition if f is an increasing function then d is an isotone derivation also.

Lemma 3.7: *Let d be an f-derivation on L, then the following hold:*

1. $dx \le fx$, for any $x \in L$
2. If L has 0, then $f0 = 0$ implies $d0 = 0$
3. $dx \wedge dy \le d(x \wedge y)$
4. $(dx \vee dy) \wedge d(x \wedge y) = d(x \wedge y)$

Proof:
(1) If $x \in L$, then $dx = d(x \wedge x) = (dx \wedge fx) \vee (fx \wedge dx) = dx \wedge fx$ (by Lemma 2.3). Therefore, $dx \le fx$.
(2) If L has 0 and $f0 = 0$, then by(1) above, $d0 \le f0 = 0$. Thus, $0 \le d0 \le 0$ and hence $d0 = 0$.
(3) Let $x, y \in L$. We have $d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy)$.
Therefore, $dx \wedge fy \le d(x \wedge y)$.
Now, by(1) above, we get $dx \wedge dy \le dx \wedge fy \le d(x \wedge y)$.
(4) For any $x, y \in L$,

$$\begin{aligned}(dx \vee dy) \wedge d(x \wedge y) &= (dx \vee dy) \wedge [(dx \wedge fy) \vee (fx \wedge dy)] \\ &= [(dx \vee dy) \wedge dx \wedge fy] \vee [(dx \vee dy) \wedge fx \wedge dy] \\ &= (dx \wedge fy) \vee [fx \wedge (dx \vee dy) \wedge dy] \\ &= (dx \wedge fy) \vee (fx \wedge dy) \\ &= d(x \wedge y).\end{aligned}$$

Lemma 3.8: *Suppose m is a maximal element of L and d is an f-derivation on L. Then we have the following,*

1. If $x \in L$, $fx \le dm$, then $dx = fx$.
2. If $x \in L$, $fx \ge dm$, then $dx \ge dm$.
3. If $dm = m$, then $fm = m$ and $d = f$.

Proof: Now,

$$\begin{aligned}dx = d(m \wedge x) &= (dm \wedge fx) \vee (fm \wedge dx) \\ &= fx \vee dx.\end{aligned}$$

Thus $fx \le dx$. Hence d=f.
(1) Let $x \in L$ and $fx \le dm$.
Then $dx = d(m \wedge x) = (dm \wedge fx) \vee (fm \wedge dx) = fx \vee (fm \wedge dx)$. Thus $fx \le dx$ and hence $dx = fx$.
(2) Let $x \in L$ and $fx \ge dm$. Then $dx = d(m \wedge x) = (dm \wedge fx) \vee (fm \wedge dx) = dm \vee (fm \wedge dx)$. Thus $dx \ge dm$.
(3) Let $dm = m$. Then $m = dm = dm \wedge fm = m \wedge fm = fm$.

Definition 3.9: *Let d be an f-derivation on L. We define $F_d = \{x \in L / dx = fx\}$.*

Lemma 3.10: *Suppose d is an f-derivation on L where f is an increasing function. If $x, y \in L$ with $y \le x$ and $x \in F_d$, then $y \in F_d$.*

Proof: Let $x, y \in L$ with $y \le x$ and $x \in F_d$. By Lemma 3.8, we have $dy \le fy$.

Thus $dy \le fy \le fx = dx$. Now, $dy = d(y \wedge x) = (dy \wedge fx) \vee (fy \wedge dx) = dy \vee fy = fy$. Hence $y \in F_d$.

Lemma 3.11: *Let d be an f -derivation on L and suppose $f(x \wedge y) = fx \wedge fy$ for all $x, y \in L$. Then $x \wedge y \in F_d$ for all $x \in F_d, y \in L$.*

Proof: Let $x \in F_d, y \in L$. Then

$$\begin{aligned} d(x \wedge y) &= (dx \wedge fy) \vee (fx \wedge dy) \\ &= (fx \wedge fy) \vee (fx \wedge dy) \\ &= fx \wedge (fy \vee dy) \\ &= fx \wedge fy \\ &= f(x \wedge y). \end{aligned}$$

Hence $x \wedge y \in F_d$.

Lemma 3.12: *If d is an isotone f -derivation on L, then $dx = d(x \vee y) \wedge fx$ for all $x, y \in L$.*

Proof: Let $x, y \in L$. Since d is an isotone f -derivation on L, $dx \le d(x \vee y) \le f(x \vee y)$.

Now,
$$\begin{aligned} dx &= d[(x \vee y) \wedge x]. \\ &= [d(x \vee y) \wedge fx] \vee [f(x \vee y) \wedge dx] \\ &= [d(x \vee y) \wedge fx] \vee dx \\ &= [d(x \vee y) \vee dx] \wedge fx \\ &= d(x \vee y) \wedge fx. \end{aligned}$$

Lemma 3.13: *Let d be an isotone f -derivation on L. If f is a decreasing function, then $x \vee y \in F_d$ for all $x, y \in F_d$.*

Proof: Let f be a decreasing function and $x, y \in F_d$. We have $x \le x \vee y$. Thus $f(x \vee y) \le fx = dx \le d(x \vee y)$. Therefore, by Lemma 3.8, $d(x \vee y) = f(x \vee y)$ and hence $x \vee y \in F_d$.

