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ALMOST SEMILATTICE

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ABSTACT

The concept of Almost Semilattice (ASL) is introduced and certain properties of $ASLs$ are derived and established set of equivalent conditions for an ASL to become semilattice. Also, introduced the concept of amicable set in ASL and certain properties of amicable set in $ASLs$ are derived. Introduce the concept of ASL with 0 and prove some properties of ASL with 0.

Key Words: *Semilattice, Almost Semilattice, Compatible Set, Maximal Set, M-amicable element, Amicable Set, unielement, unimaximal element, Almost Semilattice with zero.*

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1. INTRODUCTION

The concept of semilattice was introduced by F.Klein in (1939) [7]. He was define as a semilattice S is a partially ordered set in which any two elements α, β have a greatest lower bound $\alpha\beta$, but not necessarily a least upper bound. In Mathematical order theory, a semilattice is a partially ordered set with in which either all binary sets have a supremum (join) or all binary sets have an infimum (meet). Consequently, one speaks of either a join semilattice or meet semilattice. Semilattices provide a generalization of the more prominent concept of a lattice and as such provide a natural way to introduce this concept as partial order which is both a meet and a join semilattice. As a natural consequence of the fact that semilattices are among the most basic "Basic-like" structures, they can be characterized both in terms of order theory and Universal Algebra.

The concept of Almost Distributive Lattice (ADL) was introduced by Swamy, U. M. and Rao, G. C. [4] and they proved several properties of ADLs. Also, they introduced the concept of ideal, filter and congruences in ADLs and proved several properties of these concepts. In this paper we introduce the concept of Almost Semilattice (ASL) which is a generalization of semilattice and derive many important properties of $ASLs$. In section 2, we recall the necessary definitions and results briefly which are taken from [11]. In section 3, we introduce the concept of ASL and establish the independency of axioms in the definition. Also, we give few examples of ASL. In section 4, we prove some results in the class of ASL and obtain a few necessary and sufficient conditions for an ASL to become a semilattice. In section 5, we define amicable sets in ASL and prove that the relation between maximal sets and amicable sets in ASL. In section 6, we define unimaximal element and unielement in ASL and obtain certain properties of unimaximal elements and unielements in ASL. Finally, in section 7, we introduce the concept of ASL with 0 and prove some properties of ASL with 0.

2. PRELIMINARIES

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

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Definition 2.1: Let A and B be two nonempty sets. Then a relation R from A to B is a subset of $A \times B$. Relations from A to A are called relation on A.

A relation R on a nonempty set A may have some of the following properties:

1. R is reflexive if for all a in A we have $(a,a) \in R$.
2. R is symmetric if for all a and b in A; $(a,b) \in R$ implies $(b,a) \in R$.
3. R is antisymmetric if for all a and b in A; $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$.
4. R is transitive if for all $a,b,c \in R$; $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$.

Definition 2.2: A relation R on a nonempty set A is an equivalence relation if R is reflexive, symmetric and transitive.

Definition 2.3: A relation R on a set A is called a partial order relation if R is reflexive, antisymmetric and transitive.

In this case (A,R) is called partially ordered set or poset.

Definition 2.4: A partial order relation $\leq$ on A is called a total order or linear order if for each $a,b \in A$, either $a \leq b$ or $b \leq a$. Then $(A,\leq)$ is called a chain or totally ordered set.

Definition 2.5: Let $(P,\leq)$ be a poset. The element a in P is called a greatest (least) element of P if for all $x \in P$, we have $x \leq a \, (a \leq x)$.

Definition 2.6: Let $(P,\leq)$ be a poset. An element a in P is called a maximal (minimal) element of P if $a \leq x \, (x \leq a)$ implies $a = x$ for all $x \in P$.

Easily seen that every poset has at most one greatest (least) element. How ever, there may be none, one or several maximal (minimal) elements. Also, seen that greatest (least) element is maximal (minimal) but not converse.

Definition 2.7: Let $(P,\leq)$ be a poset and $S \subseteq P$. Then:

1. $a \in P$ is called an upper bound of S; if $s \leq a$ for all $s \in S$.
2. $a \in P$ is called a lower bound of S; if $a \leq s$ for all $s \in S$.
3. The greatest element among the lower bounds of S, whenever if exists, is called the greatest lower bound (glb) or infimum of S and is denoted by $inf\ S$.
4. The least element among the upper bounds of S, whenever if exists, is called the least upper bound (lub) or supremum of S, and is denoted by $sup\ S$.

Definition 2.8: (Zorn's Lemma): If $(P,\leq)$ is a poset such that every chain of elements in P has an upper bound in P, then P has at least one maximal element.

Definition 2.9: A semilattice is an algebra $(S,*)$ where S is nonempty set and $*$ is a binary operation on S satisfying:

1. $x*(y*z) = (x*y)*z$
2. $x*y = y*x$
3. $x*x = x$, for all $x,y,z \in S$.

In other words, a semilattice is an idempotent commutative semigroup. The symbol $*$ can be replaced by any binary operation symbol, and in fact we use one of the symbols of $\wedge, \vee, +$ or $.$, depending on the setting. The most natural example of a semilattice is $\left(P\left(X,\cap\right)\right)$, or more generally any collection of subsets of X closed under intersection. A sub semilattice of $(S,*)$ is a subset of a semilattice S which is closed under the operation $*$. Of-course that makes T a semilattice in its own right, since the equation defining a semilattice still hold in $(T,*)$. Similarly, a homomorphism between two semilattices $(S,*)$ and $(T,*)$ is a map $h:S\to T$ with the property that $h(x*y)=h(x)*h(y)$ far all $x,y\in S$. An isomorphism is a homomorphism that is $1-1$ and onto. It is worth nothing that, because the operation is determined by the order and vice versa. Also, it can be easily observed that two semilattices are isomorphic if and only if they are isomorphic as ordered sets.

