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SOME RESULTS ON K-CONTACT AND TRANS-SASAKIAN MANIFOLDS

[1] S. Yadav*, [2] P. K. Dwivedi and [3] D. L. Suthar

[1, 3] *Department of Mathematics, Alwar Institute of Engineering & Technology*
M. I. A. Alwar-301030, Rajasthan (INDIA)

Email: prof_sky16@yahoo.com,dd_suthar@yahoo.co.in

[2] *Department of Mathematics, Institute of Engineering & Technology*
M. I. A. Alwar-301010, Rajasthan (INDIA)

Email: drpkdwivedi@yahoo.co.in

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ABSTRACT

In this paper we study some properties of curvature tensor, projective curvature tensor with respect to semi-symmetric metric connection in a k-contact and trans-Sasakian manifold. Further we obtained necessary and sufficient condition for the Ricci tensor to be symmetric and skew-symmetric with respect to this connection $\tilde{\nabla}$.

Key words and phrases: *k-contact manifold, trans-Sasakian manifold, semi-symmetric metric connection, projective curvature tensor.*

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1. INTRODUCTION

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [7]. A linear connection $\tilde{\nabla}$ in a n –dimensional differentiable manifold M is said to be semi-symmetric connection if its torsion $\tilde{T}$ is of the form.

$$\tilde{T}(X,Y) = u(X)Y - u(Y)X, \tag{1.1}$$

where u is 1 -form The connection $\tilde{\nabla}$ is said to be metric connection if there is a Riemannian metric g such that $\tilde{\nabla}g = 0$, otherwise it is non-metric.

In 1932, Hyden [9] defined a semi-symmetric metric connection on a Riemannian manifolds and this was further developed by Yano [14] in 1970 and later studied by.Amur and Pujar [1], Bagewadi [2], De et al [6], Sharafuddin and Hussain [12].

Tripathi [13] studied a semi-symmetric metric connection in a Kenmotsu manifolds whereas Bagewadi et all studied k-contact and trans-Sasakian manifolds with reference to this connection.

2. PRILIMINARIES

Let M^n be an almost contact metric manifold [5] with almost contact metric structure (ϕ,ξ,η,g) , that is ϕ is a $(1,1)-$ tensor field, ξ is a vector field, η is a 1 –form and g is a compatible Riemannian metric such that

$$\text{(i)}\ \ \phi^2 = -I + \eta\otimes\xi, \quad \text{(ii)}\ \ \eta(\xi) = 1, \quad \text{(iii)}\ \ \phi(\xi) = 0, \quad \text{(iv)}\ \ \phi o\xi = 0, \tag{2.1}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

**Corresponding author:* [1] *S. Yadav*, *E-mail: prof_sky16@yahoo.com*

(i) $g(X,\phi Y) = -g(\phi X,Y)$, (ii) $g(X,\xi) = \eta(X)$, (2.3)

for all $X,Y \in TM$.

If M is a k –contact Riemannian manifold, then besides (2.1) and (2.2), the following relations hold [11]:

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.4}$$

$$\nabla_X .\xi = -\phi X \ , \tag{2.5}$$

$$(\nabla_X .\eta)(Y) = -g(\phi X,Y) \ , \tag{2.6}$$

$$S(X,\xi) = (n-1)\eta(X) \ , \tag{2.7}$$

for any vector field X,Y ,where R and S denotes respectively the curvature tensor of type $(1,3)$ and the Ricci tensor of type $(0,2)$.

An almost contact metric structure (ϕ,ξ,η,g) in M is called trans-Sasakian structure[9]if $(M \times R, J, G)$ belongs to the class W_4 ([4],[8]) where J is the almost complex structure on $M \times R$ defined by $J(X,\lambda \, d/dt) = (\phi X - \lambda\xi, \eta(X) \, d/dt)$ for all vector fields X on M and smooth functions λ on $M \times R$ and G is the product metric on $M \times R$ This may expressed by condition[4].

$$(\nabla_X .\phi)(Y) = \alpha(g(X,Y)\xi - \eta(Y)X). + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{2.8}$$

for some smooth function α and β on M and we say that the trans-Sasakian structure is of type (α,β). Let M be an n –dimensional trans-Sasakian manifold. From (2.8) it is easy to see that

$$\nabla_X .\xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{2.9}$$

$$(\nabla_X .\eta)(Y) = -\alpha g(\phi X,Y) + \beta g(\phi X,\phi Y) \tag{2.10}$$

