International Journal of Mathematical Archive-7(5), 2016, 128-133
Available online through www.ijma.info ISSN 2229 – 5046

BETWEEN REGULAR OPEN SETS AND OPEN SETS

S. PIOUS MISSIER, M. ANNALAKSHMI*[2]

[1]Assosiate Professor PG & Research Department of Mathematics,
V. O. Chidambaram College Tuticorin. India.

[2]Research Scholar, PG & Research Department of Mathematics,
V. O. Chidambaram College Tuticorin, India.

(Received On: 09-05-16; Revised & Accepted On: 31-05-16)

ABSTRACT

In this paper, we introduce a new class of sets called regular-open sets, this class of sets lies between the classes of regular open and open sets. We also study its fundamental properties and compare it with some other types of sets and we investigate further topological properties of sets.*

Mathematical Subject Classification: *540A5.*

Keywords and Phrases: *regular open set, generalized closure, regular*-open set, regular*-interior, regular*-closed set, regular*-closure.*

1. INTRODUCTION

In 1970 Levine [4] introduced generalized closed sets. In 1937, regular open sets were introduced and used to define the semi-regularization space of a topological space. Dunham [2] introduced the concept of generalized closure using Levine's generalized closed sets and defined new topology τ^* and studied some of their properties.

In this paper, we define a new class of sets, namely regular*-open sets, using the generalized closure operator *Cl** due to Dunham. We investigate fundamental properties of regular*-open sets. We also define regular*- interior point and regular*-interior of a subset. We also introduce the concept of regular*-closed sets and investigate many fundamental properties of regular*-closed sets. We also define the regular*-closure of a subset and study their properties.

2. PRELIMINARIES

Throughout this paper, (X, τ) will always denote topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X, τ), *Cl(A)* and *Int(A)* denote the closure and interior of A respectively.

Definition 2.1: A subset A of a topological space (X, τ) is said to be
- (i) **regular open**[11] if $A=Int(Cl(A))$.
- (ii) **α-open** [6] if $A \subseteq Int(Cl(Int(A)))$.
- (iii) **semi-open** [3] if there exits an open set G such that $G\subseteq A\subseteq Cl(G)$ equivalently $A \subseteq Cl(Int(A))$.
- (iv) **pre-open** [5] if $A \subseteq Int(Cl(A))$.

Definition 2.2: A subset A of a topological space (X, τ) is said to be
- (i) **regular closed** if X\A is regular open in X.(i.e) $A = Cl(Int(A))$.
- (ii) **α-closed** if $Cl(Int(Cl(A)))\subseteq A$.
- (iii) **semi-closed** if $Int(Cl(A))\subseteq A$.
- (iv) **pre-closed** if $Cl(Int(A))\subseteq A$.

Corresponding Author: M. Annalakshmi[2]*
[2]Research Scholar, PG & Research Department of Mathematics,
V. O. Chidambaram College Tuticorin, India.

Remark 2.3:
(i) The class of all regular open sets in (X, τ) is denoted by RO (X, τ).
(ii) The class of all regular closed sets in (X, τ) is denoted by RC (X, τ).

Definition 2.4: A subset A of a topological space (X, τ) is said to be
(i) **generalized closed**[4] (briefly g-closed) if *Cl(A)* $\subseteq U$ whenever $A \subseteq U$ and U is open in X.
(ii) **generalized open**[4](briefly g-open) if X\A is g-closed in X.

Definition 2.5: If A is a subset of a space (X, τ),
(i) The **generalized closure** of A [2] is defined as the intersection of all g-closed sets in X containing A and is denoted by *Cl*(A).*
(ii) The **generalized interior** of A[2] is defined as the union of all g-open sets in X contained in A and is denoted by *Int*(A).*
(iii) The **regular interior** of A is defined as the union of all regular open sets of X contained in A. It is denoted by *rInt(A).*
(iv) The **regular closure** of A is defined as the intersection of all regular open sets of X contained in A. It is denoted by *rInt(A).*

Theorem 2.6:
(i) Finite intersection of regular open sets is regular open.
(ii) Finite union of regular closed sets is regular closed.