Definition 2.10: [3] An element a of S is called a central element if there exist semigroup S_1 with 1 and S_2 with 0 and an isomorphism S onto $S_1\times S_2$ that mapes a onto $(1,0)$. The set $B(S)$ of all central elements of S is called the Birkhoff center of S.

Definition 2.11: [9] A ring R is called a p_1-ring if, to each $x\in R$, there exists a central idempotent $x^o\in R$ such that:

1. $xx^o=x$
2. For any idempotent e of R, $xe=x$ implies that $x^oe=x^o$.

Here, x^o is known as minimal idempotent duplicator of x in the center of R.

Definition 2.12: [5] A semigroup S with 0 is called a Bear-Stone semigroup if, to each $x\in S$, there exists a central idempotent $x^*\in S$ such that:

1. $x^*S=\{y\in S\mid xy=0=yx\}$
2. The map $s\mapsto(x^*s,x^{**}s)$ is an isomorphism of S onto $x^*S\times x^{**}S$.

Definition 2.13: [12] A ring R is called a regular ring if, to each $a\in R$, there exists $x\in R$ such that $axa=a$.

Definition 2.14: [6] A ring R is called a p- ring (p is prime) if, for any $x\in R$, $x^p=x$ and $px=0$.

Definition 2.15: [1] A ring R is called biregular if every principal ideal is generated by a central idempotent .

Definition 2.16: A ring R is a Bear ring if, to each $x\in R$, there exists a central idempotent $e\in R$ such that $eR=\{y\in R\mid xy=0=yx\}$.

Definition 2.17: A pseudocomplemented distributive lattice with 0 is called a Stone lattice if, for any $x\in L$, $x^*\vee x^{**}=1$.

Definition 2.18: [8] A pseudocomplemented semilattice S is called strongly admissible if:

1. For each $x\in S$, there exists a dense element $d\in S$ $(that\ is\ ,d^*=0)$ such that $x=x^{**}d$.
2. There is a mapping $f:S^{**}\times D\to D$, where S^{**} is the set of all closed elements of S and D the set of all dense elements of S, such that, for any $x\in S$, $x\le f(a,d)$ if and only if $x\wedge a\le d$ for all $a\in S^{**}$ and $d\in S$.
3. $f(a\vee b,d)=f(a,d)\wedge f(b,d)$ for all $a,b\in S^{**}$ and $a\in D$.

Definition 2.19: [10] Let S be a semigroup with 0 satisfying the hypothesis of the above definition. Then S is called a p_1-semigroup if:

1. For each $x\in S$, there exist $x^o\in B(S)$ such that $xx^o=x$
2. For any $a\in B(S)$ such that $ax=x$, must $ax^o=x^o$

3. DEFINITION AND INTERPRETATION OF THE AXIOMS

In this section we introduce the concept of an Almost Semilattice and we establish the independency of axioms in the definition. Further we give few examples of Almost Semilattice.

Definition 3.1: An Almost Semilattice is an algebra $(L,\circ)$ where L is a nonempty set and $\circ$ is a binary operation on L, satisfies:

(AS_1) $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)

(AS_2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)

(AS_3) $x \circ x = x$ (Idempotent Law)

For brevity, in future, we will refer to this Almost Semilattice as ASL. Now, we give examples to exhibit the idempotency of the axioms in the above definition.

Example 3.1: Let $L = \{1,2,3,...\}$, the set of all natural number. Define a binary operation $\circ$ on L by: $x \circ y = x.y$ for all $x, y \in L$, where . is a usual multiplication.

Here, the algebra $(L,\circ)$ satisfies the axioms (AS_1) and (AS_2). But, it fails to satisfy the axiom (AS_3), since $x \circ x = x.x \neq x$, for all $x(\neq 1) \in L$.

Example 3.2: Let L be a nonempty set. Define a binary operation $\circ$ on L by

$x \circ y = x$, for all $x, y \in L$.

Here, the algebra $(L,\circ)$ satisfies the axioms of (AS_1) and (AS_3). But, it does not satisfy the axiom (AS_2). Since for any three distinct elements $x, y, z \in L, (x \circ y) \circ z = x \circ z = x$ and $(y \circ x) \circ z = y \circ z = y$ .Therefore, $(x \circ y) \circ z \neq (y \circ x) \circ z$.

Example 3.3: Let $L = \{a,b,c\}$. Define a binary operation $\circ$ on L as follows:

$\circ$	a	b	c
a	a	a	c
b	c	b	a
c	a	a	c

Here, the algebra $(L,\circ)$ satisfies the axioms of (AS_2) and (AS_3). But, it fails to satisfy the axiom (AS_1), since $(a \circ b) \circ c = a \circ c = c \neq a = a \circ a = a \circ (b \circ c)$.

We conclude this section by exhibiting the structure of an ASL in some known algebras.

Example 3.4: Every semilattice $(S,\circ)$ is an ASL.

Example 3.5: Let $L = \{a,b,c\}$. Define a binary operation $\circ$ on L as below:

$\circ$	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

Then L is an ASL, but not a semilattice, since $b \circ c = c \neq b = c \circ b$.

The following examples shows that every nonempty set can be made into an ASL.

Example 3.6: Let L be a nonempty set. Define a binary operation $\circ$ on L by

$x \circ y = y$, for all $x, y \in L$

Then it is easy to verify that $(L,\circ)$ is an ASL, and it is called discrete ASL.

Example 3.7: Let $(R,+,.,0)$ be a commutative regular ring with unity. Let a^o be the unique idempotent element in R such that $aR = a^o R$. For any $a, b \in R$, define $a \circ b = a^o b$. Then $(R,\circ)$ is an ASL.

It is well known that the structure of P_1-semigroup is a common abstraction of P_1-rings and Baer- Stone semigroups. Thus the class of P_1-semigroups include the classes of Boolean rings, regular rings, P-rings, biregular rings, Bear rings, stone lattice, Strongly admissible semilattice etc. In the following example, we define a binary operation $\circ$ on a P_1-semigroup $(S,.)$ and with this operation, $(S,\circ)$ becomes an ASL. Thus we have an ASL structure in each of the algebras mentioned above.

