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ON GENERALIZED INVERSES OF q-k-EP MATRICES

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ABSTRACT

In this chapter, existence of the group inverse for q-k-EP matrices is investigated. Equivalent conditions for various generalized inverses of a q-k-EP$_r$ matrix to be q-k-EP$_r$ are determined. Validity of the reverse order law for the Moore-Penrose inverse of the product of q-k-EP$_r$ matrices is discussed.

Keywords: *Moore-Penrose Inverse, Quaternion matrix, Range hermitian k-EP matrices, Generalized inverses of matrices.*

1. INTRODUCTION

The algebra H of real quaternion, which is a four- dimensional non-commutative algebra over real number field R with canonical basis $1, i, j, k$ satisfying the conditions,

$$i^2 = j^2 = k^2 = ijk = -1 \text{ that implies } ij = -ji = k, jk = -kj = i \text{ and } ki = -ik = j.$$

The elements in H can be written in a unique way as, $\alpha = a + bi + cj + dk$, where a, b, c and d are real numbers, i.e., $H = \{\alpha = a + bi + cj + dk \mid a, b, c, d \in R\}$.

The conjugate of α is defined as $\bar{\alpha} = a - bi - cj - dk$, and the norm $|\alpha| = \sqrt{\alpha\bar{\alpha}}$ for $0 \neq \alpha \in H$, $\alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|^2}$.

We consider K is a permutation matrix associated with the permutation
$k(x) = (S_n)$, where S = {1,2,…,n}. Also $K^2 = I, \overline{K} = K^T = K^* = K^{-1} = K$.

A matrix has an inverse only if it is square, and even then only if it is non-singular, or in other words, if its columns (or rows) are linearly independent. By a generalized inverse of a given matrixAwe shall mean a matrix X associated in some way with A that (i) exists for a class of matrices larger than the class of non-singular matrices, (ii) has some of the properties of the usual inverse, and (iii) reduces to the usual inverse whenA is non-singular.

A generalized inverse of A is any matrix satisfying AXA=A. If A were nonsingular, multiplication by A^{-1} both on the left and on the right would give at once X = A^{-1}.

NOTATIONS AND PRELIMINARIES

In this section, the notations, definitions and Theorems used in the thesis are given. Throughout, it is concerned with complex square matrices.

$H_{n\times n}$: The space of nxn quaternion matrices of order n.
H_n : The space of quaternion n-tuples.
I_n : Identity matrix of appropriate size.
V : Permutation matrix with units in the secondary diagonal.

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For $A\in H_{n\times n}$,
dim(A) : Dimension of A.
det(A) : Determinant of A.
rk(A) : Rank of A is the maximum number of linearly independent rows or columns of A.
R(A) : Range space of A = { $y\in H_n$ / y= Ax for some $x\in H_n$ }.
N(A) : Null space of A = { $x\in H_n$ / Ax = 0 }.
A^T : The transpose of A.
A^S : The secondary transpose of A.
$\overline{A}$: The conjugate of A.
A^* : The conjugate transpose of A.
$\overline{A}^S$: The conjugate secondary transpose of A.
A^- : 1- inverse of A, is a solution of the equations AXA= A.
$A^=$: {1, 2} inverse of A, is solution of the equations XA=A and XAX = X.
A{1} : The set of all 1-inverses of A.
A{2} : The set of all 2-inverses of A.
A{1, 2} : The set of all {1,2 } inverses of A.
A{1, 2, 3}: The set of all {1,2,3} inverses of A, that is the set ofall solutions of the equations
AXA = A,XAX = X and $(AX)^*$ = (AX).
A{1,2,4} : The set of all { 1,2,4 } inverses of A, that is the set of all solutions of the equations
AXA = A, XAX =Xand$(XA)^*$ = (XA).
$A^\dagger$: Moore-Penrose inverse of A is the unique solution of the equations
AXA = A, XAX = X, $(AX)^*$ = (AX)and $(XA)^*$ = (XA). $A^\dagger$ exists is unique
$A^\#$: Group inverse of A, satisfyingthe equations AXA = A, XAX = X, XA = AX. If $A^\#$ exists, then it is unique.
$A\geq B$: A is greater than or equal to B.
$A \pm B$: Parallel sum of A and B.