Remark 2.7:
(i) The union of two regular open sets need not be regular open.
(ii) Every regular open set is open.
(iii) Every clopen set is regular open.
(iv) The intersection of two regular closed sets need not be regular closed.

Theorem 2.8: Let $\forall i \in [1 \dots n]$ and $n \in N$, A_i's are sub sets of a topological space (X, τ), then
(i) $Int(\bigcap_{i=1}^{n} A_i) = \bigcap_{i=1}^{n} Int(A_i)$,
(ii) $Cl(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} Cl(A_i)$.

Theorem 2.9: Let A be a subset of (X, τ), then
(i) *.Int(X\A) = X\Cl(A),*
(ii) *Cl(X\A) = X\Int(A),*
(iii) *X\(X\A) = A.*

Definition 2.10: A subset A of a topological space (X, τ) is called **clopen** if it is both open and closed in (X, τ).

3. REGULAR STAR OPEN SETS

Definition 3.1: A subset A of a topological space (X, τ) is said to be **regular*-open** if *A = Int(Cl*(A)).*

Notation 3.2: The set of all regular*-open sets (r*-open sets) in (X, τ) is denoted by R*O(X, τ) (or) R*O(X).

Remark 3.3: In any space (X, τ), ϕ and X are regular*-open sets.

Example 3.4: Let X = {a, b, c, d} and τ= {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, {a, b, c}, X}

In this space, regular open sets are {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, X} and regular*-open sets are {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, X}. That is RO(X) = R*O(X) ⊆τ

Example 3.5: Let X = {a, b, c, d, e} and τ = {ϕ, {a}, {b}, {a, b}, {b, c, d, e}, X}. In this space RO(X) = {ϕ, {a}, {b, c, d, e}, X}, R*O(X) = {ϕ, {a}, {b}, {a, b}, {b, c, d, e}, X}. That is RO(X) ⊆ R*O(X) = τ

Example 3.6: Let X = {a, b, c, d} and τ = {ϕ, {a}, {b, c, d}, X}. In this space, RO(X) = {ϕ, {a}, {b, c, d}, X} R*O(X) = {ϕ, {a}, {b, c, d}, X}. That is RO(X) = R*O(X) = τ

Example 3.7: Let X = {a, b, c, d, e} and τ = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, {a, b, c, d}, X} In this space, RO(X) = {ϕ, {a}, {b}, {c}, {a, b}, {a, c} {b, c}, X} and R*O(X) = { ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, X }.That is RO(X) ⊂ R*O(X) ⊂ τ

Remark 3.8: The union of two regular* open sets need not be regular*-open, as seen from the following example.

Example 3.9: Consider the space as in Example: 3.4. In this space (X, τ), the subsets A={a, b} and B={b, c} are regular*-open sets, but A∪B = {a, b, c} is not regular*- open.

Theorem 3.10: Intersection of any two regular*-open sets is regular*-open.

Proof: Let A and B be regular*-open sets then $A = Int(Cl^*(A)), B = Int(Cl^*(B))$. Consider, $Int(Cl^*(A\cap B)) = Int(Cl^*(X\backslash X\backslash(A\cap B))) = Int(X\backslash Int^*(X\backslash(A \cap B))) = Int(X\backslash Int^*((X\backslash A)\cup(X\backslash B))) = Int(Cl^*(X\backslash((X\backslash A)\cup(X\backslash B)))) = Int(Cl^*(X\backslash(X\backslash A))\cap Cl^*(X\backslash(X\backslash B))) = Int(Cl^*(A)\cap Cl^*(B)) = Int(Cl^*(A))\cap Int(Cl^*(B)) = A \cap B$. Hence A ∩ B is regular*-open set.

Theorem 3.11: R*O(X, τ) forms a topology on X if and only if it is closed under arbitrary union.

Proof: Follows from Remark 3.3 and Theorem 3.10

Theorem 3.12: Every regular open set is regular*-open.

Proof: Let A be a regular open set then $A = Int(Cl(A))$. Since A is regular open, it is clopen (i.e) A is closed and every closed set is generalized closed, hence $Cl(A) = Cl^*(A) \Rightarrow Int(Cl(A))=Int(Cl^*(A))$.Hence A is regular*-open.

