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ORTHOGONALITY OF (σ, τ)-DERIVATIONS AND BI-(σ, τ)-DERIVATIONS IN SEMIPRIME RINGS

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ABSTRACT

This paper gives the notion of orthogonality between (σ,τ)-Derivations and Bi-(σ,τ)-Derivations in Semiprime rings. In this paper, we give three conditions equivalent to the notion of orthogonality between the (σ,τ)-derivation and bi-(σ,τ)-derivation of a semiprime ring. It is shown that if R is a 2-torsion free semiprime ring, B is a bi-(σ,τ)-derivation and d is a (σ,τ)-derivation on R, then B and d are orthogonal if only if one of the following equivalent conditions holds for every $x, y \in R$: (i) dB=0 (ii) $d(x)B(x,y)=0$ or $d(x)B(y,x)=0$ (iii) dB is a bi- (σ,τ)-derivation

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***Key Words**: Semiprime ring, Derivation, Biderivation, Orthogonal, (σ,τ)-Derivation and Bi-(σ,τ)-Derivations.*

INTRODUCTION

Bresar and Vukman [2], introduced the notion of orthogonality for a pair d and g of derivations on a semiprime ring and they have proved several necessary and sufficient conditions for d and g to be orthogonal. Daif. *et al.* [4], studied the orthogonality between the derivation and biderivation of a ring and also in terms of a nonzero ideal of a 2-torsion free semiprime ring. In this section, we give three conditions equivalent to the notion of orthogonality between the (σ,τ)-derivation and bi-(σ,τ)-derivation of a semiprime ring. It is shown that if *R* is a 2-torsion free semiprime ring, *B* is a bi-(σ,τ)-derivation and *d* is a (σ,τ)-derivation on *R*, then *B* and *d* are orthogonal if only if one of the following equivalent conditions holds for every $x, y \in R$: (i) *dB*=0 (ii) $d(x)B(x,y)=0$ or $d(x)B(y,x)=0$ (iii) *dB* is a bi-(σ,τ)-derivation.

PRELIMINARIES

Throughout this paper *R* will be an associative ring. A ring *R* is said to be 2-torsion-free if $2x = 0$, $x \in R$ implies $x = 0$. *R* is called prime if $xRy = 0$ implies $x = 0$ or $y = 0$, and R is semiprime if $xRx = 0$ implies $x = 0$ for all $x, y \in R$.

We write the usual commutator $[x, y] = xy - yx$ for all $x, y \in R$, and we use the basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xz, y] = [x, y]z + x[z, y]$.

An additive mapping $d: R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for every $x, y \in R$. Let R be a semiprime ring, two derivations *d* and *g* of *R* are called orthogonal if $d(x)Rg(y) = 0 = g(y)Rd(x)$ [2]. Following Daif.*et.al* [4], a biadditive map $B: R\times R \rightarrow R$ is called a biderivation of *R* if $B(xy, z) = B(x, z)y + xB(y, z)$ for all $x, y, z \in R$. We know that an additive mapping $d: R \rightarrow R$ is called a (σ,τ) derivation if $d(xy) = \sigma(x)d(y) + d(x)\tau(y)$ for all $x, y \in R$. A biadditive mapping $B: R\times R \rightarrow R$ is called a biderivation of *R* if it is a derivation in each argument. That is for every $x \in R$, the maps $B: R\times R \rightarrow R$ and $y \rightarrow B(y,x)$ are derivations of *R* into *R*. *B* is called a bi-(σ,τ) derivation if $B(xy,z) = \sigma(x)B(y,z) + B(x,z)\tau(y)$ for all $x, y \in R$, where σ, τ are endomorphisms on *R*. A (σ,τ)-derivation *d* and bi-(σ,τ) derivation *B* of *R* are called orthogonal if $B(x,y)Rd(z) = 0 = d(z)RB(x,y)$ for all $x, y, z \in R$.

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Throughout this section *R* will denote a 2-torsion free semiprime ring.

We now consider some well known results that will be needed in the subsequent results.

Lemma 1: [[2], Lemma 1] Let *R* be a 2-torsion free semiprime ring and $a,b \in R$. Then the following are equivalent:

- $axb = 0$ for all $x \in R$
- $bxa = 0$ for all $x \in R$
- $axb + bxa = 0$ for all $x \in R$

If one of the above conditions is fulfilled, then $ab = ba = 0$, too.

Lemma 2: [[4], Lemma 2] Let *R* be a semiprime ring. Suppose that an additive mapping *h* on *R* and a biadditve mapping $f : R \times R \to R$ satisfy $f(x,y)Rh(x) = (0)$, then $f(x,y)Rh(z) = (0)$ for all $x,y,z \in R$.