TYPES OF MATRIX A DEFINITIONS

Symmetric matrix	$a_{ij} = a_{ji}$(or) $A = A^T$
Skew-Symmetric	$a_{ij} = -a_{ji}$(or) $A = - A^T$
Hermitian	$\bar{a}_{ij} = a_{ji}$(or) $A = A^*$
Skew –Hermitian	$\bar{a}_{ij} = -a_{ji}$(or) $A = - A^*$
Secondary Hermitian	$A = \overline{A}^S$
Secondary Skew –Hermitian	$A = -\overline{A}^S$
Idempotent	$A^2 = A$
EP or range hermitian	$N(A) = N(A^*)$ (or) $R(A) = R(A^*)$
EP_r	$N(A) = N(A^*)$ and rk(A) = r (or) $R(A) = R(A^*)$ and rk(A) = r

Throughout ‘V’ refers as a permutation matrix with units in the secondary diagonal and the following results.

Theorem 1.1: [1] For A, $B\in H_{n\times n}$ the following statements hold:

(i) $R(A^\dagger) = R(A^*)$ and $N(A^\dagger) = N(A^*)$.

(ii) $R(A) = R(B) \Leftrightarrow AA^\dagger = BB^\dagger$.

Theorem 1.2: [p.162, [1]] Let $A\in H_{n\times n}$.Then group inverse $A^\#$exists $\Leftrightarrow$ rk(A) = rk(A^2).

Theorem 1.3: [p.164, [1]] Let $A\in H_{n\times n}$. Then A is EP $\Leftrightarrow A^\# = A^\dagger$ when $A^\#$ exists.

2. q-k-EP GENERALIZED INVERSES

In this section, equivalent conditions for various generalized inverses of a q-k-EP_r matrix to be q-k-EP_r are determined. Generalized inverses belonging to the sets A{1,2}, A{1, 2, 3} and A{1, 2, 4} of a q-k-EP_r matrix A are characterized.

In (1), it is shown that A is q-k-EP_r and only if $A^\dagger$ is q-k-EP_r. Thus, the q-k-EP_r property of complex matrices is preserved for its Moore -Penrose inverses. However, all other generalized inverses of a q-k-EP_r matrix need not be q-k-EP_r. For instance,

$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ with $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Here A is q-k-$EP_1$.

But $A^{-} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ is 1- inverse of A, which is not q-k-EP_1.

A generalized inverse A=∈A{1, 2} is shown to be q-k-EPr whenever Ais q-k-EPr under certain conditions in the following way.

Theorem 2.1: Let $A \in H_{nxn}$, $X \in A\{1, 2\}$ and XA, AX are q-k-EPr matrices. Then A is q-k-EPr⇔ X is q-k-EPr.

Proof: Since AX and XAare q-k-EP_r, (By[6])

We have $R(AX) = R(V(AX)^*)$ and $R(XA) = R(V(XA)^*)$.

Since $X \in A\{1, 2\}$, we have AXA = A, XAX = X.

Now, $R(A) = R(AX)$
$= R(V(AX)^*)$
$= R(VX^*A^*)$
$= R(VX^*)$.

$R(VA^*) = R(VA^*X^*)$
$= R(V(XA)^*)$
$= R(XA)$
$= R(X)$.

Now, A is q-k-EP_r ⇔ $R(A) = R(VA^*)$ and rk(A) = r
⇔ $R(VX^*) = R(X)$ and rk(A) = rk(X) = r
⇔ X is q-k-EP_r.

Hence the Theorem.

Remark 2.2: In the above theorem, the conditions that both AX and XA to be q-k-EP_r are essential.