Remark 3.13: Converse of the above theorem 3.12 need not be true, as seen from the following example.

Example 3.14: Consider the space as in Example 3.5, in this space (X, τ), the set {a, b} is regular*-open but not regular open.

Theorem 3.15: Every regular*-open set is open.

Proof: Let A is regular*-open then $A = Int(Cl^*(A))$. Now $Int(A) = Int(Int(Cl^*(A))) = Int(Cl^*(A)) = A$.*(i.e)* $Int(A) = A$. Hence A is open.

Remark 3.16: The converse of the above theorem 3.15 need not be true, as seen from the following example.

Example 3.17: Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X} In this space, R*O(X) = {ϕ, {a}, {b}, {a, b}, X}. Here, the set {a, b, c} is open but not regular*open.

Theorem 3.18: In any topological space (X, τ), RO(X, τ) ⊆ R*O(X, τ) ⊆ τ. That is, the class of regular*-open sets is placed between the class of regular open sets and the class of open sets.

Proof: Follows from Theorem 3.12 and Theorem 3.15

Theorem 3.19: Every regular*-open sets are pre-open.

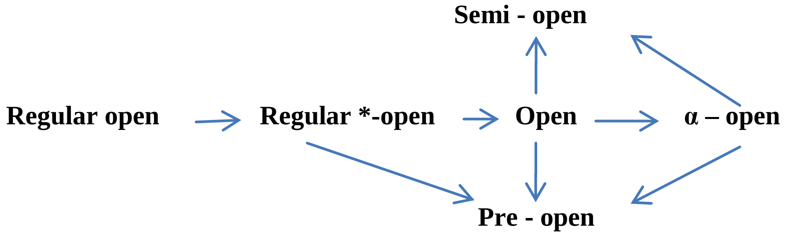
Proof: Let A be a regular*-open set, then $A = Int(Cl^*(A))$. Since $Cl^*(A) \subseteq Cl(A)$ Then $Int(Cl^*(A))\subseteq Int(Cl(A))$. Hence A is pre-open.

Remark 3.20: The converse of the above theorem 3.19 need not be true, as seen from the following example.

Example 3.21: Consider the space as in Example 3.7, in this space (X, τ), the subset {a, b, c, d} is pre-open but not regular*-open.

From the above discussions we have the following implication diagram

Diagram 3.22:



Definition 3.23: The **regular*-interior** of **A** is defined as the union of all regular*-open sets of X contained in A. It is denoted by *r*Int(A).*

Definition 3.24: Let **A** be a subset of X. A point x in X is called a **regular*-interior point** of A if A contains a regular*-open set containing x.

Remark 3.25: If A is any subset of X, *r*Int(A)* need not be regular*-open set, as seen from the following example.

Example 3.26: Consider the space as in Example 3.4, in this space (X, τ). Let A = {a, b, c} *r*Int(A)* = {a ,b, c}. But {a, b, c} is not regular*-open.

Theorem 3.27: In any topological space (X, τ), if A and B are subsets of X then the following hold:

- (i) $r^*Int(\phi) = \phi$,
- (ii) $r^*Int(X) = X$,
- (iii) $r^*Int(A) \subseteq A$,
- (iv) $A \subseteq B \Rightarrow r^*Int(A) \subseteq r^*Int(B)$,
- (v) $Int(r^*Int(A)) \subseteq Int(A)$,
- (vi) $rInt(A) \subseteq r^*Int(A) \subseteq Int(A) \subseteq A$,
- (vii) $r^*Int(A\cup B) \supseteq r^*Int(A) \cup r^*Int(B)$,
- (viii) $r^*Int(A\cap B) = r^*Int(A) \cap r^*Int(B)$.