Example 3.8: Let $(S,.)$ be a P_1-semigroup. Let us recall that, to each $x \in S$, there exists x^0 in the Birkhorff center $B(S)$ of S which is least among the elements of $B(S)$ with the property $x^0 x = x$. Since $x^0 \in B(S)$, there exists $x^{0'} \in B(S)$ such that the mapping $y \mapsto (x^0 y, x^{0'} y)$ of S onto $x^0 S \times x^{0'} S$ is an isomorphism. Now, define, for any $x, y \in S, x \circ y = x^0 y$. Then it can be easily verified that $(S,\circ)$ is an ASL.

4. PROPERTIES OF $ASLs$

In this section, we prove some results in the class of $ASLs$. Further, for any ASL L, we define a partial ordering $\leq$ on L and prove that, with this partial ordering, the poset is directed above if and only if L is a semilattice. We also obtain a few more necessary and sufficient conditions for an ASL to become a semilattice.

Throughout the remaining of this section, by L we mean an ASL $(L,\circ)$ unless otherwise specified. Using the axioms of ASL, we have the following.

Lemma 4.1: For any $a, b \in L$, we have

1. $a \circ (a \circ b) = a \circ b$
2. $(a \circ b) \circ b = a \circ b$
3. $b \circ (a \circ b) = a \circ b$

Proof: Suppose $a, b \in L$. Then,

1. $a \circ (a \circ b) = (a \circ a) \circ b$
   $= a \circ b$.
2. $(a \circ b) \circ b = a \circ (b \circ b)$
   $= a \circ b$.
3. $b \circ (a \circ b) = (b \circ a) \circ b$
   $= (a \circ b) \circ b$
   $= a \circ (b \circ b)$
   $= a \circ b$.

We introduce a partial ordering on L in the following.

Definition 4.2: For any $a, b \in L$, we say that a is less or equal to b and write $a \leq b$, if $a \circ b = a$.

Now, we prove the following results which depends on $\leq$.

Lemma 4.3: For any $a, b \in L, a \circ b \leq b$.

Proof: Let $a, b \in L$. Then we have, $(a \circ b) \circ b = a \circ b$ since by (2) of $Lemma$ 4.1. Hence $a \circ b \leq b$.

Lemma 4.4: For any $a, b \in L, a \circ b = b \circ a$ whenever $a \leq b$.

Proof: Let $a, b \in L$ such that $a \leq b$.Now, $a \circ \mathrm{b} = \mathrm{a} = \mathrm{a} \circ \mathrm{a} = (\mathrm{a} \circ \mathrm{b}) \circ \mathrm{a} = (\mathrm{b} \circ \mathrm{a}) \circ \mathrm{a} = \mathrm{b} \circ (\mathrm{a} \circ \mathrm{a}) = \mathrm{b} \circ \mathrm{a}$. Therefore $a \circ \mathrm{b} = \mathrm{b} \circ \mathrm{a}$.

In the following, we prove that $\leq$ is a partially ordered relation.

Theorem 4.5: The relation $\leq$ is a partial ordering on L.

Proof: The reflexivity of $\leq$ follows from (AS_3). Let $a,b \in L$ be such that $a \leq b$ and $b \leq a$. Then $a \circ b = a$ and $b \circ a = b$. Therefore by *lemma* 4.4, $a = b$. Thus $\leq$ is antisymmetric. Finally, suppose $a,b,c \in L$ such that $a \leq b$ and $b \leq c$. Then $a \circ b = a$ and $b \circ c = b$. Now, $a \circ c = (a \circ b) \circ c = a \circ (b \circ c) = a \circ b = a$. Therefore $a \leq c$. Thus, $\leq$ is transitive. Hence $\leq$ is a partial ordered relation on L.

Remark 4.6: If we define a relation θ on L by $a\theta b$ if $a \circ b = b$, then θ is reflexive and transitive. But, θ is not in general antisymmetric. For, consider the ASL $(L,\circ)$ defined in *example* 3.6. In this example, if L contains at least two elements, say a and b. Then we have $a\theta b$ and $b\theta a$. But, $a \neq b$ and hence θ is not antisymmetric.

However, in the following, we prove that θ is antisymmetric equivalent to an ASL to become a semilattice. Also, give a set of equivalent conditions for an ASL to become a semilattice.

Theorem 4.7: Let L be an ASL. Then the following are equivalent:

1. L is a Semilattice
2. The relation $\theta := \{(a,b) \in L \times L \mid a \circ b = b\}$ is antisymmetric
3. The relation θ defined above is a partial ordering on L

Proof:

(1) ⇒(2): Assume (1). Suppose $(a,b),(b,a) \in \theta$. Then $a \circ b = b$ and $b \circ a = a$.
Now, $a = b \circ a = a \circ b = b$. Thus $a = b$. Therefore θ is antisymmetric.

(2) ⇒(3): Suppose θ is anti-symmetric. We shall prove that θ is both reflexive and transitive. Since $a \circ a = a$ for all $a \in L$, $(a,a) \in \theta$. Therefore θ is reflexive. Suppose $(a,b),(b,c) \in \theta$. Then $a \circ b = b$ and $b \circ c = c$. Now, $a \circ c = a \circ (b \circ c) = (a \circ b) \circ c = b \circ c = c$. Thus $(a,c) \in \theta$. Hence θ is transitive. Therefore θ is a partial ordered relation on L.