For instance, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

A is q-k-EP_1. $X = A^{=} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in A\{1, 2\}$

$AX = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $XA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

AX and XA are not q-k-EP_1 .Also X is not q-k-EP_1.

Now, we show that generalized inverses belonging to the sets A{1, 2, 3} and A{1, 2, 4} of a q-k-EP_r matrix A is alsoq-k-EP_r under certain conditions in the following Theorems.

Theorem 2.3: Let $A \in H_{n\times n}$, $X \in A\{1, 2, 3\}$, R(X) = R(A*). Then A is q-k-EPr⇔X is q-k-EPr.

Proof: Since $X \in A\{1, 2, 3\}$, we have AXA = A, XAX = X, (AX)* = AA. Therefore,
$R(A) = R(AX) = R((AX)^*) = R(A^*A^*) = R(X^*)$.

$R(X) = R(A^*) \Rightarrow XX^{\dagger} = A^*(A^*)^{\dagger}$ [By Theorem (1.1)]
$\Rightarrow XX^{\dagger} = A^*(A^{\dagger})^*$
$\Rightarrow XX^{\dagger} = (A^{\dagger}A)^*$
$\Rightarrow XX^{\dagger} = A^{\dagger}A$
$\Rightarrow VXX^{\dagger}V = VA^{\dagger}AV$
$\Rightarrow (VX)(VX)^{\dagger} = (AV)^{\dagger}(AV)$

$\Rightarrow (VX)(VX)^{\dagger} = (AV)^{*}((AV)^{*})^{\dagger}$
$\Rightarrow (VX) = R((AV)^{*})$
$\Rightarrow R(VX) = R(VA^{*})$.

A is q-k-$EP_r \Leftrightarrow$ $R(A) = R(VA^{*})$ and $rk(A) = r$.
$\Leftrightarrow R(X^{*}) = R(VX)$ and $rk(A) = rk(X) = r$.
$\Leftrightarrow$X is q-k-EP_r. (By[6])

Hence the Theorem.

Theorem 2.4: Let $A\in H_{n\times n}$, $X\in A\{1, 2, 4\}$, $R(A) = R(X^*)$. Then A is q-k-EPr$\Leftrightarrow$X is q-k-EPr.

Proof: Since $X\in A\{1, 2, 4\}$, we have $AXA = A$, $XAX = X$, $(X A)^* = XA$.

Also $R(A) = R(X^{*})$.

Now, $R(VA^{*}) = R(VA^{*}X^{*})$
$= R(V(XA)^{*})$
$= R(V(XA))$
$= R(VX)$.

A is q-k-$EP_r \Leftrightarrow R(A) = R(VA^{*})$ and $rk(A) = r$
$\Leftrightarrow R(X^{*}) = R(VX)$ and $rk(A) = rk(X) = r$
$\Leftrightarrow$X is q-k-EP_r (By[6])

Hence the Theorem.

Remark 2.5: In particular, if $X = A^{\dagger}$ then $R(A^{\dagger}) = R(A^{*})$ holds, Hence A is q-k-EP_r is equivalent to $A^{\dagger}$ is q-k-EP_r.

3. GROUP INVERSE OF q-k-EPMATRICES

In this section, the existence of the group inverse for q-k-EP matrices under certain condition is derived.

It is well kwon that, for an EP matrix, group inverse exists and coincides with its Moore-Penrose inverse. However, this is not the case for a q-k-EP matrix. For example,

Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

A is q-k-EP_1 matrix, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $rk(A) = rk(A^2)$.

Therefore, [By Theorem 1.2], group inverse $A^{\neq}$ does not exists for A.

Here, it is proved that for a q-k-EP matrix A, if the group inverse exists, it is also a q-k-EP matrix.

Theorem 3.1: Let $A\in H_{n\times n}$ be q-k-EPr and $rk(A) = rk(A^2)$. Then A# exists and is q-k-EPr.

Proof: Since $rk(A) = rk(A^2)$, [By Theorem 1.2], $A^{\#}$ exists for A. To show that $A^{\#}$ is q-k-EP_r, it is enough to show that $R(A^{\#}) = R(V(A^{\#})^{*})$.