Lemma 3: Let *R* be a 2-torsion free semiprime ring. A bi-(σ,τ)- derivation *B* and a (σ,τ)-derivation *d* are orthogonal iff

$$B(x,y)d(z) + d(x)B(z,y) = 0 \text{ for all } x,y,z \in R.$$

Proof: Suppose *B* and *d*, such that

$$B(x,y)d(z) + d(x)B(z,y) = 0 \text{ for all } x,y,z \in R. \quad (1)$$

By taking $z = zx$, we obtain

$$B(x,y)d(zx) + d(x)B(zx,y) = 0.$$ Thus

$$B(x,y)\sigma(z)d(x) + B(x,y)d(z)\tau(x) + d(x)\sigma(z)B(x,y) + d(x)B(z,y)\tau(x) = 0$$

It implies, $$B(x,y)\sigma(z)d(x) + d(x)\sigma(z)B(x,y) + \{B(x,y)d(z) + d(x)B(z,y)\}\tau(x) = 0 \quad (2)$$

Using (1) and (2) gives

$$B(x,y)\sigma(z)d(x) + d(x)\sigma(z)B(x,y) = 0 \text{ for all } x,y,z \in R. \quad (3)$$

It can be written as $B(x,y)Rd(x) + d(x)RB(x,y) = 0$.

Using *Lemma* 1 in (3) gives

$$d(x)RB(x,y) = (0) \text{ for all } x,y \in R.$$

Hence by *Lemma* 2, we get

$$d(x)RB(z,y) = (0) \text{ for all } x,y,z \in R. \quad (4)$$

Using *Lemma* 1, again in (4) gives

$$d(x)RB(z,y) = (0) = B(z,y)Rd(x).$$

So *B* and *d* are orthogonal.

Conversely, if *B* and *d* are orthogonal then

$$d(x)B(z,y) = (0) = B(x,y)d(z),$$ by *Lemma* 1.

Thus $$d(x)B(z,y) + B(x,y)d(z) = 0$$

From the definitions of *d* and *B*, we have

Lemma 4: Let *d* be a (σ,τ)- derivation and *B* a bi-(σ,τ)-derivation of a ring *R*. The following identity holds for all $x,y,z \in R$.

$$dB(xy,z) = d\{\sigma(x)B(y,z) + B(x,z)\tau(y)\}$$
$$= \sigma^2(x)dB(y,z) + d\sigma(x)\tau B(y,z) + \sigma B(x,z)d\tau(y) + dB(x,z)\tau^2(y).$$

Theorem 5: Let *R* be a 2-torsion free semiprime ring. A bi-(σ,τ)-derivation B and a (σ,τ)-derivation *d* are orthogonal iff *dB*=0.

Proof: Let *B* and *d* be such that *dB*=0. According to *Lemma* 4,

$d\sigma(x)\tau B(y,z)+\sigma B(x,z)d\tau(y)=0$ Then

$d(x_1)B(y_1,z_1)+B(x_1,z_1)d(y_1)=0$, where $\sigma(x)=x_1,\ \tau B=B\tau$,

$\tau(y,z)=(y_1,z_1)$, $\sigma B=B\sigma$ and $\tau(y)=y_1$.

By using *Lemma* 3, *d* and *B* are orthogonal.

Conversely, if *d* and *B* are orthogonal, then $d(x)sB(y,z)=0$ for all $x,y,z,s\in R.$

Hence $0=d(d(x)sB(y,z))=\sigma(d(x)s)dB(y,z)+d(d(x)s)\tau B(y,z)$

$0=\sigma d(x)\sigma(s)dB(y,z)+\sigma d(x)d(s)\tau B(y,z)+d^2(x)\tau(s)\tau B(y,z)$.

$0=d(x_1)\sigma(s)dB(y,z),+d(x_1)RB(y_1,z_1)+d(x_1)RB(y_1,z_1)$

where $\sigma d=d\sigma$ and $\sigma(x)=x_1,\tau B=B\tau$ and $\tau(y,z)=(y_1,z_1)$.

The sum of the last two summands is zero as *d* and *B* are orthogonal. So the above relation becomes

$$d(x_1)rdB(y,z)=0, \tag{5}$$

where $\sigma(s)=r\in R$ and x,y,z,r are arbitrary elements in *R*. In 5, let $x_1=B(y,z),$ then

$dB(y,z)RdB(y,z)=(0$ for all $y,z\in R.$

Since *R* is semiprime,

$dB(y,z)=0$ for all $y,z\in R.$

Hence *dB*=0.