Proof: (i), (ii), (iii) and (iv) follows from Definition 3.22 and (v) follows from Definition 3.22 and (iv) above. (vi) follows from Theorems 3.12 and 3.15. (vii) follows from (iv) above.(viii) follows from Theorem 2.8

Remark 3.28: In (vi) of Theorem 3.27, each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 3.29: In the space (X, τ), where X = {a, b, c, d, e} and τ = {ϕ, {a}, {b, c}, {a, b, c}, {a, b, c, d}, X}. Here RO(X) = {ϕ, {a}, {b, c}, X } and R*O(X) = {ϕ, {a}, {b, c}, {a, b, c}, X}. Let A = {a, b, c} then *rInt(A) = r*Int(A) = Int(A)* = {a, b, c} = A. Here *rInt(A) = r*Int(A) = Int(A)* = A. Let B = {a, b} then *Int(B)* = {a, b} and *rInt(B) = r*Int(B)* = {a}. Here $rInt(B) = r^*Int(B) \subsetneq Int(B) = B$.Let C = {a, b, c, e} then *rInt(C) = r*Int(C) = Int(C)* = {a, b, c}. Here $rInt(C) = r^*Int(C) = Int(C) \subsetneq C$.

Example 3.30: In the space (X, τ), where X = {a, b, c, d, e} and τ = {ϕ, {a}, {b}, {a, b}, {b, c, d, e}, X}. Here RO(X) = {ϕ, {a}, {b, c, d, e}, X} and R*O(X) = {ϕ, {a}, {b}, {a, b}, {b, c, d, e}, X}. Let A = {b, c} then *Int(A) = r*Int(A)* = {b} and *rInt(A)* = ϕ. Here $rInt(A) \subsetneq r^*Int(A) = Int(A) \subsetneq A$.

Remark3.31: The inclusion in (vii) of Theorem 3.27 may be strict and equality may also hold. This can be seen from the following examples.

Example 3.32: Consider the space (X, τ) as in Example 3.29. Let A = {a} and B = {b, c} then A∪B = {a, b, c}, *r*Int(A)*= {a}, *r*Int(B)* = {b, c}, *r*Int(A∪B)* ={a, b, c}. Here $r^*Int(A\cup B) = r^*Int(A) \cup r^*Int(B)$.Let C = {a,b} and D = {c, e} then C∪D = {a,b,c,e}, *r*Int(C)*={a}, $r^*Int(D)=\phi$,*r*Int(C∪D)*={a, b, c}. Here $r^*Int(C\cup D) \supsetneq r^*Int(C) \cup r^*Int(D)$.

4. REGULAR* CLOSED SET

Definition 4.1: The complement of regular*-open set is called **regular*-closed** set (r*-closed) *(i.e)* $A = Cl(Int^*(A))$

Notation4.2: The set of all regular*-closed sets in (X, τ) is denoted by R*C(X, τ) (or) R*C(X).

Remark 4.3: In any space (X, τ), ϕ and X are regular*-closed sets.

Theorem 4.4: Union of any two regular*-closed sets is regular*-closed.

Proof: Let A and B be regular*-closed sets then $A = Cl(Int^*(A))$ and $B = Cl(Int^*(B))$.Now

consider, $Cl(Int^*(A\cup B)) = Cl(Int^*(X\setminus X\setminus(A\cup B))) = Cl(X\setminus Cl^*(X\setminus(A \cup B))) = Cl(X\setminus Cl^*((X\setminus A)\cap(X\setminus B)))$ $=Cl(Int^*(X\setminus((X\setminus A) \cap(X\setminus B))))= Cl(Int^*(X\setminus(X\setminus A)) \cup Int^*(X\setminus(X\setminus B))) =Cl(Int^*(A)\cup Int^*(B)) = Cl(Int^*(A))\cup Cl(Int^*(B))= A\cup B$. Hence A∪B is regular*-closed.

Remark 4.5: The intersection of two regular*-closed sets need not be regular*-closed, as seen from the following example.

Example 4.6: Consider X = {a, b, c, d} and τ = {ϕ,{a},{b},{c},{a, b},{b, c},{a, c},{a, b, c}, X}. In this space, R*C(X) = {ϕ, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}, X}. Here A = {a, d} and B = {b, d} are regular*-closed but A∩B = {d} is not regular*-closed.

Theorem 4.7: Every regular*-closed set is closed.

Proof: Let A be regular*-closed, then X\A is regular*-open. By theorem 3.15, X\A is open.⇒ A is closed.

Remark 4.8: The converse of the above theorem 4.7 need not be true, as seen from the following example.

