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DIVISIBILITY OF HYPER RINGS

K. GOHILA*
M. Phil Scholar, PG and Research Department of Mathematics,
Sri S. Ramasamy Naidu Memorial College, Sattur-626203, Tamil Nadu, INDIA.

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ABSTRACT

In this paper we introduce the divisible hyper rings as a generalization of the usual hyper rings. We investigate some of their properties and examples are constructed for divisible hyper rings.

Key words: *Divisible Semi hyper ring, Divisible left hyper ring, Divisible sub semi hyper ring.*

AMS Subject Classification: *20N20.*

INTRODUCTION

The theory of algebraic hyper structures which is a generalization of ordinary algebraic structures was first introduced by Marty [3].Since then, many researchers have studied the theory of hyper structures and develop it. Krasner has studied the notion of hyper ring Davvaz [2] (2009) has defined some relations in hyper rings.

The hyper ring have appeared as a new class of algebraic hyper structures more general than that of hyper fields, introduced by Krasnar in the theory o valued fields. Several types of hyper rings have been proposed, but the most general one is that introduced by Spartalis[4], used also in the content of P -hyper rings or (H, R) -hyper rings [5]. A comprehensive review of hype rings theory is covered in Nakassis, Vougioklis and in the book written by Davvaz, Leoreanu-Fotea. R.Ameri, H. Hedayati defined k -hyper ideals in semi hyper rings in [1].

The aim of this paper is to initiate the study of divisibility in semi hyper rings and we construct some of their properties, theorems and examples. First, we present the basic definitions.

PRELIMINARIES

In this section, some of the basic definitions are summarized which are needed in sequel.

Definition 2.1: A hyper groupoid (H, ×) is called a semi hyper group if for all a, b, c of H, (a × b) × c = a × (b × c) which means that $\cup_{u\in a\times b}\, u\times c = \cup_{v\in b\times c}\, a\times v$.

Definition 2.2: A hyper groupoid (H, ×) which is both a semi hyper group and a quasi hyper group is called a hyper group.

Definition 2.3: A hyper structure (H,⊕, ⊙)is called a Ringoid if both ⊕, ⊙ are binary operations.

Definition 2.4: A Semi hyper ring is an algebraic structure (R, +, ·) which satisfies the following axioms:

(1) (R, +) is a commutative hyper monoid, That is:
- (a) (x + y) + z = x + (y + z) for all x, y, z ∈ R.
- (b) There is 0 ∈ R, such that x + 0 = 0 + x = x for all x ∈ R.
- (c) x + y = y + x for all x, y ∈ R.

(2) (R,.) is semi group, that is x.(y.z) = (x.y).z for all x, y, z ∈ R.
(3) The multiplication is distributive with respect to hyper operation '+' that is; x.(y + z) = x.y + x.z and
(4) (x + y).z = x.z + y.z for all x, y, z ∈ R.
(5) The element 0 ∈ R, is an absorbing element, that is; x.0 = 0.x = 0 for all x ∈ R.

Corresponding Author: K. Gohila*
M. Phil Scholar, PG and Research Department of Mathematics,
Sri S. Ramasamy Naidu Memorial College, Sattur-626203, Tamil Nadu, INDIA.

Example 2.5: Consider the set of 2×2 matrix on W: S $=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in W\right\}$ where W is a set of whole numbers. Then (S, +, ·) is a semi hyper ring under the hyper operation of addition and multiplication is defined below:

For, A $=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and B $=\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ be taken from S.

Then A+B $=\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \left\{\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}\right\} \subseteq S$ and A.B $=\begin{bmatrix} a & b \\ c & d \end{bmatrix} . \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$

Then the additive identity being null matrix and multiplicative identity being identity matrix.

Example 2.6: The set R = {0, 1} with the following hyper operations is a semi hyper ring.

⊕	0	1
0	{0}	{1}
1	{1}	{0,1}

⊙	0	1
0	{0}	{0}
1	{0}	{0,1}

Definition 2.7: An algebraic system (R, +, .) is said to be general hyper ring, if
(1) (R, +) is hyper group.
(2) (R, .) is a semi hyper group.
(3) '.' is distributive with respect to '+' .

