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CURVATURE RELATED PROBLEMS FROM COMBINATORIAL VIEW POINT

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ABSTRACT

The aim of this paper, we discuss about curvature related problems from the combinatorial point of view. Some of the basic problems are lying to combinatorial nature of that underlying the space with geometry and PDEs to such evolutionary process. Later we come to know yields related problems to combinatorial and algebraic structures of manifolds.

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1. INTRODUCTION

In this paper we discuss how problems are related to curvature in geometry from the combinatorial point of view. We know that curvature is a geometric attribute if the underlying space is Euclidean then we know that the curvature of the space is zero. In other wards all Euclidean spaces are flat. Otherwise the curvature comes to fore geometry is all about curvature. If the entity is a differentiable curve then we would like to know how this curve is curved in space. For surfaces, it is to know how the curvature of the surface gives its geometric picture. Differential geometry developed over the years from Gauss, Riemannian, Weyl, Chern and many more over the times have delved upon the issue of curvature while studying the problems arising from analysis and topology. Gravity as the manifestation of curvature enabled Einstein to develop his theory of relativity by regarding space time as a continuum. Thus, curvature dominated investigating the problems associated with the space and methods employed to understand its nature came from the methods of calculus. The study in this regard also emphasized the importance of topology. Henri Poincare around 1900's initiated topological studies for the problems in celestial mechanics as a result algebraic topology took birth. The classification themes in topology under taken in 40's was a grand success. It enable us to see that up to homeomorphism any compact simple surface is a copy of S^2 the sphere. Further, classifying higher dimensional spaces based on two dimensional spaces encountered several problems. Infact, the classification of three manifolds up to homemorphism as three sphere S^3 came to be known as Poincare conjecture. This conjecture was recently settled by G.Perlman by employing the methods of geometric flows (Ricci flow). Infact, these investigations carried out by S.S.Chern and it students like S.T.Yau generated a good deal of topological and geometrical ideas. Thurston's geometrization conjecture is more significant in understanding the geometry of 3-manifolds. Infact, S.T.Yau does an excellent work in relating PDE's and geometry. Some of the findings are the most note worthy developments that have taken place in the past few decades. Guiding many researchers to continue their study on problems arising from these backgrounds. Here, we combine the combinatorial nature of the underlying space and study evolution of the space itself. We are drawn to relate geometry and PDEs to such evolutionary processes. In the next two sections we give some basic terminologies associated with the combinatorial and algebraic structures of the underlying manifolds (or space). Subsequent sections deal with our work.

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2. COMBINATORIAL AND ALGEBRAIC STRUCTURES

Combinatorial structures arise quite naturally. For instance in the decomposition of an entity into components, factorization of a positive integer into its factors, permutations on n symbol, mappings from a finite set into itself, polynomials over a finite field, partition of integers, sets and graphs some of the well known examples . In all these aspects what we notice is the components related to their decompositions. In case of cycle decomposition of permutations cycle lengths, partition of an integer the size of each factor, like wise the other discrete structure decompositions and the size of their parts provide information about the entity under consideration which are by and large are similar in nature so far as their structural properties are concerned. what really matters us is the underlying process associated with such parts when picked randomly. For, geometry we are definitely concerned about the geometric attributes in particular the curvature of the manifold. The decomposition is only to facilitate the nature of the problem associated with the flow, [7]. We intend to develop a model which arise when one selects at random a combinatorial structure of size n. We assume that such a selection is equally likely. By this we mean, if the set is decomposed into n parts then picking any one from these parts has probability $\frac{1}{n}$. The most and well know model is to pick and integer uniformly from {1, 2……,n}. The models thus arising from these probabilistic background lead to a stochastic process (random process) that count the number of components of all conceivable size. Moreover, it also leads to investigate the common features of these processes, [1], [3], [4], [6].

Before we give some examples we notice the following general description of a combinatorial structure.

Suppose that the combinatorial structure decomposes into components. Let $p(n)$ denote the number of instance of size n. For a given instance of size n, the obvious description is number of its components. If K is the number of components then we would like to know how many of these K components are sizes one, two etc.

Finally, it boils down to assume that for n, n fixed to count the instances $p(n)$ of size n. This is same as, picking up an instance at random from the uniform distribution over all $p(n)$ possibiabilities and ask for the probability of each component structure.

In this random formulation the counts of components size become random variables. By writing C_i or $C_i(n)$ for the number of components of size i then $(c_1(n), \dots, c_n(n))$ gives the entire component size counting process and $K(n) = \sum_{i=1}^{n} c_i(n)$, the total components of the random variables where $\{c_1(n), \dots, c_n(n)\}$ are dependent and with weighted sums,

$$c_1(n) + 2c_2(n) \dots + nc_n(n) = n.$$

Following two examples would give us enough ideas about the above descriptions.