Example 4.9: Consider X={a, b, c, d}, τ = {ϕ, {a},{b}, {a, b}, {a, b, c}, X}. In this space, R*C(X)={ϕ,{c, d}, {a, c, d}, {b, c, d}, X}. Here {d} is closed but not regular*closed.

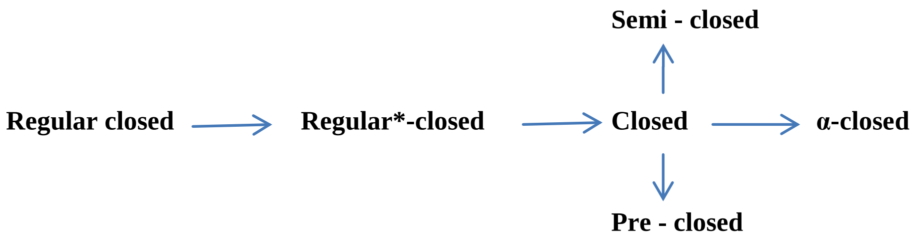
Theorem 4.10: Every regular closed set is regular*closed.

Proof: Let A be a regular closed set, then X\A is regular open. By theorem 3.12, X\A is regular*-open⇒ A is regular*-closed.

Remark 4.11: The converse of the above theorem 4.10 need not be true, as seen from the following example.

Example4.12: Consider X= {a, b, c, d, e} and τ = {ϕ, {a}, {b}, {a.b}, {b, c, d, e}, X}. In this space, RC(X) ={ϕ, {a},{b, c, d, e}, X} and R*C(X) ={ϕ, {a}, {c, d, e}, {a, c, d, e}, {b, c, d, e}, X}. Here {c, d, e}, {a, c, d, e} are regular*-closed but not regular closed. From the above discussions we have the following implication diagram

Diagram 4.13:



Definition 4.14: The **regular*closure** of A is defined as the intersection of all regular*closed sets of X containing A. It is denoted by *r*Cl(A).*

Definition 4.15: Let A⊆X. An element $x \in X$ is called **regular*-adherent point** of A if every regular*-open set in X containing x intersects A.

Definition 4.16: Let A⊆X. An element $x \in X$ is called a **regular*-limit point** of A if every regular*-open set in X containing x intersects A in a point different from x.

Definition 4.17: The set of all regular*-limit points of A is called the **regular*-Derived set** of A. It is denoted by $D_{r*}[A]$.

Remark 4.18: If A is any subset of X, *r*Cl(A)* need not be a regular*closed set, as seen from the following example.

Example 4.19: In the space (X, τ), X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, {a, b, c}, X}. Here R*C(X) = {ϕ, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}, X}. Let A = {d} then *r*Cl(A)* = {d} but {d} is not regular*-closed.

Theorem 4.20: In any topological space (X,τ), the following result hold:
- (i) $r^*Cl(\phi) = \phi$,
- (ii) $r^*Cl(X) = X$,
- (iii) $A \subseteq r^*Cl(A)$,
- (iv) $A \subseteq B \Rightarrow r^*Cl(A) \subseteq r^*Cl(B)$, if A and B are subsets of X,
- (v) $A \subseteq Cl(A) \subseteq r^*Cl(A) \subseteq rCl(A)$,
- (vi) $r^*Cl(A \cup B) = r^*Cl(A) \cup r^*Cl(B)$, if A and B are subsets of X,
- (vii) $r^*Cl(A \cap B) \subseteq r^*Cl(A) \cap r^*Cl(B)$, if A and B are subsets of X,
- (viii) $Cl(A) \subseteq Cl(r^*Cl(A))$.

Proof: (i), (ii), (iii) and (iv) follow from Definition 4.14. (v) follows from Theorem 4.10 and Theorem 4.7. (vi) follows from Theorem 2.8. (vii) follow from (iv) above. From (iii) above, we have $A \subseteq r^*Cl(A)$ and hence $Cl(A) \subseteq Cl(r^*Cl(A))$.