Since, $AA^{\#} = A^{\#}A$, we have , $R(A) = R(AA^{\#})$
$= R(A^{\#}A)$
$= R(A^{\#})$.

$AA^{\#}A = A \Rightarrow A^{*} = A^{*}(A^{\#})^{*}A^{*}$
$\Rightarrow VA^{*} = VA^{*}(A^{\#})^{*}A^{*}$

Therefore, $R(VA^{*}) = R(VA^{*}(A^{\#})^{*}A^{*})$
$= R(VA^{*}(A^{\#})^{*})$
$= R(V(A^{\#}A)^{*})$

$= R(V(AA^{\#})^{*})$
$= R(V(A^{\#})^{*} A^{*})$
$= R(V(A^{\#})^{*})$.

Now, A is q-k-EP_r $\Rightarrow R(A) = R(V A^{*})$ and $rk(A) = r$
$\Rightarrow R(A^{\#}) = R(V(A^{\#}))$ and $rk(A) = rk(A) = r$
$\Rightarrow A^{\#}$ is q-k-EP_r.

Hence the Theorem.

Remark 3.2: In the above Theorem the condition that $rk(A) = rk(A^2)$ is essential.

Example 3.3:
Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$VA = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ is $EP_1 \Rightarrow$ A is q-k-EP_1.

$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $rk(A^2) = 0 \Rightarrow rk(A) \neq rk(A^2)$.

Therefore, $A^{\#}$ does not exist for a q-k-EP matrix A.

Thus, for a q-k-EP matrix A, if $A^{\#}$ exists then it is also q-k-EP_r.

Theorem 3.4: For $A \in H_{n\times n}$, if A# exists then, A is q-k-EP $\Leftrightarrow (VA)^{\#} = A^{\dagger} V$.

Proof:
A is q-k-EP $\Leftrightarrow$ V A is EP (By[6])
$\Leftrightarrow (V A)^{\#} = (V A)^{\dagger}$ [By Theorem (1.3)]
$\Leftrightarrow (V A)^{\#} = A^{\dagger} V$ (By [6])

Hence the Theorem.

Theorem 3.5: For $A \in H_{n\times n}$, A is q-k-EPr $\Leftrightarrow A^{\dagger}$ = V(Polynomial in AV) $\Leftrightarrow A^{\dagger}$ = (Polynomial in VA)V.

Proof: It is clear that if $(VA)^{\dagger} = f(VA)$ for some polynomial f(X), then VA commutes with $(VA)^{\dagger}$
$\Rightarrow (V A)(V A)^{\dagger} = (V A)^{\dagger} (V A)$
$\Rightarrow (V A)(A^{\dagger} V) = (A^{\dagger} V)(V A)$
$\Rightarrow V AA^{\dagger} V = A^{\dagger} A$
$\Rightarrow V AA^{\dagger} = A^{\dagger} AV$
$\Rightarrow$ A is q-k-EP_r .

Conversely, Let A be q-k-EP_r, then $V AA^{\dagger} = A^{\dagger} AV$ and $V A^{\dagger} A = AA^{\dagger} V$.

Now, we will prove: $A^{\dagger}$ can be expressed as V(Polynomial in AV) and (Polynomial in VA)V

Let, $(VA)^{s} + \lambda_1(VA)^{s+1} + \lambda_2 (VA)^{s+2} + \dots + \lambda_q(VA)^{s+q} = 0$, be the minimum polynomial of VA. Then s=0 or s = 1.

For suppose that $s \geq 2$, then
$(VA)^{\dagger} [(VA)^{s} + \lambda_1(VA)^{s+1} + \dots + \lambda_q(VA)^{s+q}] = 0$;

Hence
$[(VA)(VA)^{\dagger} (VA)](VA)^{s-2}+\lambda_1[(VA)(VA)^{\dagger} (VA)](VA)^{s-1}+\dots+\lambda_q[(VA)(VA)^{\dagger} (VA)](VA)^{s+q-2} = 0$.