(3) ⇒(1): Suppose θ is a partial ordered relation on L. We shall prove that L is a semilattice. Let $a, b \in L$. Then $(a \circ b) \circ (b \circ a) = a \circ (b \circ (b \circ a)) = a \circ ((b \circ b) \circ a) = a \circ (b \circ a) = (a \circ b) \circ a = (b \circ a) \circ a = b \circ (a \circ a) = b \circ a$ and $(b \circ a) \circ (a \circ b) = b \circ (a \circ (a \circ b)) = b \circ ((a \circ a) \circ b) = b \circ (a \circ b) = (b \circ a) \circ b = (a \circ b) \circ b = a \circ (b \circ b) = a \circ b$.
Therefore $(a \circ b, b \circ a),(b \circ a, a \circ b) \in \theta$. Since θ is antisymmetric, $a \circ b = b \circ a$. Thus L is a semilattice.

Theorem 4.8: For any $a,b \in L$, the following are equivalent:

1. L is a semilattice
2. $a \circ b \leq a$
3. $a \circ b$ is the glb of a and b in $(L,\circ)$
4. $b \circ a \leq b$
5. $b \circ a$ is the glb of a and b in $(L,\circ)$

Proof:

(1) ⇒(2): Suppose L is a semilattice. Then $a \circ b = b \circ a$ for any $a,b \in L$. Since $b \circ a \leq a$ (by leema 4.3)., $a \circ b \leq a$

(2) ⇒(3): Suppose $a \circ b \leq a$. But, we have $a \circ b \leq b$ (by Leema 4.3). Therefore, $a \circ b$ is a lower bound of a and b. Let $c \in L$ such that c is a lower bound of a and b. Then $c \leq a$ and $c \leq b$.

Now,

$$\begin{aligned} c \circ (a \circ b) &= (c \circ a) \circ b \\ &= c \circ b \quad (\because c \leq a) \\ &= c \quad (\because c \leq b) \end{aligned}$$

Therefore $c \leq a \circ b$. Hence $a \circ b$ is the glb of a and b.

(3) ⇒(1): Assume (3) We shall prove that L is a semilattice. Let $a,b \in L$. Then we have $a \circ b$ is the glb of a and b. Therefore $a \circ b \leq a$. Hence by lemma 4.4, $(a \circ b) \circ a = a \circ (a \circ b)$. This implies $(b \circ a) \circ a = (a \circ a) \circ b$ and hence $b \circ (a \circ a) = (a \circ a) \circ b$. It follows that $b \circ a = a \circ b$. Thus L is a semilattice. Similarly we can prove that conditions (1), (4) and (5) are also equivalent.

Theorem 4.9: For any $a \in L$, the set $L_a = \{x \circ a \mid x \in L\}$ is a semilattice under the induced operation $\circ$, with a as its greatest element.

Proof: Let $a \in L$. Then, clearly L_a is closed under the operation $\circ$. Hence $(L,\circ)$ is an ASL. let $t,s \in L_a$. Then $t = x \circ a$ and $s = y \circ a$ for some $x, y \in L$. Now, $t \circ s = (x \circ a) \circ (y \circ a) = ((x \circ a) \circ y) \circ a = (y \circ (x \circ a)) \circ a = y \circ ((x \circ a) \circ a) = y \circ ((a \circ x) \circ a) = y \circ (a \circ (x \circ a)) = (y \circ a) \circ (x \circ a) = s \circ t$. Then $\circ$ is commutative. Hence $(L_a,\circ)$ is a semilattice.

In the following, we prove that the operation $\circ$ is isotone.

Theorem 4.10: Let $a,b \in L$ with $a \leq b$. Then $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$ for all $c \in L$.

Proof: Suppose $a,b \in L$ such that $a \leq b$. Then $a \circ b = a$. Now,
$(a \circ c) \circ (b \circ c) = ((a \circ c) \circ b) \circ c = (b \circ (a \circ c)) \circ c = ((b \circ a) \circ c) \circ c = (b \circ a) \circ (c \circ c) = (a \circ b) \circ c = a \circ c$.
Thus $(a \circ c) \circ (b \circ c) = a \circ c$. Therefore $a \circ c \leq b \circ c$. Also, consider
$(c \circ a) \circ (c \circ b) = ((c \circ a) \circ c) \circ b = (c \circ (c \circ a)) \circ b = ((c \circ c) \circ a) \circ b = (c \circ a) \circ b = c \circ (a \circ b) = c \circ a$. Thus $(c \circ a) \circ (c \circ b) = c \circ a$. Hence $c \circ a \leq c \circ b$.

5. AMICABLE SETS

If $(S,.)$ is a P_1- semigroup, then S is an ASL as described in *example* 3.8 and the Birkhoff center $B(S)$ of S has the following property; "given $x \in S$ there is an element in $B(S)$, $(via.x^0)$ which is least among the elements a of $B(S)$ such that $a \circ x = ax = x$ ".

In this section, we introduce compatible set, maximal set M, M-amicable element and amicable set in ASL L. Also, we prove some results on these concepts. We establish a relation between maximal sets and amicable sets. First we introduce definition of compatible set.

Throughout the remaining of this section, by L we mean an ASL $(L,\circ)$ unless otherwise specified.

Definition 5.1: Let L be an ASL. Then for any $a,b \in L$, we say that a is compatible with b and write $a \sim b$ if $a \circ b = b \circ a$. A subset S of L is said to be compatible set if $a \sim b$, for all $a,b \in S$.

If L is an ASL, then it can be easily seen that for any $a \in L$, $\{a\}$ is a compatible set. Also, seen that, the set of all compatible sets in an ASL L is a poset with respect to set inclusion.

Definition 5.2: Let L be an ASL. Then a maximal compatible set of L is called a *maximal* set.

It is clear that if L is a semilattice, then L itself a maximal set. This clearly that $\sim$ is reflexive and symmetric. But, ingeneral $\sim$ is not transitive in L. For, consider the ASL in *example* 3.5. In this example, we have $b \sim a$ and $a \sim c$. But, $b \nsim c$ since $b \circ c = c \neq b = c \circ b$. Hence $\sim$ is not transitive in L.

Now, we prove the following:
Theorem 5.3: For any $a, b \in L, a \sim b$ if and only if $a \circ b \sim b \circ a$.