In a n –dimensional trans-Sasakian manifold, we have

$$R(\xi,X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X) \ , \tag{2.11}$$

$$2\alpha\beta + \alpha\xi = 0, \tag{2.12}$$

$$S(X,\xi) = ((n-1)(\alpha^2 - \beta^2 - \xi\beta)X) - (n-2)X\beta - (\phi X)\alpha, \tag{2.13}$$

If a n –dimensional trans-Sasakian manifold of type (α,β)., we have [3].

$$\phi(grad\ \alpha) = (n-2)grad\beta. \tag{2.14}$$

Then equation (2.11) and (1, 13), reduces to

$$R(\xi,X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X) \ , \tag{2.15}$$

$$S(X,\xi) = ((n-1)(\alpha^2 - \beta^2)\eta(X). \tag{2.16}$$

In the whole paper we study trans-Sasakian manifold under the condition (2.14).

Let (M^n, g) be an n –dimensional Riemannian manifold of class C^∞ with metric tensor g and let ∇ be the Levi-Civita connection on M^n. A Linear connection $\tilde{\nabla}$ on (M^n, g) is said to be semi-symmetric [12] if the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ satisfies

$$\tilde{T}(X,Y) = \pi(X)Y - \pi(Y)X, \tag{2.17}$$

where π is a 1 –form on (M^n, g) with ρ as associated vector field,i.e. $\pi(X) = g(X,\rho)$ for any differentiable vector field X on (M^n, g).

A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection [7] if it further satisfies $\tilde{\nabla} g = 0$ .In a almost contact manifold the semi-symmetric metric connection is defined by identifying 1 -form π of (2.17) with contact 1 -form η , i.e. setting [12]

$$\tilde{T}(X,Y) = \eta(X)Y - \eta(Y)X, \tag{2.18}$$

with ξ associated vector field i.e. $g(X,\xi) = \eta(X)$

The relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M^n, g) has been obtained by .Yano [14], which is given by

$$\tilde{\nabla}_X .Y = \nabla_X .Y + \eta(Y)X - g(X,Y)\xi \tag{2.19}$$

Further, a relation between the curvature tensor R and $\tilde{R}$ of type $(1,3)$ of the connection ∇ and $\tilde{\nabla}$ respectively is given by [14]:

$$\tilde{R}(X,Y)Z = R(X,Y)Z - K(Y,Z)X + K(X,Z)Y - g(Y,Z)FX + g(X,Z)FY \ . \tag{2.20}$$

where K is a tensor of type $(0,2)$ defined as

$$K(Y,Z) = g(FY,Z) = (\nabla_X .\eta)(Z) - \eta(Y)\eta(Z) + \tfrac{1}{2}\eta(\xi)g(Y,Z) \tag{2.21}$$

for any vector field X and Y .From (2.20), it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) - (n-2)K(Y,Z) - a \cdot g(Y,Z) \tag{2.22}$$

where $\tilde{S}$ denotes the Ricci tensor with respect $\tilde{\nabla}$ to and $a = TrK$.

3. MAIN RESULTS

Lemma: 3.1. Let (M^n, g) be an $n-$dimensional trans-Sasakian manifold with semi-symmetric metric connection $\tilde{\nabla}$ Then

(i) $\left(\tilde{\nabla}_X .\phi\right)(Y) = (\nabla_X .\phi)(Y) - \eta(Y)\phi X - g(X,\phi Y)\xi$

(ii) $\tilde{\nabla}_X \xi = -\alpha\phi X - (1-\beta)\phi^2 X$

(iii) $\left(\tilde{\nabla}_X .\eta\right)(Y) = \alpha g(\phi Y, X) + (1+\beta) g(\phi X, \phi Y)$

Proof: From (2.1)-i and(2.19) we get (i). Iin view of (2.9) and (2.19) and by using(2.3)-(ii),we get(ii). Finally taking covariant derivative of (2.3)-ii and using(2.3)-i and (2.9) we get (iii).
This proves the lemma3.1.

Corollary: 1 In a $n-$dimensional trans-Sasakian manifold (M^n, g) with semi-symmetric metric connection $\tilde{\nabla}$ the tensor field K of type $(0,2)$ satisfies

$2F + I = 0$, for all X.