Theorem 6: Let *R* be a 2-torsion free semiprime ring. A bi-(σ,τ)-derivation *B* and a (σ,τ)-derivation *d* are orthogonal iff $d(x)B(x,y)=0$ or $d(x)B(y,x)=0$ for all $x,y\in R.$

Proof: We assume *B* and *d* such that

$$d(x)B(x,y)=0 \text{ for all } x,y\in R. \tag{6}$$

A linearization on *x* for 6 gives,

$$d(x)B(x,y)+d(x)B(z,y)+d(z)B(x,y)+d(z)B(z,y)=0 \tag{7}$$

for all $x,y,z\in R.$

Using (6) and (7), we obtain

$$d(x)B(z,y)+d(z)B(x,y)=0 \text{ for all } x,y,z\in R. \tag{8}$$

By taking $z=zs$ in *(*8) gives

$d(x)B(zs,y)+d(zs)B(x,y)=0.$ This implies

$$d(x)\sigma(z)B(s,y)+d(x)B(z,y)\tau(s)+\sigma(z)d(s)B(x,y)+d(z)\tau(s)B(x,y)=0 \tag{9}$$

By (8), we get

$d(x)B(z,y)=-d(z)B(x,y)$ and

$d(s)B(x,y)=-d(x)B(s,y).$

So 9 becomes,

$$d(x)\sigma(z)B(s,y)-d(z)B(x,y)\tau(s)-\sigma(z)d(x)B(s,y)+d(z)\tau(s)B(x,y)=0 \tag{10}$$

By replacing $\sigma(z)=d(x)$ in (10), gives

$$d(x)^2B(s,y)-d(z)B(x,y)\tau(s)-d(x)^2B(s,y)+d(z)\tau(s)B(x,y)=0.$$

This implies $d(z)[\tau(s),B(x,y)]=0.$

By taking $\tau(s)=r\in R,$ we have $d(z)[r,B(x,y)]=0.$ (11)

By assuming $r=rw$ in (11), we get

$$d(z)[rw,B(x,y)]=0.$$

So $$d(z)r[w,B(x,y)]+d(z)[r,B(x,y)]w=0.$$

By (11), it reduces to

$$d(z)r[w,B(x,y)]=0.$$

It can be written as $d(z)R[w,B(x,y)]=0$ for all $x,y,z,w\in R.$

But $[d(z),B(x,y)]R[d(z),B(x,y)]=(0)$ for all $x,y,z\in R.$

Hence, $d(z)B(x,y)=B(x,y)d(z)$ for all $x,y,z\in R.$

Therefore, (8) can be written as

$$d(x)B(z,y)+B(x,y)d(z)=0 \text{ for all } x,y,z\in R.$$

Thus, using *Lemma* 3, we see that *d* and *B* are orthogonal..

Similarly, we can prove that if $d(x)B(y,x)=0$ then *d* and *B* are orthogonal.

Conversely, if *d* and *B* are orthogonal, then $d(x)RB(x,y)=(0)$ for all $x,y\in R.$
Therefore, $d(x)B(x,y)=(0),$ by *Lemma* 1.
Similarly $d(x)B(y,x)=0.$

Theorem 7: Let *R* be 2-torsion free semiprime ring. Then a bi-(σ,τ)-derivation *B* and a (σ,τ)-derivation *d* are orthogonal iff dB is a bi-(σ,τ)-derivation.

Proof: Let *B* and *d* be such that dB is a bi-(σ,τ) derivation. Then

$$dB(xy,z)=\sigma(x)dB(y,z)+dB(x,z)\tau(y) \text{ for all } x,y,z\in R. \quad (12)$$

In *Lemma* 4, by taking $\sigma^2=\sigma$ and $\tau^2=\tau$, we get

$$dB(xy,z)=\sigma(x)dB(y,z)+d\sigma(x)\tau B(y,z)+\sigma B(x,z)d\tau(y)+dB(x,z)\tau(y). \quad (13)$$

From (12) and (13), we get

$$d\sigma(x)\tau B(y,z)+\sigma B(x,z)d\tau(y)=0.$$

By taking $\sigma(x)=x_1, B\tau=\tau B$, $\tau(y,z)=(y_1,z_1)$, $\tau(y)=y_1$ and $\sigma(x,z)=(x_1,z_1)$, the above relation reduces to

$$d(x_1)B(y_1,z_1)+B(x_1,z_1)d(y_1)=0 \text{ for all } x_1,y_1,z_1\in R.$$

So, by *Lemma* 3, we have that *d* and *B* are orthogonal.

Conversely, let *d* and *B* are orthogonal. Then Lemma 3 implies that

$$d(x)B(y,z)+B(x,z)d(y)=0 \text{ for all } x,y,z\in R. \quad (14)$$

Again by using *Lemma* 4 to the relation (13), and using $\sigma(x) = x_1$, $B\tau = \tau B$ and $\tau(y,z) = (y_1, z_1)$, $\tau(y) = y_1$ and $\sigma(x,z) = (x_1, z_1)$, we get

$$dB(xy,z) = \sigma(x)dB(y,z) + d(x_1)B(y_1,z_1) + B(x_1,z_1)d(y_1) + dB(x,z)\tau(y)$$

for all $x_1, y_1, z_1 \in R.$

By 14, it reduces to,

$$dB(xy,z) = \sigma(x)dB(y,z) + dB(x,z)\tau(y) \text{ for } x, y, z \in R.$$

Thus *dB* is a bi-(σ,τ)-derivation.

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