Proof: Suppose $a \circ b \sim b \circ a$. Then $(a \circ b) \circ (b \circ a) = (b \circ a) \circ (a \circ b)$. Now,
$a \circ b = (a \circ b) \circ (a \circ b) = (b \circ a) \circ (a \circ b) = (a \circ b) \circ (b \circ a) = ((a \circ b) \circ b) \circ a = (a \circ (b \circ b)) \circ a = (a \circ b) \circ a = (b \circ a) \circ a = b \circ (a \circ a) = b \circ a$. Hence $a \sim b$. The converse is trivial.

In the following, we prove that, any maximal set in ASL, is a semilattice. For this first we need the following lemmas.

Lemma 5.4: For any $a,b,c \in L$, $a \sim b$ and $a \sim c$ imply that $a \sim b \circ c$.

Proof: Suppose $a \sim b$ and $a \sim c$ in L. Now, $a \circ (b \circ c) = (a \circ b) \circ c = (b \circ a) \circ c = b \circ (a \circ c) = b \circ (c \circ a) = (b \circ c) \circ a$. Therefore $a \sim b \circ c$.

Lemma 5.5: Let M be a maximal set in L and $x \in L$ be such that $x \sim a$, for all $a \in M$. Then $x \in M$.

Proof: Suppose M is a maximal set and $x \in L$ with $x \sim a$ for all $a \in M$. Thus $M \cup \{x\}$ is a compatible set and $M \subseteq M \cup \{x\}$. It follows that $M = M \cup \{x\}$. Therefore $x \in M$.

Now, we prove the following theorem.

Theorem 5.6: If M is a maximal set in L, then M is a semilattice under the induced operation $\circ$ on L.

Proof: Let M be a maximal set in L. Then, by lemma 5.4 and 5.5, M is closed under the operation $\circ$. It follows that $(M, \circ)$ is a semilattice.

We immediately have the following corollary, whose proof is straight forward.

Corollary 5.7: The following are equivalent in an ASL L.
1. L is a semilattice
2. L is a compatible set
3. L is a maximal set

In the following, we prove some more properties of maximal set in L.

Theorem 5.8: Let M be a maximal set in L and $a \in M$. Then for any $x \in L, x \circ a \in M$.

Proof: Suppose M is a maximal set in L and $a \in M$. Then for any $x \in L$ and $b \in M$, consider $(x \circ a) \circ b = x \circ (a \circ b) = x \circ (b \circ a) = (x \circ b) \circ a = (b \circ x) \circ a = b \circ (x \circ a)$. Thus $x \circ a \sim b$. Therefore by *lemma* 5.5, $x \circ a \in M$.

Corollary 5.9: Let M be a maximal set. Then M is an initial segment in the poset $(L, \circ)$. That is, for any $x \in L$ and $a \in M, x \leq a$ implies $x \in M$.

Proof: Suppose M is a maximal set and $a \in M$ such that $x \leq a$ $(x \in L)$. Then $x \circ a = x$. Since $x \in L$ and $a \in M$, by the above *theorem* 5.8, $x \circ a \in M$. Therefore $x = x \circ a \in M$.

Now, we give the definition of M-amicable element in L.

Definition 5.10: Let M be a maximal set in L. Then an element $x \in L$ is said to be $M-amicable$ if there exists $a \in M$ such that $a \circ x = x$.

Theorem 5.11: Let M be a maximal set and $x \in L$ be M- amicable. Then there exists an element $a \in M$ with the following properties:
1. $a \circ x = x$
2. For any $b \in L$ with $b \circ x = x$. Then $b \circ a = a$

Proof: Let $x \in L$ be M-amicable. Then by the definition of M-amicable, there exists an element $c \in M$ such that $c \circ x = x$. Thus by theorem 5.8, $x \circ c \in M$.

Now, put $a = x \circ c$. Then, $a \circ x = (x \circ c) \circ x = (c \circ x) \circ x = c \circ (x \circ x) = c \circ x = x$. Therefore (1) is proved. Now, suppose $b \in L$ with $b \circ x = x$. Then $b \circ a = b \circ (x \circ c) = (b \circ x) \circ c = x \circ c = a$. Hence (2) is proved.

Corollary 5.12: If M is a maximal set and $x \in L$ is M-amicable, then there is a smallest element $a \in M$ with the property $a \circ x = x$.

Proof: Suppose M is a maximal set and $x \in L$ is M-amicable. Then by (1) of theorem 5.11, there exists an element a of M such that $a \circ x = x$. It remains to show that a is the smallest element of M . Suppose $b \in M$ such that $b \circ x = x$. Since M is a maximal set and $a, b \in M$, we have $a \sim b$. Now, by leema 5.11(2), we have $a = b \circ a = a \circ b$. Therefore $a \circ b = a$. Thus $a \leq b$. Hence a is the smallest element of M , with property that $a \circ x = x$.

We denote the element a of M in the above $corollary$ 5.12 by x^M . Observe that x^M depends on M as well as on x .

Corollary 5.13: Let M be a maximal set in L and $x \in L$. Then x is M-amicable and $x = x^M$ if and only if $x \in M$.

Proof: Suppose $x \in M$. Since $x, x^M \in M$, we have $x = x^M \circ x = x \circ x^M$. Therefore $x \leq x^M$. On the other hand, we have x^M is the smallest element in M with the property that $x^M \circ x = x$. It follows that $x^M \leq x$ since $x \circ x = x, x \in M$. Thus $x = x^M$. Converse is clear.

Corollary 5.14: Let M be a maximal set and $x \in L$ be M-amicable. If $a \in L$ such that $x \circ a = a$, then a is M-amicable and $a^M \leq x^M$.

Proof: Suppose $x \in L$ is M-amicable and $a \in L$ such that $x \circ a = a$. We have x^M is the smallest element in M such that $x^M \circ x = x$. Now, $x^M \circ a = x^M \circ (x \circ a) = (x^M \circ x) \circ a = x \circ a = a$. Hence a is M-amicable. Also, since a is M-amicable, we have a^M is the smallest element in M such that $a^M \circ a = a$. It follows that $a^M \leq x^M$.