Let $\tilde{R}'$ be the curvature tensor of $(0,4)$ type given by

$$\tilde{R}'(X,Y,Z,U) = g(\tilde{R}(X,Y)Z,U)$$

Theorem: 3.1 In a $n-$dimensional trans-Sasakian manifold (M^n, g) with semi-symmetric metric connection $\tilde{\nabla}$ we have

(i)
$$\tilde{R}(\xi,Y)Z = \left\{(\alpha^2 - \beta^2)g(Y,Z) + 2\alpha\beta g(\phi Y,Z) - K(Y,Z)\right\}\xi - (\xi\beta(g(Y,Z) - \eta(Y)\eta(Z))$$
$$-((\alpha^2 - \beta^2) + 1)\eta(Z)Y + \tfrac{1}{2} g(Y,Z)$$

(ii) $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$

(iii) $\tilde{R}'(Y,X)Z,W) + \tilde{R}'(Y,X)Z,W) = 0$

Proof: From (2.15) and (2.20) we get

$$\tilde{R}(\xi,Y)Z=(\alpha^2-\beta^2)[g(Y,Z)\xi-\eta(Z)Y]-K(Y,Z)\xi+K(\xi,Z)Y-g(Y,Z)F\xi+g(\xi,Z)FY \tag{3.1}$$

Using (2.10) and (2.21) in(3.1) ,we get(i) i.e.

$$\begin{aligned}\tilde{R}(\xi,Y)Z=&\{(\alpha^2-\beta^2)g(Y,Z)+2\alpha\beta g(\phi Y,Z)-K(Y,Z)\}\xi-(\xi\beta(g(Y,Z)-\eta(Y)\eta(Z))\\&-((\alpha^2-\beta^2)+1)\eta(Z)Y+\tfrac{1}{2}g(Y,Z)\end{aligned} \tag{3.2}$$

By using (2.20) and first Bainchi identity

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0 .$$

with respect to Levi-Civita connection ∇ .we get

$$\begin{aligned}\tilde{R}(X,Y)Z+\tilde{R}(Y,Z)X+\tilde{R}(Z,X)Y=&-K(Y,Z)X+K(X,Z)Y-g(Y,Z)FX+g(X,Z)FY\\&-K(Z.X)Y+K(Y,X)Z-g(Z,X)FY-K(X,Y)Z+K(Z,Y)X-g(X,Y)FZ-g(Z,Y)FX\end{aligned} \tag{3.3}$$

Therefore K is symmetric then equation (3.3) reduces to

$$\tilde{R}(X,Y)Z+\tilde{R}(Y,Z)X+\tilde{R}(Z,X)Y=0$$

Finally (2.20), we get

$$\begin{aligned}\tilde{R}'(X,Y)Z,W)=&R(X,Y)Z,W)-K(Y,Z)g(X,W)+K(X,Z)g(Y,W)\\&-g(Y,Z)g(FX,W)+g(X,Z)g(FY,W)\end{aligned} \tag{3.4}$$

By changing the role of X and Y in (3.4), we get

$$\begin{aligned}\tilde{R}'(Y,X)Z,W)=&R(Y,X)Z,W)-K(X,Z)g(Y,W)+K(Y,Z)g(X,W)\\&-g(X,Z)g(FY,W)+g(Y,Z)g(FX,W)\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5) and using (2.21) we get (iii)i.e.

$$\tilde{R}'(Y,X)Z,W)+\tilde{R}'(Y,X)Z,W)=0$$

This proves the theorem3.1.

Corollary: 2 If the curvature tensor $\tilde{R}$ with respect to semi-symmetric metric connection $\tilde{\nabla}$ vanishes in trans-Sasakian manifold the scalar curvature of the manifold is that $r=-2a.$.

4. SYMMETRIC AND SKE-SYMMETRIC CONDITION

Theorem: 4.1 The Ricci tensor $\tilde{S}$ with respect to semi-symmetric metric connection $\tilde{\nabla}$ in a trans-Sasakians manifold is symmetric if and only if $K(Y,Z)=\dfrac{a}{n-2}g(Y,Z)$.

Proof: From (2.22), we get

$$\tilde{S}(Y,Z)=S(Y,Z)-(n-2)K(Y,Z)-a\cdot g(Y,Z) \tag{4.1}$$

and

$$\tilde{S}(Z,Y)=S(Z,Y)-(n-2)K(Z,Y)-a\cdot g(Z,Y) \tag{4.2}$$

Subtracting (4.2) from (4.1), we get

$$\tilde{S}(Y,Z)-\tilde{S}(Z,Y)=-2(n-2)K(Y,Z)-2a\cdot g(Y,Z)$$

If $\tilde{S}$ is symmetric then left hand side of above equation vanishes then, we get

$$K(Y,Z)=\frac{a}{n-2}g(Y,Z)$$

This prove the theorem 4.1.

