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ADJACENT EDGE GRACEFUL LABELING OF CERTAIN GRAPHS

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ABSTRACT

Let G(V, E) be a graph with n vertices and m edges. The graph G is said to be an adjacent edge graceful graph if there exists a bijection $f: E \rightarrow \{1, 2, \ldots, m\}$ such that the induced mapping $f^*: V \rightarrow \{1,2,\ldots\}$ b $f^*(u) = \sum_i f(e_i)$ taken over all edges e_i incident to adjacent vertices of u is an injection. The resulting edge and vertex labels are distinct. Then the function f is called an adjacent edge graceful labeling of G. In this chapter we developed an adjacent edge graceful labeling of certain graphs such as $P_5 \cup P_r$, Friendship F_r, the graph $f_r \cup P_5$, the graph P_n^2, dumbbell graph (P_2, C_{2r+1}), graph $P_{r,s}$ and graph (P_r, C_3) are adjacent edge graceful labeling of the graphs.

Keywords: *Edge graceful graph, Edge graceful labeling, friendship graph, dumbbell graph.*

AMS Classification (2010): *05C78.*

1. INTRODUCTION

In this paper, we consider finite, undirected, simple graph G(V,E) with n vertices and m edges. For notations and terminology we follow Bondy and Murthy [1].The concept of edge graceful labeling was first introduced by Lo in [3]. A detailed survey of graph labeling can be found in [2]. Tharmaraj and Sarasija [4, 5] introduced the concept of adjacent edge and strongly adjacent edge graceful labeling of graphs. We extended the work of adjacent edge graceful labeling for some different types of graphs such as $P_5 \cup P_r$, Friendship F_r, the graph $f_r \cup P_5$, the graph P_n^2, dumbbell graph (P_2 ,C_{2r+1}), graph $P_{r,s}$ and graph (P_r,C_3)

Definition: Let G(V, E) be a graph with n vertices and m edges. The graph G is said to be an **adjacent edge graceful graph** if there exists a bijection $f: E \rightarrow \{1,2,.,m\}$such that the induced mapping $f^*: V \rightarrow \{1,2,\ldots\}$ by $f^*(u) = \sum_i f(e_i)$ taken over all edges e_i incident to adjacent vertices of u is an injection. The resulting edge and vertex labels are distinct. The function f is called an adjacent edge graceful labeling of G.

2. MAIN RESULTS

Theorem 2.1: Every path $P_5 \cup P_r$, where $r \geq 5$ and r is not a multiple of 4, is an adjacent edge graceful graph.

Proof: Let P_r be a path with r vertices and $r - 1$ edges.

Let P_5 be a path with 5 vertices and 4 edges.

Let $V(P_5 \cup P_r) = \{v_i, 1 \leq i \leq r + 5\}$

Let $E(P_5 \cup P_r) = \{e_i = v_i v_{i+1}, 1 \leq i \leq 4;\ e_{j-1} = v_j v_{j+1}, 6 \leq j \leq r + 4\}$.

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Hence the number of vertices and edges of the graph $P_5 \cup P_n$ are $n = (r + 5)$ and $m = (r + 3)$ respectively.

Define a bijection function $f : E(P_5 \cup P_r) \to \{1,2,\ldots, m\}$ by $f(e_i) = i$, $1 \leq i \leq r + 3$.

Let f^* be the induced vertex labeling of f.

Define an injective function $f^*: V(P_5 \cup P_r) \to \{1,2,\ldots\}$ by
$f^*(v_1) = 3$, $f^*(v_2) = 6$, $f^*(v_3) = 10$, $f^*(v_4) = 9$, $f^*(v_5) = 7$,
$f^*(v_6) = 11$, $f^*(v_7) = 18$, $f^*(v_i) = 4i - 6$, $8 \leq i \leq r + 3$,
$f^*(v_{r+4}) = 3r + 6$, $f^*(v_{r+5}) = 2r + 5$.

The edge labels and vertex labels are distinct.

Hence the path $P_5 \cup P_r$, where $r \geq 5$ and r is not a multiple of 4, is an adjacent edge graceful graph.

Theorem 2.2: Every Friendship graph F_r $(r \geq 1)$ is an adjacent edge graceful graph.

Proof: Let F_r be a friendship graph with r triangle.

Hence the number of vertices and edges of the graph F_r are $n = (2r + 1)$ and $m = 3r$ respectively.

Let the vertices $F_r = \{v_i, 1 \leq i \leq 2r, v_{2r+1}\}$

Let $E(F_r) = \{v_{2r+1} v_i, 1 \leq i \leq 2r, v_{2i-1} v_{2i}, 1 \leq i \leq r\}$.