Corollary 5.15: Let M be a maximal set and $a \in L$ be M- amicable. Then $a \circ a^M = a^M$.

Proof: Suppose $a \in L$ is M-amicable. Then a^M is the smallest element in M such that $a^M \circ a = a$ and $a \circ a = a$. Now, by $theorem$ 5.11(2) we get $a \circ a^M = a^M$.

Corollary 5.16: Let M be a maximal set and $x \in M$ be M-amicable. Then x^M is the largest element of M with the property $x \circ x^M = x^M$.

Proof: Suppose M is a maximal set and $x \in M$ is M-amicable. Then by $corollary$ 5.13 we get $x = x^M$. Hence $x \circ x^M = x^M$. Now, we show that x^M is the largest element. Suppose $b \in M$ such that $x \circ b = b$. Since both $x, b \in M$ and M is maximal set, $b \circ x = x \circ b = b$. Thus $b \leq x$. Hence $b \leq x^M$. Therefore x^M is the largest element of M with the property $x \circ x^M = x^M$.

Corollary 5.17: Let M be a maximal set and $x \in L$ be M-amicable. Then, for any $a \in L, a \circ x = x$ and $x \circ a = a$ if and only if a is M-amicable and $x^M = a^M$.

Proof: Let $x \in L$ be M-amicable and $a \in L$. Suppose $a \circ x = x$ and $x \circ a = a$. Since x is M-amicable, there exists a smallest element $x^M \in L$ such that $x^M \circ x = x$.

Now, $x^M \circ a = x^M \circ (x \circ a) = (x^M \circ x) \circ a = x \circ a = a$. Thus a is M-amicable. It remains to show that $x^M = a^M$. Now, $a^M \circ x = a^M \circ (a \circ x) = (a^M \circ a) \circ x = a \circ x = x$. It follows that $x^M \leq a^M$. Similarly we get $a^M \leq x^M$. Therefore, $a^M = x^M$.

Conversely, suppose a is M-amicable and $a^M = x^M$. Then
$a \circ x = a \circ (x^M \circ x) = a \circ (a^M \circ x) = (a \circ a^M) \circ x = a^M \circ x = x^M \circ x = x$. And also,
$x \circ a = x \circ (a^M \circ a) = x \circ (x^M \circ a) = (x \circ x^M) \circ a = x^M \circ a = a^M \circ a = a$. Therefore $a \circ x = x$ and $x \circ a = a$.

Corollary 5.18: Let M be a maximal set and $x \in L$ be M-amicable. Then x^M is the unique element of M such that $x^M \circ x = x$ and $x \circ x^M = x^M$.

Proof: Suppose M is a maximal set and $x \in L$ is M-amicable. Since $x^M \in M$, we have x^M is M-amicable and $x^M = (x^M)^M$. Put $a = x^M$, then $a = a^M$. Therefore, $x^M = a^M$. Since a is M-amicable and $x^M = a^M$, Corollary 5.17 imply that $a \circ x = x$ and $x \circ a = a$. Therefore, $x^M \circ x = x$ and $x \circ x^M = x^M$. Now, it remains to show that x^M is a unique element of M. Suppose $b \in M$ such that $b \circ x = x$ and $x \circ b = b$. Now, we show that $b = x^M$. Since $b \circ x = x$, $x \circ b = b$ and $b \in M$, by Corollary 5.13 and 5.17, $b = b^M = x^M$. Thus $b = x^M$ and hence x^M is a unique element of M satisfying the given condition.

Corollary 5.19: Let M be a maximal set in L and $x, y \in L$ be M-amicable such that $x \sim y$. Then $x^M = y^M$ if and only if $x = y$.

Proof: Suppose $x^M = y^M$. Then by colollary 5.17 we have $x \circ y = y$ and $y \circ x = x$. Thus $x = y \circ x = x \circ y = y$, since $x \sim y$. Conversely, suppose $x = y$. Then $x \circ y = y \circ y = y$ and $y \circ x = x \circ x = x$. Therefore by corollary 5.17, $x^M = y^M$.

If M is a maximal set in L, then we denote the set of all M-amicable elements of L by $A_M(L)$. Now we prove that $A_M(L)$ is an ASL with the induced operation on L.

Theorem 5.20: Let M be a maximal set. Then $(A_M(L), \circ)$ is an ASL. Moreover, for any $x, y \in A_M(L)$, we have $(x \circ y)^M = x^M \circ y^M$.

Proof: Suppose M is a maximal set of L and $A_M(L)$ is the set of all M-amicable elements of L. Now, we shall prove that $A_M(L)$ a sub ASL of L. Let $a, b \in A_M(L)$. Then there exists $x, y \in M$ such that $x \circ a = a$ and $y \circ b = b$. Now, $(x \circ y) \circ (a \circ b) = ((x \circ y) \circ a) \circ b) = (x \circ (y \circ a)) \circ b = x \circ ((y \circ a) \circ b) = x \circ ((a \circ y) \circ b) = x \circ (a \circ (y \circ b)) = (x \circ a) \circ (y \circ b) = a \circ b$.