1. Integer partitions [5,8]. Partition the integer n as $n = l_1 + l_2 + \cdots + l_k$ with $l_1 \geq l_2 \geq \cdots \geq l_k \geq 1$. For integer partitions, $p(n)$ is the traditional notation for the number of such partitions, and $\sum p(n)x^n = \prod_{i\geq 1}(1 - x^i)^{-1}$. We write $c_i(n)$ for the number of parts which are i, and the component counting structure $(c_1(n), \dots, c_n(n))$ is an encoding of the partition. An instance for $n = 10$ is $10 = 5 + 3 + 1 + 1$.
   In this instance $C_1(10) = 2, C_3(10) = C_5(10) = 1$, the other $C_i(n)$ being zero.
2. Permutations [9,2]. Consider the cycle decomposition of a permutation of the set $\{1,2, \dots, n\}$ $with$ $C_i(n)$ being the number of cycles of length i. The total number of instances of size n is $p(n) = n!$, and $C_1(n)$ is the number of fixed points.An instance for $n = 10$ is the function π with $\pi(1) = 9, \pi(2) = 1, \pi(3) = 7,$ $\pi(4) = 4,$ $\pi(5) = 3, \pi(6) = 2,$ $\pi(7) = 5, \pi(8) = 8, \pi(9) = 10, \pi(10) = 6,$
   whose cycle decomposition is $\pi = (1\ 9\ 10\ 6\ 2\ )(\ 3\ 7\ 5)(4)(8)$.
   In instance $C_1(10) = 2, C_3(10) = C_5(10) = 1$.

3. GRAPHS AS CELL COMPLEXES

In this section we are giving a cell complex exposition to a graph. Let $G = (V, E, W)$, be a graph we shall interpret it as a cell complex and describe the combinatorial structure in order to do this we need to know some algebraic basics.

Let Γ be a simple closed curve in $\mathbb{R}^2$ then it is homeomorphic to S^1. A well known theorem in topology states that any point not lying on the boundary of the curve is either an interior point of the curve or an exterior point of the curve. This subtle difference divides the plane into two components one component will contain all the interior points of it and the other component will contain points exterior to it. Thus, Γ divides $\mathbb{R}^2$ into two components $\mathbb{R}^2$–Γ and Γ interior.

Further, we can map Γ homeomorphicaly onto S^1.

In fact this is a very important observation. Also, this is an important theorem in topology. Because, Γ is not convex where as S^1 is convex.

An operation called convex hull operation on Γ denoted conv(Γ) (and read as convex hull of Γ) makes it convex. The algebraic notions creep into the realm of topology in this fashion. Few definitions are in order.

3.1 Definition: A set $S = \{v_{0,\dots\dots\dots} v_n\}$ of vector in $\mathbb{R}^N$ i.e., each $v_i \in \mathbb{R}^N$, as an N-tuple is said to be general position if the set $\{v_0 - v_n, v_1 - v_n, \dots., v_{n-1} - v_n\}$ is linearly independent

i.e., $\sum_{i=1}^{n} t_i(v_{i-1} - v_n) = 0\,, t_i = 0 \; for\; i = 1,2, \dots\dots\dots, n.$

A set $S = \{v_0, \dots.., v_n\}$ of vectors in general position is said to be a n-simplex. The n^{th} constitutes the dimension of the simplex. An n-simplex determines a subset of $\mathbb{R}^N$.

If $E \subset \mathbb{R}^N$, can we relate E as a subset (just as we saw in case of Γ) with this idea and there by connect some interesting problems underlying the geometry of the space. This can be done and in fact that is what we are going to do now.

The n-simplex determines a subset of $\mathbb{R}^N$ and is given by

$$\Delta^n[S] = \{t_o v_0 + \cdots + t_n v_n \epsilon\, \mathbb{R}^N / t_i \geq 0,\ \textstyle\sum_{i=1}^{n} t_i = 1\}$$

This is also a convex hull $(\{v_0, \dots\dots, v_n\})$. If the set $S = \{v_0, \dots\dots, v_n\}$ of vector is not in general position then we say that the n-simplex determined by S is degenerate.

A triple $\{v_0, v_1, v_2\}$ is in general position, if the points are not collinear.

A 0-simplex $\Delta^0[\{v_0\}]$ is simply the point $v_0 \in \mathbb{R}^N$.

A 1-simplex $\{v_0, v_1\}$ determine the line segment $\Delta^1[\{v_0\,, v_1\}]$.