Define a bijection $f: E(F_r) \to \{1, 2,\ldots,3r\}$ by
$f(v_{2r+1} v_{2i-1}) = 3i - 2$, $1 \leq i \leq r$, $f(v_{2r+1} v_{2i}) = 3i$, $1 \leq i \leq r$ and
$f(v_{2i-1} v_{2i}) = 3i - 1$, $1 \leq i \leq r$.

Let f^* be the induced vertex labeling of f.

Define an injective function $f^*: V(F_r) \to \{1, 2,\ldots\}$ by
$f^*(v_1) = 5 + r(3r + 1)$; $f^*(v_2) = 3 + r(3r + 1)$,
$f^*(v_{2i}) = 6i - 3 + k$, $2 \leq i \leq r$ and $f^*(v_{2i-1}) = 6i + k - 1$, $2 \leq i \leq r$, where $k = 3r^2 + r$

Thus the resulting vertex and edge labels should be distinct.

Hence the Friendship graph F_r is an adjacent edge graceful graph.

Theorem 2.3: Every dumbbell graph (P_2, C_{2r+1}), $r \geq 1$, is an adjacent edge graceful graph.

Proof: The dumbbell graph (P_2, C_{2r+1}) $(r\geq 1)$ is the join sum of the cycle C_{2r+1}, $r \geq 1$ and the path graph P_2.

Let v_1 and v_2 be the vertices of the path P_2 and $\{u_i, 1 \leq i \leq 4r\}$ be the vertices of the cycle C_{2r+1} $(r \geq 1)$ and the edges of this graphs is
$\{v_1 v_2, v_1 u_1, v_1 u_{2r}, v_2 u_{2r+1}, v_2 u_{4r}, u_i u_{i+1}, 1 \leq i \leq 2r - 1, u_j u_{j+1}, 2r + 1 \leq j \leq 4r - 1\}$.

Therefore $|V(P_2, C_{2r+1})| = n = 4r + 2$ and $|E(P_2, C_{2r+1})| = m = 4r + 3$.

Define a bijection $f: E(G) \to \{1, 2,\ldots,4r + 3\}$ by $f(v_1 u_1) = 1$, $f(u_i u_{i+1}) = i + 1$, $1 \leq i \leq 2r - 1$, $f(u_{2r} v_1) = 2r + 1$, $f(v_1 v_2) = 2r + 2$, $f(v_2 u_{2r+1}) = 2r + 3$, $f(u_j u_{j+1}) = j + 3$, $2r + 1 \leq j \leq 4r - 1$, $f(v_2 u_{4r}) = 4r + 3$.

Let f^* be the induced vertex labeling of f.

The induced vertex labels $f^*: E((P_2, C_{2r+1})) \to \{1, 2,\ldots\}$ by
$f^*(v_1) = 12(1 + r)$, $f^*(v_2) = 16(1 + r)$, $f^*(u_1) = 9 + 4r$, $f^*(u_i) = 2(2i + 1)$, $2 \leq i \leq 2r - 1$, $f^*(u_{2r}) = 8r + 3$, $f^*(u_{2r+1}) = 12r + 17$, $f^*(u_{2r+i}) = 8r + 4i + 10$, $2 \leq i \leq 2r - 1$, $f^*(u_{4r}) = 16r + 11$.

The edge labels and vertex labels are distinct.

Then f becomes an adjacent edge graceful labeling of the graph (P_2, C_{2r+1}) $(r \geq 1)$.

Theorem 2.4: The graph $(f_r \cup P_5)$ is an adjacent edge graceful graph, where f_r is a fan graph, r is an odd integer and $r \geq 5$.

Proof: Let f_r be a fan graph with $r + 1$ vertices and $2r - 1$ edges, $r \geq 5$ and odd number.

Let P_5 be a path graph with 5 vertices and 4 edges. Then the number of vertices and edges of the graph $(f_r \cup P_5)$ are $n = r + 6$ and $m = 2r + 3$ respectively.

The vertices of the fan graph $f_r = \{v_i, 1 \leq i \leq r + 1\}$ and v_{r+1} is the apex vertex.