Example 2.8: Hyper ring in the example 2.6 it is a general hyper ring and hence it satisfies the distributive property.
0 ⊙ (0 ⊕ 1) = (0 ⊙ 0) ⊕ (0 ⊙1)
(0 ⊕ 0) ⊙ 1 = (0 ⊙ 1) ⊕ (0 ⊙ 1)
0 ⊙ (1 ⊕ 1) = (0 ⊙ 1) ⊕ (0 ⊙1)
(0 ⊕ 1) ⊙ 1 = (0 ⊙ 1) ⊕ (1 ⊙1) and The other cases can be proved similar.

Example 2.9: The set R = {0, 1, 2} with the following hyper operations is a general hyper ring

⊕	0	1	2
0	{0}	{1}	{2}
1	{1}	{2}	{1,2}
2	{2}	{1,2}	R

⊙	0	1	2
0	{0}	{0}	{0}
1	{0}	{1}	{2}
2	{0}	{2}	{0,1}

It satisfies the distributive property.
0 ⊙ (1 ⊕ 2) = (0 ⊙ 1) ⊕ (0 ⊙ 2)
(0 ⊕ 1) ⊙ 2 = (0 ⊙ 2) ⊕ (1 2)
1 ⊙ (0 ⊕ 2) = (1 ⊙ 0) ⊕ (1 ⊙ 2)
(1 ⊕ 0) ⊙ 2 = (1 ⊙ 2) ⊕ (0 ⊙ 2) and The other cases can be proved similar.

Definition 2.10: A hyper ring (R, +, ·) is called commutative if (R, ·) is commutative.

Definition 2.11: Let R_1 and R_2 be hyper rings. A mapping f: $R_1 \rightarrow R_2$ is said to be good homomorphism if for all a, b ∈ R_1, f(a + b) = f(a) + f(b) and f (a.b) = f(a).f (b)

Example 2.12: Consider the hyper ring defined in the example 2.6, f: P → H where P = {0, 1}, for f(x) = 1.x

Then f(0) = 1.0 = {0}
f(1) = 1.1 = {0, 1}
f(1⊕ 0) = 1 ⊙ (1 ⊕ 0) = {0, 1}
f(1) ⊕ f(0) = {0, 1} and
f(1 ⊙ 0) = 1 ⊙ (1 ⊙ 0) = {0}
f(1) ⊙ f(0) = {0}

Hence f is a good homomorphism for f(x) = 1.x

DIVISIBILITY OF HYPER RINGS

Definition 3.1: A semi hyper ring (R,+, .) is called divisible if for any x ∈ R and n ∈ N, there is an element y ∈ R such that x ∈ $(y, .)^n$ denotes the subset y.y.y.....y (n times) of R.

Example 3.2: Consider the semi hyper ring H = {0, 1} defined in the Example 2.6. It is divisible.

Since, $0 \in H$, $n \in N$ and $1 \in H$
$0 \in (1, \cdot)^n$
$n = 2$, $0 \in 1 \odot 1 = \{0, 1\}$
$1 \in (1, \cdot)^n$
$n = 2$, $1 \in 1 \odot 1 = \{0, 1\}$
Hence H is divisible.

Definition 3.3 [2]: A non-empty subset S of a semi hyper ring (R, +, .) is called sub semi hyper ring of R if
- (i) $(a + b) \subseteq S$ for all $a, b \in S$.
- (ii) $(a.b) \subseteq S$ for all $a, b \in S$.

Example 3.4: In the semi hyper ring (R, +, ·). Consider the example 2.9 is a sub semi hyper ring. Hence S = {0, 1}, S = {0, 2} and S = {1, 2} and all.

Now, to prove S is a sub semi hyper ring
$0 \oplus 1 = 1 \subseteq S$ for all $0, 1 \in S$.
$0 \odot 1 = 0 \subseteq S$ for all $0, 1 \in S$.
Hence S satisfies the sub semi hyper ring property.

Now, to prove S = {0, 2} is a sub semi hyper ring
$0 \oplus 2 = 2 \subseteq S$ for all $0, 2 \in S$.
$0 \odot 2 = 0 \subseteq S$ for all $0, 2 \in S$.
Hence S satisfies the sub semi hyper ring property.

Now, to prove S = {1, 2} is a sub semi hyper ring
$1 \oplus 2 = \{1, 2\} \subseteq S$ for all $1, 2 \in S$.
$1 \odot 2 = 2 \subseteq S$ for all $1, 2 \in S$.
Hence S satisfies the sub semi hyper ring property.