A 2-simplex $\Delta^2[\{v_0\,, v_1, v_2\}]$ determine a triangle (with its interior) and a 3-simplex $\Delta^3[\{v_0\,, v_1, v_2, v_3\}]$ determines a tetrahedron and in general $\Delta^n[\{v_0\,, \dots, v_n\}]$, $\Delta^n = \Delta^n[S]$ is the n-simplex, S here means the vertex set. When the vertex is repeated, the simplex is degenerate. Degenerate simplices are important when one studies mappings between simplicial complexes (we define later in our discussion). In what follows the vertices play important rule in the combinatories of sets the topological properties associated with the simplicials from a core theme of algebraic topology basically developed to compute the topological invariant which is not in the scope of our study. We shall study the n-simplicial complex $\Delta^n[S]$ as subset of $\mathbb{R}^N$, for its characterization. A point $p \in \Delta^n$ may be uniquely specified by the co-efficient $(t_0, t_1, \dots\dots, t_n)$. This can be noticed easily for,

$$t_o v_0 + t_1 v_1 + \cdots + t_n v_n = t'_o v_0 + t'_1 v_1 + \cdots + t'_n v_n$$

$$(t_o - t'_0)v_0 + (t_1 - t'_1)v_1 + \cdots + (t_n - t'_n)v_n = 0.$$

Since, $\sum_{i=0}^{n} t_i = 1$ and $\sum_{i=0}^{n} t'_i = 1$ (by definition)

$\sum_{i=0}^{n} (t_i - t'_i) = 0,$ would mean $t_i = t'_i$ for each $i = 0,1, \dots\dots.., n$

Uniqueness follows.

In particular,

$$(t_o - t'_0)v_0 + (t_1 - t'_1)v_1 + \cdots + (t_n - t'_n)v_n = 0.$$

$$(t_o - t'_0)(v_0 - v_1) + \cdots + (t_n - t'_n)(v_0 - v_n) = 0.$$

where $\{v_0, \dots\dots, v_n\}$ are linearly independent.

So the co-efficient are uniquely determine by p. the co-efficient $(t_0, t_1, \dots\dots., t_n)$ is called the barycentric co-ordinates of $p \in \Delta^n$.

Remark: Although $\Delta^n[\{v_0\,, \dots, v_n\}]$ is a subspace of $\mathbb{R}^N$ as a topological space it is determined by the barycentric co-ordinates.

3.2 Proposition: Let Δ^n denote the subspace of $\mathbb{R}^{N+1}$ given by

$$\Delta^n[S] = \{t_o, \dots, t_n \epsilon\, \mathbb{R}^{N+1} / \, t_o + \cdots + t_n = 1, t_i \geq 0\}.$$

If $S = \{v_0, \dots \dots, v_n\}$ is a set of vectors in general position in $\mathbb{R}^N$, then $\Delta^n[S]$ is homomorphic to Δ^n.

Note that the topological properties of Δ^n are shared with the $\Delta^n[S]$, where $\Delta^n[S]$ is any other simplex.

For example: As a subspace of $\mathbb{R}^N$, $\Delta^n[S]$ is compact because Δ^n is closed and bounded in $\mathbb{R}^{N+1}$.

3.3 Proposition: The point $p \in \Delta^n[S]$ with barycentric coordinates and satisfying $t_i > 0$ for all i from an open subset of $\Delta^n[s]$ (as a subspace of $\mathbb{R}^N$), p is in the boundary of $\Delta^n[s]$ if and only if $t_i = 0$, for some i.

Proof: Let $\Delta^n \in \mathbb{R}^N$ then the subset of points with barycentric co-ordinates $t_i > 0$ is the intersection of the open sets.

$$U_i = \{t_o, \dots, t_n \epsilon\, \mathbb{R}^{N+1} / t_i \geq 0\} \text{ with } \Delta^n.$$

Therefore, it is an open subset of Δ^n. Its homomorphic image in $\Delta^n[S]$ is also open in $\Delta^n[s]$.

We can extended the mapping

$$\phi : \Delta \to \Delta^n[S] \quad \text{to the subset } \prod \text{ and } \mathbb{R}^{N+1},$$

where $\prod = \{t_o, \dots, t_n \epsilon\, \mathbb{R}^{N+1} / t_o + \cdots + t_n = 1\}$

The higher plane containing Δ^n in $\mathbb{R}^{N+1}$. The mapping $\tilde{\phi}: \prod \to \mathbb{R}^N$ defined by $\tilde{\phi}(t_o, \dots, t_n) = t_o v_0 + t_1 v_1 + \cdots + t_n v_n$ takes point on the boundary of Δ^n to point on the boundary of Δ^n.The point on the boundary have some $t_i = 0$. Since openset in $\mathbb{R}^{N+1}$ containing such points must continuous point with $t_i > 0$ which map by $\tilde{\phi}$ to points outside $\Delta^n[S]$.

Conversely, if a point p is on a boundary of $\Delta^n[S]$. Then any point set containing p would intersect with complimentary $\Delta^n[S]$ and thus point in the image of $\prod$ under fee Δ with negative co-ordinate. This implies some $t_j = 0$.

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