International Journal of Mathematical Archive-7(5), 2016, 78-80
Available online through www.ijma.info ISSN 2229 – 5046

MATRIX EXPONENTIAL AND INEQUALITIES

MOHAMMED ABDULLAH SALMAN[1*], V. C. BORKAR[2]

[1,2]Department of Mathematics & Statistics,
Yeshwant Mahavidyalaya, Swami Ramamnand Teerth Marthwada University, Nanded, India.

(Received On: 02-05-16; Revised & Accepted On: 17-05-16)

ABSTRAC

In this paper we discussed inequalities involving quantities associated with one or more positive definite matrices, and prove some inequalities involving commuting exponential matrices and we give some example as application.

Keywords: *Matrix Exponential, Commuting Matrix, Non-commuting Matrix, inequalities.*

INTRODUCTION

The matrix exponential is a very important subclass of functions of matrices that has been studied extensively in the last 50 years. Throughout this paper we define $A \in C^{n \times n}$ as a square matrix, we write $trac\ A$ to denote the trace of A and $\det\ A$ for the determinant of A. If A is positive definite we write it $A > 0$, and when A is semi positive definite we write it $A \geq 0$. These matrices appear in many problems of physics, such as in applications of the theory of small oscillations of a mechanical system, in relativistic quantum mechanics or in the theory of algebraic models in quantum physics [3] Recently, there has been substantial interest in matrix inequalities. Inequalities are useful in many applications. For example, trace inequalities naturally arise in control theory and in communication systems with multiple input and multiple output and Some matrix inequalities used in statistical mechanics

MAIN RESULTS

The main results here are to show that this inequality is true when we apply the exponential matrix; we want to prove the following theorems.

Theorem 1: Let A, B be two positive definite matrices. Then

$$n(\det A.\det B)^{\frac{m}{n}} \leq trace(A^m B^m)\ iff\ e^{n(\det A.\det B)} \leq e^{trace(A^m B^m)}$$

for any positive integer m.

Proof: we start from the first part if it is true then the second part directly true, since A is square matrix and positive definite then it is diagonalizable, then there exists an orthogonal matrix S where the columns of this matrix is the eigenvectors corresponding eigenvalues of A and there exists a diagonal matrix Λ such that $\Lambda = S^T AS$ .Now the matrix A has the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots.., \lambda_n$ then we can write it as a diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2 \lambda_3, \ldots., \lambda_n)$

Corresponding Author: Mohammed Abdullah Salman[1]*
[1,2]Department of Mathematics & Statistics, Yeshwant Mahavidyalaya,
Swami Ramamnand Teerth Marthwada University, Nanded, India.

Now, let $b_{11}(m),............,b_{nn}(m)$ denote to the elements of this expression $(SBS^T)^m$, then by using these expressions we get

$$\frac{1}{n}trace(A^m B^m)=\frac{1}{n}trace(S\Lambda^m S^T B^m)$$

$$=\frac{1}{n}trace(\Lambda^m S^T B^m S)$$

$$=\frac{1}{n}trace(\Lambda^m (S^T B^m S))$$

$$=\frac{1}{n}trace(\Lambda^m (S^T BS)^m)$$

whrer $A=S\Lambda S^T$ and replacing Λ ,$S^T BS$ by quantities we get

$$=\frac{1}{\mathrm{n}}[\lambda_1^m b_{11}(m)+\lambda_2^m b_{22}(m)+.......+\lambda_n^m b_{nn}(m)] \tag{1}$$

From [9] we use the arithmetic-mean geometric- mean inequality property to get

$$\frac{1}{n}trace(A^m B^m)\geq\left[\lambda_1^m\lambda_2^m.......\lambda_n^m\right]^{\frac{1}{n}}\ \left[b_{11}(m)b_{22}(m).....b_{nn}(m)\right]^{\frac{1}{n}} \tag{2}$$

and we know from [2] that the determinant is $\det A\leq a_{11}a_{22}.......a_{nn}$ for any positive definite matrix A (Hadamard's inequality) If A is positive semi definite, then $\det A\leq\prod_{i=1}^{n}a_{ii}$

Furthermore, when A is positive definite, then equality holds if and only if A is diagonal.

To conclude that $\det(S^T BS)^m\leq b_{11}(m)b_{22}(m).......b_{nn}(m)$ and we get

$\det\Lambda^m\leq\lambda_1^m.....,\lambda_n^m$. By return to (2) we get

$$\frac{1}{n}trace(A^m B^m)\geq\left[\det\Lambda^m\right]^{\frac{1}{n}}.\left[\det(S^T BS)^m\right]^{\frac{1}{n}}$$

$$=[\det(S^T AS)]^{\frac{\mathrm{m}}{\mathrm{n}}}.[\det(S^T BS)]^{\frac{\mathrm{m}}{\mathrm{n}}}$$

$$=[\det(A).\det(B)]^{\frac{\mathrm{m}}{\mathrm{n}}}$$

Here we use that fact $A>0$, $B>0$, this proof the first part. Now we apply exponential matrix to another part we get

$$e^{n(\det A.\det B)}\leq e^{trace(A^m B^m)}$$

this means the proof complete.