Definition 3.5 [2]: A left hyper ideal of a semi hyper ring R is non-empty subset I of R, satisfying
- (i) For all $x, y \in I$, $x + y \subseteq I$.
- (ii) For all $a \in I$ and $x \in R$, $x.a \in I$.

Example 3.6: Consider the semi hyper ring R = {0, 1} and I = {0} ⊂ R defined in the example 2.6. Hence I = {0} is a left hyper ideal of R.

Now, to prove I is the left hyper ideal of R.

Let 0. $0 \in I$, $0 \oplus 0 = \{0\} \subseteq I$ for all $x, y \in I$

For all $0 \in I$ and $1 \in R$, $0.1 = 0 \in I$.
Hence we have I is the left hyper ideal of R.

Example 3.7: Consider the semi hyper ring R = {0, 1, 2} and I = {1, 2} ⊂ R.

Now, to prove I is the left hyper ideal of R.

Let 1. $2 \in I$, $1 \oplus 2 = \{1, 2\} \subseteq I$ for all $x, y \in I$

For all $1 \in I$ and $2 \in R$, $1.2 = 2 \in I$.
Hence we have I is the left hyper ideal of R.

Theorem 3.8: A sub semi hyper ring of a semi hyper ring is divisible and vice versa.

Proof: Let S be a sub semi hyper ring. That is
- (i) $(a \oplus b) \subseteq S$ for all $a, b \in S$.
- (ii) $(a \odot b) \subseteq S$ for all $a, b \in S$.

To prove S is divisible.

That is to prove $a \in S$, $n \in N$ and $b \in S$ such that $a \in (b, \cdot)^n$

Let us consider a ∈ S, n ∈ N

Then a ∈ $(b, \cdot)^n$ for some b ∈ S is divisible.

Conversely let us assume that S is divisible.

To prove S is sub semi hyper ring.

Since, S is divisible, we can get S is a semi hyper ring.

Then it also satisfies the sub semi hyper ring properties. That is
- (i) (a + b) ⊆ S for all a, b ∈ S.
- (ii) (a.b) ⊆ S for all a, b ∈ S.

Now, let a ∈ S and b ∈ S, then it Satisfies the sub semi hyper ring property.

Hence S is a sub semi hyper ring.

Example 3.9: Let S = {0, 1} is a sub semi hyper ring. Since from the above Example 3.4 is divisible.

Conversely suppose that S = {0, 1} is divisible. To prove S is a sub semi hyper ring. That is to prove
- (i) (a + b) ⊆ S for all a,b ∈ S.
- (ii) (a.b) ⊆ S for all a, b ∈ S.

$0 \oplus 1 = \{1\} \subseteq S$.
$0 \odot 1 = \{0\} \subseteq S$
This implies that S is a sub semi hyper ring.
Hence S is a sub semi hyper ring.

Theorem 3.10: A left hyper ideal of a semi hyper ring is divisible and vice versa.

Proof: Let I be a hyper ideal of a semi hyper ring R. Then for any a, b ∈ I, we have $a \oplus b \subseteq I$.

Also, for any a, b ∈ I, since a ∈ I ⊆ R, we have a, b ∈ I.

Claim I is divisible. That is to prove that a ∈ $(b, \cdot)^n$, where n ∈ N, a, b ∈ I.

Let a ∈ I, and n ∈ N.

Since R is divisible, then a ∈ $(b, \cdot)^n$ for some b ∈ R and I ⊆ R, we have a ∈ $(b, \cdot)^n$ for some b∈I.

Hence I is divisible in R.

Conversely let us assume that, I is divisible in R.

To prove I is a left hyper ideal of R. That is to prove
- (i) For all x, y ∈ I, x + y ⊆ I.
- (ii) For all a ∈ I and x ∈ R, x.a ∈ I.

Obviously I is a hyper ideal of R.

Example 3.11: From the example 3.6 is divisible and vice versa.

Theorem 3.12: Let (R, +, ·) is a semi hyper ring and $\{I_i\}_{i\in\wedge}$ be a family of hyper ideals of R is divisible. Then $\cap_{i\in\wedge} I_i$ is also a hyper ideal of R which is divisible.

Proof: Let a, b ∈ $\cap_{i\in\wedge} I_i$, then a, b ∈ I_i for all i ∈ ∧.

Since, each I_i is a hyper ideal of R, so $a \oplus b \subseteq I_i$ for all i ∈ ∧.