The vertices of the path graph P_5 are u_1, u_2, u_3, u_4 and u_5. Thus

Let $V(f_r \cup P_5) = \begin{cases} v_i, & 1 \leq i \leq r+1 \\ u_i, & 1 \leq i \leq 5 \end{cases}$

Let $E(f_r \cup P_5) = \begin{cases} e_j & = v_j v_{j+1}, \; 1 \leq j \leq (r-1) \\ e_{r-1+i} & = v_i v_{r+1}, \; 1 \leq j \leq r \\ e_k & = u_k u_{k+1}, \; 1 \leq k \leq 4 \end{cases}$

Define a bijection $f: E(P_5 \cup f_r) \rightarrow \{1, 2, \ldots, 2r + 3\}$ by
$f(e_j) = j, 1 \leq j \leq r - 1$, $f(e_{r-1+i}) = r - 1 + i$, $1 \leq i \leq r$, $f(e_i) = 2r - 1 + i$, $1 \leq i \leq 4$.

Then f induces an injective function $f^*: V(f_r \cup P_5) \rightarrow \{1, 2, \ldots\}$ by
$f^*(v_1) = \frac{3r^2 + r + 8}{2}$ $f^*(v_{r-1}) = \frac{3r^2 + 13r - 20}{2}$ $f^*(v_r) = \frac{3r^2+7r-10}{2}$,
$f^*(v_{r+1}) = \frac{r(5r-3)}{2}$, $f^*(v_{i+1}) = \frac{3r^2+3r+16}{2} + 6(i-1)$, $1 \leq i \leq r - 3$.
$f^*(u_1) = 4r + 1$, $f^*(u_2) = 6r + 3$, $f^*(u_3) = 8r + 6$, $f^*(u_4) = 6r + 6$, $f^*(u_5) = 4r + 5$.

Thus the vertex and edge labels are distinct.

Hence the graph $(f_r \cup P_5)$ admits an adjacent edge graceful graph.

Theorem 2.5: Every graph P_n^2 ($n \geq 5$, $n \neq 9$) is an adjacent edge graceful graph.

Proof: The square of a simple path graph P_n is the graph P_n^2 and $n \neq 9$. The vertices of the graph P_n^2 is same as the vertices of the path graph P_n. Every pair of vertices with distance two or less in P_n is connected by an edge.

Let P_n be a path with n vertices and $n - 1$ edges and $n \geq 5$, $n \neq 9$.

Let $V(P_n^2) = \{v_i, 1 \leq i \leq n\}$
$E(P_n^2) = \{v_i v_{i+1}, 1 \leq i \leq n - 1; v_i v_{i+2}, 1 \leq i \leq n - 2\}$
$|V(P_n^2)| = n$ and $|E(P_n^2)| = m = 2n - 3$

Define a bijection $f: E[P_n^2] \rightarrow \{1, 2, \ldots, m\}$ by $f(v_i v_{i+1}) = i$, $1 \leq i \leq n - 1$, $f(v_i v_{i+2}) = n + (i - 1)$, $1 \leq i \leq n - 2$.

Define the induced vertex labels as the injective function $f^*: V(P_n^2) \rightarrow \{1, 2, \ldots\}$.

Case-(i): If $n \geq 10$ then the induced vertex labels are
$f^*(v_1) = 3n + 11$, $f^*(v_2) = 5n + 19$, $f^*(v_3) = 6n + 31$, $f^*(v_4) = 7n + 45$,
$f^*(v_i) = 8n - 20 + 16i$, $5 \leq i \leq n - 4$, $f^*(v_n) = 10n - 20$, $f^*(v_{n-1}) = 15n - 34$,
$f^*(v_{n-2}) = 19n - 49$, $f^*(v_{n-3}) = 22n - 66$.

Case-(ii): If $n = 5$ then the vertex labels are
$f^*(v_1) = 26$, $f^*(v_2) = 36$, $f^*(v_3) = 39$, $f^*(v_4) = 37$ and $f^*(v_5) = 30$.

Case-(iii): If $n = 6$ then the induced vertex labels are as follows:
$f^*(v_1) = 29$, $f^*(v_2) = 49$, $f^*(v_3) = 57$, $f^*(v_4) = 60$, $f^*(v_5) = 56$, $f^*(v_6) = 40$.

Case-(iv): If $n = 7$ then the vertex labels are
$f^*(v_1) = 32$, $f^*(v_2) = 54$, $f^*(v_3) = 73$, $f^*(v_4) = 82$, $f^*(v_5) = 84$, $f^*(v_6) = 71$, $f^*(v_7) = 50$.

Case (v): If $n = 8$ then the induced vertex labels are:
$f^*(v_1) = 35$, $f^*(v_2) = 59$, $f^*(v_3) = 79$, $f^*(v_4) = 101$, $f^*(v_5) = 110$, $f^*(v_6) = 103$,
$f^*(v_7) = 86$, $f^*(v_8) = 60$.

Hence P_n^2, $n \geq 5$ and $n \neq 9$ is an adjacent edge graceful.