Corollary: Let A and X be positive definite $n\times n$ matrices such that $\det X=1$. Then

$$n(\det A)^{\frac{1}{n}}\leq trace(AX)\ iff\ e^{n(\det A)^{\frac{1}{n}}}\leq e^{trace(AX)}$$

Proof: Take $B=X$ and $m=1$ in the previous theorem and the proof will be done as we desired

Theorem 2: Let A, B be two positive definite matrices and $AB=BA$ are commute. Then

$$if\ \ 2^{(m-1)n}\ \det(A^m+B^m)\geq(\det(A+B))^m\ \text{then}\ e^{[2^{(m-1)n}\det(A^m+B^m)]}\geq e^{[(\det(A+B)^m]}$$

iff

$$if\ \ 2^{(m-1)}\ trace(A^m+B^m)\geq(trace(A+B))^m\ \text{then}\ \mathrm{e}^{[2^{(m-1)}\ trace(A^m+B^m)]}\geq e^{[(trace(A+B))^m]}$$

for any positive integer m.

Proof: we start to prove the first one (a), to prove this part we it is enough to prove

$$\frac{A^m+B^m}{2} \geq \left(\frac{A+B}{2}\right)^m \quad (*)$$

Now for any pair of commuting positive definite matrices A, B. We use induction method to prove (*). it is clear that inequality (*) holds true for $m=2$. let us assume that (*) is true for $m=k$. We have to prove (*) for $m=k+1$. Indeed, since

$$\frac{A^k+B^k}{2}\cdot\frac{A+B}{2}=\frac{A+B}{2}\cdot\frac{A^k+B^k}{2}$$

it follows that

$$\begin{aligned}\left(\frac{A+B}{2}\right)^{k+1} &\leq \frac{A+B}{2}\cdot\frac{A^k+B^k}{2}\\ &=\frac{A^{k+1}+B^{k+1}}{2}-\frac{A^{k+1}+B^{k+1}}{4}+\frac{BA^k+AB^k}{4}\\ &=\frac{A^{k+1}+B^{k+1}}{2}-\frac{A^{k+1}+B^{k+1}-BA^k-AB^k}{4}\\ &=\frac{A^{k+1}+B^{k+1}}{2}-\frac{(A^k-B^k)(A-B)}{4} \quad (**)\end{aligned}$$

Now, the equality

$(A^k-B^k)(A-B)=(A^{k-1}+A^{k-2}B+.....+B^{k-1})(A-B)^2$ for A, B two positive definite matrices and $AB=BA$ are commute this implies that $AB \geq 0$.

Now we suppose that $T=A^{k-1}+A^{k-2}B+.....+B^{k-1} \geq 0\left[0,1\right]$, then $T(A-B)^2=(A-B)^2T$ then we have $(A^k-B^k)(A-B)\geq 0$, therefore from(**) we get

$$\left(\frac{A+B}{2}\right)^{k+1} \leq \frac{A+B}{2}\cdot\frac{A^k+B^k}{2}$$

this mean the proof vomplete for the first part from (a) and the second part is directly.

For second part (b) we can prove it directly from (*).

Remark: The condition $AB=BA$ in inequality (*) is essential.

REFERENCES

1. G.H. Hardy, J.E. Littlewood And G. Polya, 1952, Inequalities, Cambridge Univ, Press.
2. R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, 1985, ISBN 0-521-30586-2, xiii+561 pp.
3. E.H. LIEB AND W.E. THIRRING, Inequalities for the moments of the eigenvalues of the Schrodinger Hamiltonian and their relation to Sobolev inequalities, *Studies in Mathematical Physics, Essays in Honor of Valentine*, (1976), Bartmann, Princeton, N.J., 269–303.
4. J.R. Magnus And H. Neudecker, John Wiley & Sons, 1990, *Matrix Differential alculus with Applications in scitsitatSand Econometrics*.
5. M. Marcus And H. Minc, 1964, Boston, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Inc.
6. N. J. Higham, Functions of Matrices Theory and Computation, SIAM, Society for Industrial and Applied Mathematics, Philadelphia, 2008. ISBN 978-0- 89871- 646-7. xx+425 pp.

Source of support: Nil, Conflict of interest: None Declared