Thus, $(VA)^{s-1} + \lambda_1(VA)^s + \ldots + \lambda_q(VA)^{s+q-1} = 0$

which is a contradiction.

If s = 0 then $(VA)^{\dagger} = (VA)^{-1} = -\lambda_1 I - \lambda_2(VA) - \ldots - \lambda_q(VA)^{q-1}$

$A^{\dagger} = A^{-1} = -\lambda_1 V - \lambda_2 V(AV) - \ldots - \lambda_q V(AV)^{q-1}$

$= V[-\lambda_1 I - \lambda_2(AV) - \ldots - \lambda_q(AV)^{q-1}]$

= V(Polynomial in AV).

Thus, $A^{\dagger}$ = V(Polynomial in AV).

If s = 1, then $(VA)^{\dagger}[(VA) + \lambda_1(VA)^2 + \ldots + \lambda_q(VA)^{q+1}] = 0$ and it follows that

$(VA)^{\dagger}(VA) = -\lambda_1(VA) - \lambda_2(VA)^2 - \ldots - \lambda_q(VA)^q$ is a Polynomial in A.

However, $(V A)^{\dagger} = [(V A)^{\dagger}(V A)](V A)^{\dagger} = -\lambda_1(V A)^{\dagger}(V A) - \lambda_2(V A) - \ldots - \lambda_q(V A)^{q-1}$

$A^{\dagger} V = -\lambda_1 A^{\dagger} VV A - \lambda_2(V A) - \ldots \lambda_q(V A)^{q-1}$

$A^{\dagger} = -\lambda_1 A^{\dagger} AV - \lambda_2(V A)V - \ldots - \lambda_q(V A)^{q-1}V = [-\lambda_1 I - \lambda_2(V A) - \ldots \lambda_q(V A)^{q-1}]V$

Thus, $A^{\dagger}$ = (Polynomial in V A)V.

Hence the Theorem.

4. REVERSE ORDER LAW FOR q-k-EP MATRICES

For any two non singular matricesA, $B \in C_{n\times n}$, $(AB)^{-1} = B^{-1} A^{-1}$ holds. However, it is not true for generalized inverses of matrices **[2].** In general, $(AB)^{\dagger} \neq B^{\dagger} A^{\dagger}$, for any two matrices A and B. For example,

$$A = \begin{bmatrix} 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ AB = \begin{bmatrix} 1 \end{bmatrix},\ (AB)^{\dagger} = \begin{bmatrix} 1 \end{bmatrix}.$$

$(AB)^{\dagger} \neq B^{\dagger} A^{\dagger}$. We say that reverse order law holds for Moore-Penrose inverse of the product of A and B, if $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$.

It is well known that **[p.181 [1]]**, (AB)= BA if and only if R(BB* A*) =R(A*) and R(A*AB) =R(B).

In this section, for a pair of q-k-EP_r matrices A and B, necessary and sufficient condition for $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$ is given.

Theorem 4.1: If A and B are q-k-EP_r matrices with $R(A) = R(B^*)$ then$(AB)^{\dagger} = B^{\dagger} A^{\dagger}$.

Proof: Since A is q-k-EP_r, $R(A) = R(V A^*)$

$\Rightarrow R(B^*) = R(V A^*)$ [By hypothesis]

$\Rightarrow R(VB) = R(V A^*)$ [Since B is q-k-EP_r]

$\Rightarrow R(B) = R(A^*)$ [Since $R(VA) = R(VB) \Rightarrow R(A)=R(B)$]

$\Rightarrow R(B) = R(A^{\dagger})$ [By Theorem (1.1)]

That is, given $x \in C_n$, there exists a $y \in C_n$ such that Bx = Ay.