Theorem 2.6: The graph $P_{r,s}$ ($r \geq 3$, $s \geq 3$) is adjacent edge graceful.

Proof: Let u and v be two fixed vertices of the graph $P_{r,s}$.

Let $u_{i1}, u_{i2}, \ldots, u_{i\,r-1}$ be the vertices of the i^{th} path.

Let $V(P_{r,s}) = \{u, v, u_{ij}, 1 \leq i \leq s, 1 \leq j \leq r-1\}$ and

$$E(P_{r,s}) = \begin{cases} uu_{i1}, & 1 \leq i \leq s \\ u_{ij}u_{ij+1}, & 1 \leq i \leq s, 1 \leq j \leq r-2 \\ u_{i(r-1)}v, & 1 \leq i \leq s \end{cases}$$

$|V(P_{r,s})| = n = (r-1)s + 2$ and $|E(P_{r,s})| = m = rs$

Define a bijection $f: E(P_{r,s}) \rightarrow \{1, 2, \ldots, rs\}$ by $f(uu_{i1}) = r(i-1) + 1$, $1 \leq i \leq s$
$f(u_{ij}u_{ij+1}) = r(i-1) + j + 1$, $1 \leq i \leq s$, $1 \leq j \leq r-2$, $f(u_{i\,r-1}v) = ri$, $1 \leq i \leq s$

Let f^* be the induced vertex labeling of f.

Define a injection $f^*: E(P_{r,s}) \rightarrow \{1, 2, \ldots.\}$ by
$f^*(u) = s[r(s-1) + 3]$, $f^*(v) = s[r(s+1) - 1]$,
$f^*(u_{i1}) = r[\frac{s(s-1)}{2} + 2(i-1)] + s + 5$, $1 \leq i \leq s$
$f^*(u_{ij}) = 4[(i-1)r + j] + 2$, $1 \leq i \leq s$, $2 \leq j \leq r-2$,
$f^*(u_{i\,r-1}) = r[2i + \frac{s(s+1)}{2}] - 3$, $1 \leq i \leq s$.

Here the edge label and vertex labels are distinct.

Hence we obtained the graph $P_{r,s}$ is an adjacent edge graceful graph.

Theorem 2.7: The graph $P_{2,s}$ is not an adjacent edge graceful graph.

Proof: Let u and v be the adjacent vertices.

Here $f^*(u) = 1 + 2 + \ldots + 2s = \dfrac{2s(2s+1)}{2} = s(2s+1) = f^*(v)$

Hence the adjacent vertices u and v have same label as $s(2s+1)$.

Therefore $P_{2,s}$ is not adjacent edge graceful .

Theorem 3.24: The graph (P_r, C_3), $r \geq 2$ is an adjacent edge graceful graph.

Proof: Let $u_1, u_2, \ldots, u_r$ be the vertices of path P_r and v_i, v_{i+1}, v_{i+2} be the vertices in the i^{th} copy of cycle C_3.

$$\text{Let } V(P_r, C_3) = \begin{cases} u_i, 1 \leq i \leq r \\ v_i, 1 \leq i \leq 3r \end{cases}$$

$$\text{Let } E(P_r, C_3) = \begin{cases} u_iu_{i+1}, & 1 \leq i \leq (r-1) \\ u_iv_{3i-2}, & 1 \leq i \leq r \\ v_{3i-2}\, v_{3i-1}, v_{3i-1}\, v_{3i}, v_{3i}\, v_{3i-2}, & 1 \leq i \leq r \end{cases}$$

Then $|V(P_r,C_3)| = n = 4r$ and $|E(P_r,C_3)| = m = 5r - 1$.

Define a bijection $f: E[P_r,C_3] \rightarrow \{1, 2,\ldots,m\}$ by $f(u_i u_{i+1}) = 5i$, $1 \le i \le r-1$,
$f(u_1 v_1) = 4$, $f(u_i v_{3i-2}) = 5i - 4$, $2 \le i \le r$, $f(v_1 v_2) = 1$, $f(v_2 v_3) = 2$, $f(v_3 v_1) = 3$,
$f(v_{3i-2} v_{3i-1}) = 5i - 3$, $2 \le i \le r$, $f(v_{3i-1} v_{3i}) = 5i - 2$, $2 \le i \le r$, $f(v_{3i} v_{3i-2}) = 5i - 1$, $2 \le i \le r$.

The induced vertex label is an injection function $f^*: V(P_r,C_3) \rightarrow \{1, 2,\ldots\}$

The vertex and edge labels are distinct.

Hence we obtained that the (P_r, C_3) graph is an adjacent edge graceful graph.

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