Let $t \in M$. Then $(x \circ y) \circ t = x \circ (y \circ t) = x \circ (t \circ y) = (x \circ t) \circ y = (t \circ x) \circ y = t \circ (x \circ y)$. This imply that $(x \circ y) \sim t$, for all $t \in M$. Thus, by leema 5.5 $x \circ y \in M$ and hence $a \circ b$ is M-amicable. Therefore $a \circ b \in A_M(L)$. Hence $(A_M(L), \circ)$ is a sub ASL and hence is ASL. It remains to show that $(x \circ y)^M = x^M \circ y^M$. Now, consider $(x^M \circ y^M) \circ (x \circ y) = (y^M \circ x^M) \circ (x \circ y) = y^M \circ (x^M \circ (x \circ y)) = y^M \circ ((x^M \circ x) \circ y) = y^M \circ (x \circ y) = (y^M \circ x) \circ y = (x \circ y^M) \circ y = x \circ (y^M \circ y) = x \circ y$. Also, $(x \circ y) \circ (x^M \circ y^M) = (y \circ x) \circ (x^M \circ y^M) = y \circ (x \circ (x^M \circ y^M)) = y \circ ((x \circ x^M) \circ y^M) = y \circ (x^M \circ y^M) = (y \circ x^M) \circ y^M = (x^M \circ y) \circ y^M = x^M \circ (y \circ y^M) = x^M \circ y^M$. Hence $(x^M \circ y^M)^M = (x \circ y)^M$. Now, we show that $(x^M \circ y^M)^M = x^M \circ y^M$. But, we have $x^M \circ y^M = (x^M \circ y^M)^M$, since $x^M \circ y^M \in M$. Therefore, $(x \circ y)^M = x^M \circ y^M \in M$.

It can be easily seen that for any maximal set M of L, $M \subseteq A_M(L) \subseteq L$. Now we prove the following:

Theorem 5.21: Let M be a maximal set in L. Then the following are equivalent:

1. $M = A_M(L)$
2. $M = L$
3. L is a semilattice

Proof:

(1) ⇒(2): Assume (1). Let $a \in M$ and $x \in L$. Then by *theorem* 5.8, $x \circ a \in M$. We need to show that $a \circ x \in M$ Now, consider

$(x \circ a) \circ (a \circ x) = ((x \circ a) \circ a) \circ x = (x \circ (a \circ a)) \circ x = (x \circ a) \circ x = (a \circ x) \circ x = a \circ (x \circ x) = a \circ x$. Hence $a \circ x \in A_M(L) = M$. Therefore, $x \circ a, a \circ x \in M$ and hence $a \circ x \sim x \circ a$. It follows that $(a \circ x) \circ (x \circ a) = (x \circ a) \circ (x \circ a)$. Hence, we get $x \circ a = a \circ x$. Therefore $x \sim a$. Thus $x \in M$ since M is a maximal set. Therefore $L \subseteq M$ and hence $M = L$.

(2) ⇒(3): Suppose $M = L$. Since M is a maximal set and hence is a compatible set, it follows that $a \circ b = b \circ a$ for all $a, b \in M(= L)$. Thus L is a semilattice.

(3) ⇒(1): Assume (3). Clearly $M \subseteq A_M(L)$. Conversely, let $a \in A_M(L)$. Then, for any $t \in M$, we have $t \circ a = a \circ t$, since L is a semilattice. Hence $t \sim a$. It follows that $a \in M$. Therefore, $A_M(L) \subseteq M$. Hence $A_M(L) = M$.

Definition 5.22: A maximal set M in L is said to be *amicable* if $A_M(L) = L$. That is, every element in L is M-amicable.

Note that, in a discrete ASL, every singleton set is amicable. Now we prove the following theorem:

Theorem 5.23: If $(S,.)$ is a P_1- semigroup, then $B(S)$, the Birkhoff center of S is an amicable set in S.

Proof: Let $(S,.)$ be a P_1- semigroup. Let us recall that, for any $x \in S$, there is a smallest idempotent element $x^0 \in B(S)$ such that $x^0 x = x$. It is enough if we prove that $B(S)$ is a maximal set and $x^0 \circ x = x$ and $x \circ x^0 = x^0$. Let $x \in S$ such that $x \sim a$ for all $a \in B(S)$. In particular, $x \sim x^{\circ}$ so that $x^0 = x^0 x^0 = x \circ x^0$ $= x^0 \circ x = x^0 x = x$ and hence $x \in B(S)$. Thus $B(S)$ is a maximal set. Now, by the definition of the operation $\circ$ on S, we have $x^0 \circ x = x^0 x = x$ and $x \circ x^0 = x^0 x^0 = x^0$ for all $x \in S$. Hence $B(S)$ is an amicable set where, for any $x \in S$, $x^{B(S)} = x^0$.

In the following example we describe an $ASL\ L$ and exhibit a maximal set M of L for which $A_M(L) \subsetneq L$. That is, M is a maximal set but not amicable.

Example 5.1: Let L be the set of all sequences $\{a_n\}$ of nonnegative integers whose range is finite. Define a binary operation $\circ$ on L as follows: For any $\{a_n\}, \{b_n\} \in L$,

$$\{a_n\} \circ \{b_n\} = \{c_n\} \text{ where } c_n = \begin{cases} b_n, & \text{if} a_n \neq 0 \\ 0, & \text{if} a_n = 0 \end{cases} \tag{1}$$

Then it can be verified that $(L,\circ)$ is an ASL. Also observe that, for any $\{a_n\}, \{b_n\} \in L$, $\{a_n\} \sim \{b_n\}$ if and only if $a_n \neq 0 \neq b_n$ implies $a_n = b_n$. Write $M = \{\{a_n\} \in L |\ a_n = n\ or\ a_n = 0, for\ all\ n\}$. Observe that every sequence in M has only a finite number of nonzero entries. Clearly, M is a compatible set in L. Now, we prove that M is a maximal set. Let $\{c_n\} \in L$ and let $\{c_n\} \sim \{a_n\}$ for all $\{a_n\} \in M$. Suppose for some m, $c_m \neq 0$.

Now consider the sequence $\{a_n\}$ where
$$a_n = \begin{cases} m, & \text{if} n = m \\ 0, & \text{if} n \neq m \end{cases} \tag{2}$$

Then $\{c_n\} \in M$ so that $\{c_n\} \sim \{a_n\}$. Hence $c_m = a_m = m$ since $c_m \neq 0 \neq a_m$. Thus $\{a_n\} \in M$. Therefore, M is a compatible set.