Now, $Bx = A^{\dagger} y \Rightarrow (B^{\dagger} A^{\dagger} A)Bx = (B^{\dagger} A^{\dagger} A) A^{\dagger} y$

$\Rightarrow B^{\dagger} A^{\dagger} ABx = B^{\dagger} A^{\dagger} AA^{\dagger} y$

$\Rightarrow B^{\dagger} A^{\dagger} ABx = B^{\dagger} A^{\dagger} y$

$\Rightarrow B^{\dagger} A^{\dagger} ABx = B^{\dagger} Bx$

Since $B^{\dagger} B$ is hermitian , it follows that $B^{\dagger} A^{\dagger} AB$ is hermitian.

Similarly, $A^{\dagger} y = Bx \Rightarrow (ABB^{\dagger}) A^{\dagger} y = (ABB^{\dagger} B)x$

$\Rightarrow ABB^{\dagger} A^{\dagger} y = A(BB^{\dagger} B)x$

$\Rightarrow ABB^{\dagger} A^{\dagger} y = A(Bx)$

$\Rightarrow ABB^{\dagger} A^{\dagger} y = A(A^{\dagger} y)$

$\Rightarrow ABB^{\dagger} A^{\dagger} y = AA^{\dagger} y.$

Since $AA^{\dagger}$ is hermitian, it follows that $ABB^{\dagger} A^{\dagger}$ is hermitian.

Further, **[By Theorem (1.1)],**

$R(A) = R(B) \Rightarrow AA^{\dagger} = BB^{\dagger}$

$R(A^{\dagger}) = R(B) \Rightarrow A^{\dagger}(A^{\dagger})^{\dagger} = BB^{\dagger}$

$\Rightarrow A^{\dagger} A = BB^{\dagger}$.

Hence,

$$
\begin{aligned}
(AB)(B^{\dagger} A^{\dagger})(AB) &= ABB^{\dagger}(A^{\dagger} A)B \\
&= ABB^{\dagger}(BB^{\dagger})B \\
&= (AB)(B^{\dagger} BB^{\dagger})B \\
&= (AB)(B^{\dagger})(B) \\
&= A(BB^{\dagger} B) \\
&= A(B) \\
&= AB.
\end{aligned}
$$

$$
\begin{aligned}
(B^{\dagger} A^{\dagger})(AB)(B^{\dagger} A^{\dagger}) &= B^{\dagger}(A^{\dagger} A)(BB^{\dagger}) A^{\dagger} \\
&= B^{\dagger}(BB^{\dagger})(BB^{\dagger}) A^{\dagger} \\
&= (B^{\dagger} B)(B^{\dagger} BB^{\dagger}) A^{\dagger} \\
&= (B^{\dagger} B)(B^{\dagger})(A^{\dagger}) \\
&= (B^{\dagger} BB^{\dagger}) A^{\dagger} \\
&= B^{\dagger} A^{\dagger}.
\end{aligned}
$$

Thus, $B^{\dagger} A^{\dagger}$ satisfies the definition of the Moore-Penrose inverse,

Thus, $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$.

Hence the Theorem.

Remark 4.2: In the above Theorem, the condition that $R(A) = R(B^*)$ is essential.

Example 4.3:

Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ A and B are q-k-EP_1 matrices.

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A^{\dagger} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B^{\dagger} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$(AB)^{\dagger} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B^{\dagger} A^{\dagger} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Here, $R(A) = R(B^*)$.

Thus, $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$.

Example 4.4:

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ A and B are q-k-EP_1 matrices.

$AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $rk(AB) = 1$, $R(A) \neq R(B^*)$.

$A^\dagger = (1/4)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$B^\dagger A^\dagger = (1/4)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$(AB)^\dagger = (1/2)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Thus $(AB)^\dagger \neq B^\dagger A^\dagger$.

Remark 4.5: The converse of the Theorem (4.1) need not be true in general. For let

$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ A and B are q-k-EP_1 matrices.

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

$(AB)^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B^\dagger A^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $(AB)^\dagger = B^\dagger A^\dagger$.

But $R(A) \neq R(B^*)$.

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