Remark 4.21: In (v) of Theorem 4.20, each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 4.22: In the space (X, τ), X = {a, b, c, d, e} and τ = {ϕ, {a}, {b, c}, {a, b, c}, {a, b, c, d}, X}. Here RC(X) = {ϕ, {a, d, e}, {b, c, d, e}, X} and R*C(X) = {ϕ, {d, e}, {a, d, e}, {b, c, d, e}, X}. Let A = {a, d, e} then *Cl(A)* = {a, d, e}, *rCl(A)* = {a, d, e} and *r*Cl(A)* = {a, d, e}. Here $A = Cl(A) = r^*Cl(A) = rCl(A)$. Let B = {d, e} then *Cl(B)* = {d, e}, *rCl(B)* = {a, d, e}, and *r*Cl(B)* = {d, e}. Here $B = Cl(B) = r^*Cl(B) \subsetneq rCl(B)$. Let C = {b, c, d} then *Cl(C)* = {b, c, d, e}, *rCl(C)* = {b, c, d, e}, and *r*Cl(C)* = {b, c, d, e}. Here $C \subsetneq Cl(C) = r^*Cl(C) = rCl(C)$. Let D = {c, d} then *Cl(D)* = {c, d, e}, *rCl(D)* = {b, c, d, e}, and *r*Cl(D)* = {b, c, d, e}. Here $D \subsetneq Cl(D) \subsetneq r^*Cl(D) = rCl(D)$.

Remark4.23: The inclusion in (vii) of Theorem 4.20 may be strict and equality may also hold. This can be seen from the following examples.

Example 4.24: In the space (X, τ), X = {a, b, c, d, e, f, g} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, e}, {a, c, d},{b, f, g}, {a, b, c, d}, {a, b, f, g}, {a, b, c, d, e}, {a, b, e, f, g}, {a, b, c, d, f, g}, X}. Here R*C(X) = {ϕ, {c, d}, {f, g}, {a, c, d, e}, {b, e, f, g}, {c, d, f, g}, {a, c, d, e, f, g}, {b, c, d, e, f, g}, X}. Let A = {a, d, e}, B = {a, d, f} then $A \cap B$= {a, d}, *r*Cl(A)* = {a, c, d, e}, *r*Cl(B)* = {a, c, d, e, f, g} and $r^*Cl(A \cap B)$ = {a, c, d, e}. Here $r^*Cl(A \cap B) = r^*Cl(A) \cap r^*Cl(B)$. Let C = {a, d}, D = {d, e} then $C \cap D$ = {d}, *r*Cl(C)* = {a, c, d, e}, *r*Cl(D)* = {c, d, e} and $r^*Cl(A \cap B)$ = {c, d}. Here $r^*Cl(A \cap B) \subsetneq r^*Cl(A) \cap r^*Cl(B)$.

REFERENCES

1. Chandrasekhara Rao. K, General Topology (Part II), *Viswanathan Printers, Madras, 1990.*
2. Dunham.W., A New Closure Operator for Non- T_1 Topologies, *Kyungpook Math., J.22 (1982), 55-60.*
3. Levine.N, Semi-open sets and semi-continuity in topological space, *Amer. Math. Monthly, 70(1) (1963), 36-41.*
4. Levine.N, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo, 19(2) (1970), 89-96.*
5. Mashhour.A.S., Abd El-Monsef.M.E., and El-Deeb.S.N., On Precontinuous and weak precontinuous mappings. *Proc. Math, Phys, Soc. Egypt, 53 (1982), 47-53.*
6. Njastad.O., On Some Classes of Nearly Open sets, *Pacific J. Math. 15(1965) No. (3), 961 970.*
7. Palaniappan. N and K. Chandrasekara Rao, Regular generalized closed sets, *Kyungpook Math., J.33 (1993), 211-219.*
8. Pious Missier.S and Robert.A, A New class of nearly open sets, *IJMA., 3(7) (2012), 2575-2582.*
9. Pious Missier.S and Robert.A, On Semi*-Closed Sets, *Asian journal of current Engineering and Maths1:4 Jul-Aug (2012) 173-176.*
10. Pious Missier.S and Robert.A, New notions via semi-star-alpha-open sets (communicated)
11. Stone. M, Application of the theory of Boolean rings to general topology, *Trans.Amer.Math.Soc., 41(1937), 374-481.*
12. Willard.S, General Topology, *Addison-wesley Publishing Company.Inc, 1970.*

Source of support: Nil, Conflict of interest: None Declared