Theorem: 4.2.The necessary and sufficient condition for the Ricci tensor $\tilde{S}$ with respect to connection $\tilde{\nabla}$ in trans-Sasakian manifold is skew-symmetric is that

$$S(Y,Z)=(n-2)K(Y,Z)+a\cdot g(Y,Z)$$

Proof: From (4.1) and (4.2), we get

$$\tilde{S}(Y,Z)+\tilde{S}(Z,Y)=2S(Y,Z)-2(n-2)K(Y,Z)-2a\cdot g(Y,Z) \tag{4.3}$$

If $\tilde{S}$ is skew-symmetric then left hand side of the equation(4.3) vanishes then, we get

$$S(Y,Z)=(n-2)K(Y,Z)+a\cdot g(Y,Z) \tag{4.4}$$

Moreover if using (4.4.) in(4.3),we get

$$\tilde{S}(Y,Z)+\tilde{S}(Z,Y)=0$$

This proves the theorem4.2.

Corollary: 3 In a k-contact manifold with semi-symmetric metric connection $\tilde{\nabla}$ we have

(i) $$\tilde{S}(X,\xi)=\frac{1}{2}(3n-4-a)\eta(X)$$

(ii) $$\tilde{S}(\phi X,\phi Y)=S(X,Y)+\left(\frac{1}{2}-a\right)g(X,Y)+(n-2)g(\phi X,Y)-\left(\frac{2n-2a-1}{2}\right)\eta(X)\eta(Y)$$

5. PROJECTIVE CURVATURE TENSOR

Let M be an n –dimensional k-contact manifold. The projective curvature tensor $\tilde{P}$ of type $(1,3)$ of with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is defined by[10].

$$\tilde{P}(X,Y)Z=\tilde{R}(X,Y)Z+\frac{1}{n+1}\left\{\tilde{S}(X,Y)Z-\tilde{S}(Y,X)Z\right\}-\frac{n}{n^2-n}\left\{\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y\right\}$$
$$-\frac{1}{n^2-1}\left\{\tilde{S}(Z,Y)X-\tilde{S}(Z,X)Y\right\} \tag{5.1}$$

Theorem: 5.1 An n -dimensional ξ -projectively flat k-contact manifold is locally ξ -projectively flat with semi-symmetric metric connection $\tilde{\nabla}$.

Proof: Let $\tilde{P}$ and P be the projective curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively
By using (2.20) and (2.22) equation (5.1) reduces to

$$\tilde{P}(X,Y)Z=R(X,Y)Z+\frac{1}{n+1}\{S(X,Y)Z-S(Y,X)Z\}-\frac{n}{n^2-n}\{S(Y,Z)X-S(X,Z)Y\}$$
$$-\frac{1}{n^2-1}\{S(Z,Y)X-S(Z,X)Y\}+\left(1-\frac{n(n-2)}{n^2-n}-\frac{(n-2)}{n^2-1}\right)[K(X,Z)Y-K(Y,Z)X] \tag{5.2}$$
$$+\frac{na}{n^2-n}\left(g(X,Z)Y-g(Y,Z)X\right)+g(X,Z)FY-g(Y,Z)FX$$

Equation (5.2), yields

$$\tilde{P}(X,Y)Z=P(X,Y)Z+\left(1-\frac{n(n-2)}{n^2-n}-\frac{(n-2)}{n^2-1}\right)[K(X,Z)Y-K(Y,Z)X]$$
$$+\frac{na}{n^2-n}\left(g(X,Z)Y-g(Y,Z)X\right)+g(X,Z)FY-g(Y,Z)FX \tag{5.3}$$

Taking $Z=\xi$ in (5.3) and using (2.10) and(2.21),we get

$$\tilde{P}(X,Y)\xi=P(X,Y)\xi+\left(\frac{1}{2}-\frac{(n-2)}{2(n-1)}-\frac{(n-2)}{2(n^2-1)}+\frac{an}{n^2-n}\right)[\eta(Y)X-\eta(X)Y]$$
$$+\eta(X)FY-\eta(Y)FX \tag{5.4}$$

Taking X and Y orthogonal to ξ,we get $\tilde{P}(X,Y)\xi=0$

This prove of the theorem5.1.

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