Thus, $a \oplus b \subseteq \cap_{i\in\wedge} I_i$

Now, for x ∈ R and a ∈ $\cap_{i\in\wedge} I_i \Rightarrow$ a ∈ I_i for all i ∈ ∧, so x · a ∈ I_i for all i ∈ ∧ $\Rightarrow$ x · a ∈ $\cap_{i\in\wedge} I_i$

Similarly $a \cdot x \in \cap_{i\in\wedge} I_i$

Then $\cap_{i\in\wedge} I_i$ is a hyper ideal of R.

Claim $\cap_{i\in\wedge} I_i$ is divisible.

Let $a \in \cap_{i\in\wedge} I_i$, then $a \in I_i$ for all $i \in \wedge$ and $n \in N$.

Each I_i is divisible, then $a \in (b, \cdot)^n$ for all $b \in I_i$.

Thus $b \in \cap_{i\in\wedge} I_i$

Hence $\cap_{i\in\wedge} I_i$ is divisible.

Theorem 3.13: If S is a sub semi hyper ring of a semi hyper ring R is divisible and I is a hyper ideal of R, then
(i) S + I is a sub semi hyper ring of R which is divisible.
(ii) S ∩ I is a hyper ideal of S is divisible.

Proof: (1) $S + I = \cup_{s\in S\, i\in I}(s + i)$

Let $x, y \in S + I$, then there exists $s_1, s_2 \in S$ and $i_1, i_2 \in I$ such that $x \in s_1 + i_1$ and $y \in s_2 + i_2$, then
$x + y \subseteq (s_1 + s_2) + (i_1 + i_2) \subseteq S + I$
$x \cdot y \subseteq (s_1 + s_2) \cdot (i_1 + i_2) \subseteq S + I$

Hence S + I is a sub semi hyper ring of R.

Claim S + I is divisible.

Let $a \in S + I$ then there exists $s_1 \in S$, $i_1 \in I$ such that $a \in s_1 + i_1$.

Since, S + I is a sub semi hyper ring of R then, $a.b \in (s_1 + i_1) \cdot (s_2 + i_2)$ for some $b \in s_2 + i_2$.

That is $a \in (b, \cdot)^n$ for $n \in N \Longrightarrow (b, \cdot)^n \subseteq s_1 + i_1$.

Thus, $b \in s_2 + i_2$ where $s_2 \in S$, $i_2 \in I$.

Hence S + I is divisible.

(2) Let $a, b \in S \cap I$, implies $a, b \in S$ and $a, b \in I$.

Since, S is a sub semi hyper ring and I is a hyper ideal of R, then $a+b \subseteq S$ and $a + b \subseteq I$, which implies $a + b \subseteq S \cap I$.

Also, for $s \in S$, then obviously $a \cdot s \subseteq S$, also $a \cdot s \subseteq I$, since I is a hyper ideal and S is sub semi hyper ring.

Hence S ∩ I is a hyper ideal of R.

Claim S ∩ I is divisible.

Let $a \in S \cap I \Longrightarrow a \in S$ and $a \in I$.

Since both S and I are divisible, then $a \in (b, \cdot)^n$ for some $b \in S$, also $a \in (b, \cdot)^n$ for some $b \in I$.

Hence $a \in (b, \cdot)^n$ for some $b \in S \cap I$. Hence S ∩ I is divisible.

Example 3.14 Consider the sub semi hyper ring S = {0, 1} and the hyper ideal I = {0, 2}. S + I = {0, 1, 2}

Now, let $0 \in (2, \cdot)^n$
$n = 2, 0 \in 2 \cdot 2 = \{0, 1\}$
$n = 3, 0 \in 2 \cdot 2 \cdot 2 = \{0, 1\} \cdot 2 = \{0, 2\}$

Hence S + I is divisible.

Similarly we can prove S ∩ I is divisible.
S ∩ I = {0}
$0 \in (0, \cdot)^n$
$0 \in 0$

Hence S ∩ I is divisible.

Example 3.15: Let us consider S = {a, b} and I = {a, b}.Then S + I = {a, b}

Now, $a \in (b, \cdot)^n = b \cdot b = \{a, b\}$ and $a \in b \cdot b \cdot b = \{a, b\} \cdot b = \{a, b\}$

Hence S + I is divisible.

Now, to prove S ∩ I is divisible. S ∩ I = {a, b}

Similarly S ∩ I is divisible.

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