From Lemma 3.11 and Lemma 3.13, we get

Corollary 3.14: *Let d be an isotone f -derivation on L. If f is a constant function on L, then F_d is an ideal of L.*

Theorem 3.15: *Let m be a maximal element in L and d be an f -derivation on L. If $fm = m$ and $f(x \wedge y) = fx \wedge fy$ for all $x, y \in L$, then the following are equivalent.*

1. d is an isotone f -derivation on L.
2. $dx = dm \wedge fx$ for all $x \in L$.
3. $d(x \wedge y) = dx \wedge dy$ for all $x, y \in L$.

Proof:
(1) $\Rightarrow$ (2): Supose d is an isotone f -derivation on L and $x \in L$. Then $dx = d(m \wedge x) = (dm \wedge fx) \vee (fm \wedge dx)$. Thus $dm \wedge fx \le dx$. We have $x \wedge m \le m$. So that $dx \wedge fm \le (dx \wedge fm) \vee (fx \wedge dm) = d(x \wedge m) \le dm$.

Therefore, $dx = dx \wedge m \wedge fx = dx \wedge fm \wedge fx \le dm \wedge fx$. Hence $dx = dm \wedge fx$.

(2)$\Rightarrow$(3): For any $x, y \in L$,

$$\begin{aligned} dx \wedge dy &= dm \wedge fx \wedge dm \wedge fy \\ &= dm \wedge fx \wedge fy \\ &= dm \wedge f(x \wedge y) \\ &= d(x \wedge y). \end{aligned}$$

(3)$\Rightarrow$(1) is trivial.

Theorem 3.16: *Let d be an f-derivation on L. Then the following are equivalent.*

1. d is isotone f - derivation on L.
2. $dx \wedge d(x \wedge y) = dx \wedge dy \quad for \quad all \quad x, y \in L.$

Proof:

(1)$\Rightarrow$(2): Suppose d is an isotone f - derivation on L and $x, y \in L$. Then

$$\begin{aligned} d(x \wedge y) &= d(x \wedge y) \wedge dy \\ &= [(dx \wedge fy) \vee (fx \wedge dy)] \wedge dy \\ &= (dx \wedge fy \wedge dy) \vee (fx \wedge dy \wedge dy) \\ &= (dx \wedge dy) \vee (fx \wedge dy) \\ &= (dx \vee fx) \wedge dy \\ &= fx \wedge dy \end{aligned}$$

Hence $dx \wedge d(x \wedge y) = dx \wedge fx \wedge dy = dx \wedge dy$.

(2)$\Rightarrow$(1): Let $x, y \in L$ with $x \leq y$. By (1), $dx \wedge dy = dx \wedge d(x \wedge y) = dx \wedge dx = dx$. Therefore, $dx \leq dy$ and hence d is an isotone f-derivation .

Finally, we conclude this paper with the following theorem.

Theorem 3.17: *Let d be an f-derivation on L. If $f(x \vee y) = fx \vee fy$ for all $x, y \in L$, then the following are equivalent.*

1. $d(x \wedge y) = dx \wedge dy$ for all $x, y \in L$.
2. $d(x \vee y) = dx \vee dy$ for all $x, y \in L$.

Proof:

(1)$\Rightarrow$(2): Let $x, y \in L$. Then

$$\begin{aligned} dx &= d[(x \vee y) \wedge x] \\ &= [d(x \vee y) \wedge fx] \vee [f(x \vee y) \wedge dx] \\ &= [d(x \vee y) \wedge fx] \vee [(fx \wedge dx) \vee (fy \wedge dx)] \\ &= [d(x \vee y) \wedge fx] \vee [dx \vee (fy \wedge dx)] \\ &= [d(x \vee y) \wedge fx] \vee dx \\ &= [d(x \vee y) \vee dx] \wedge fx \end{aligned}$$

By (1), we get that $d(x \vee y) \wedge dx = d[(x \vee y) \wedge x] = dx$ and thus $d(x \vee y) \vee dx = d(x \vee y)$.

Hence $dx = d(x \vee y) \wedge fx$.

Again,

$$\begin{aligned} dy &= d[(x \vee y) \wedge y] \\ &= [d(x \vee y) \wedge fy] \vee [f(x \vee y) \wedge dy] \\ &= [d(x \vee y) \wedge fy] \vee [(fx \wedge dy) \vee (fy \wedge dy)] \\ &= [d(x \vee y) \wedge fy] \vee dy \\ &= [d(x \vee y) \vee dy] \wedge fy. \end{aligned}$$

Again by (1), we get that $d(x\vee y)\wedge dy = d[(x\vee y)\wedge y] = dy$ and thus $d(x\vee y)\vee dy = d(x\vee y)$ and hence $dy = d(x\vee y)\wedge fy$.

Now,

$$\begin{aligned} dx\vee dy &= [d(x\vee y)\wedge fx]\vee[d(x\vee y)\wedge fy] \\ &= d(x\vee y)\wedge(fx\vee fy) \\ &= d(x\vee y)\wedge f(x\vee y) \\ &= d(x\vee y). \end{aligned}$$

(2)$\Rightarrow$(1): Let $x, y\in L$.

Then

$$\begin{aligned} dx\wedge dy &= d[x\vee(x\wedge y)]\wedge d[(x\wedge y)\vee y] \\ &= [dx\vee d(x\wedge y)]\wedge[d(x\wedge y)\vee dy] \\ &= [d(x\wedge y)\vee dx]\wedge[d(x\wedge y)\vee dy] \\ &= d(x\wedge y)\vee(dx\wedge dy). \end{aligned}$$

Thus $d(x\wedge y)\le dx\wedge dy$. Hence by Lemma 3.7, we get that $d(x\wedge y) = dx\wedge dy$.

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