Now consider the constant sequence $\{1\}$. Here, $\{1\} \in L$, but $\{1\} \notin A_M(L)$. For, if $\{1\} \in A_M(L)$, then there exists $\{a_n\} \in M$ such that $\{a_n\} \circ \{1\} = \{1\}$ which means $a_n \neq 0$ for all n, which is a contradiction. Hence M is a maximal set in L which is not amicable.

Definition 5.24: Let $(L_1, \circ)$ and $(L_2, \circ)$ be two $ASLs$. Then a mapping $\psi : L_1 \to L_2$ is said to be a $homomorphism$, if for any $x, y \in L,\ \psi(x \circ y) = \psi(x) \circ \psi(y)$. A homomorphism ψ is said to be a $monomorphism(epimorphism)$ if ψ is $1-1$ (onto) and ψ is said to be an $isomorphism$ if ψ is bijection.

Finally, we conclude this section with the following theorem which explains the relation between the maximal sets in L and the amicable sets in L.

Theorem 5.25: Let M be a maximal set and M' be an amicable set in L. Then the mapping $a \mapsto a^{M'}$ is a monomorphism of the semilattice $(M, \circ)$ into the semilattice $(M', \circ)$. Further, if M is also amicable, then the above mapping is a surjection.

Proof: Suppose M is a maximal set and M' is an amicable set in L. Define $\psi : M \to M'$ by $\psi(a) = a^{M'}$. Then for any $a, b \in M$, we have $a \circ b \in M$. Now, by theorem 5.20. we get $\psi(a \circ b) = (a \circ b)^{M'} = a^{M'} \circ b^{M'}$. $= \psi(a) \circ \psi(b)$ Hence ψ is a homomorphism. Suppose $a, b \in M$ such that $\psi(a) = \psi(b)$. Then $a^{M'} = b^{M'}$. Therefore by corollary 5.19 $a = b$ since every element in M is M-amicable. Hence ψ is an injection. Therefore, ψ is a monomorphisim. Suppose M is an amicable. Then $A_M(L) = L$. But, we have M' is amicable. Therefore $A_{M'}(L) = L$. Hence $A_M(L) = L = A_{M'}(L)$. Let $b \in M'$. Then $b \in A_{M'}(L) = L = A_M(L)$. Thus b is M-amicable. Now, by corollary 5.17, there exists a unique element $b^M \in M$ such that $b \circ b^M = b^M$ and $b^M \circ b = b$. On the other hand $b^M \in M$ is M-amicable. Since M' is amicable, $b^M \in A_M(L) = L = A_{M'}(L)$. Thus b^M is M'-amicable. Now, by $corollary$ 5.17, there exists a unique element $(b^M)^{M'} \in M'$ such that $b^M \circ (b^M)^{M'} = (b^M)^{M'}$ and $(b^M)^{M'} \circ b^M = b^M$. Hence by uniqueness we get $b = (b^M)^{M'}$. Now, put $a = b^M \in M$. Then $\psi(a) = \psi(b^M) = (b^M)^{M'} = b$. Hence ψ is a surjection.

6. UNIELEMENT AND UNIMAXIMAL ELEMENT

In this section, we introduce the concept of unielement and unimaximal element in ASL L and prove some properties of these concept. First we begin this section with the following definition:

Definition 6.1: Let M be a maximal set in L. An element u of L is said to be a unimaximal of M if $a \leq u$ for all $a \in M$.

Observe that a unimaximal of a maximal set, if it exists, then it is unique and is in M. As usual, we say that an element $x \in L$ is a maximal if, for any $y \in L, x \leq y$ implies $x = y$.

Definition 6.2: An element $m \in L$ is said to be unimaximal if $m \circ x = x$ for all $x \in L$. Observe that every unimaximal element is maximal and also, in discrete ASL, every element is unimaximal and hence are maximal elements.

Theorem 6.3: If x and y are elements of L which are unimaximal, then we have the following:

1. $x \circ y$ is unimaximal
2. $y \circ x$ is unimaximal

Proof: Suppose x and y are unimaximal elements of L. Then $x \circ a = a$ and $y \circ a = a$ for all $a \in L$. Now, $(x \circ y) \circ a = x \circ (y \circ a) = x \circ a = a$. Therefore $x \circ y$ is unimaximal. Similarly, $y \circ x$ is unimaximal.

It is clear that for any $x, y \in L$, $x \circ y$ is a unimaximal if and only if $y \circ x$ is a unimaximal.

Theorem 6.4: Let M be a maximal set in L with unielement u. Then $M = \{x \circ u \mid x \in L\}$.

Proof: Put $H = \{x \circ u \mid x \in L\}$. Now, we shall prove that $M = H$. Suppose M is a maximal set with the unielement u. Then we have $x \le u$ for all $x \in M$. Hence $x = x \circ u$ for all $x \in M$. Therefore $M \subseteq H$. Let $t \in H$. Then $t = x \circ u$ for some $x \in L$. Now, let $s \in M$. Then $s \circ t = s \circ (x \circ u) = (s \circ x) \circ u = (x \circ s) \circ u =$. $x \circ (s \circ u) = x \circ (u \circ s) = (x \circ u) \circ s = t \circ s$ Therefore $s \sim t$ for all $s \in M$. It follows that $t \in M$ since M is a maximal set. Thus $H \subseteq M$. Hence $M = H$.

Corollary 6.5: Let m be a unimaximal element of L. Then the set $M_m := \{x \circ m \mid x \in L\}$ is a maximal set in L, with m as its unielement.

Proof: Clearly M_m is a compatible set. Let $y \in L$ be such that $y \sim x \circ \mathrm{m}$ for all $x \in L$. In particular, $y \sim m$, so that $y \circ m = m \circ y = y$, since m is unimaximal. Hence $y \in M_m$. Thus M_m is a maximal set. Clearly m is the unielement of M_m.

Theorem 6.6: Suppose L has a unimaximal element. If a maximal set M of L is amicable, then M has a unielement.

Proof: Suppose M is amicable. Let m be a unimaximal element of L. Since $m \in L = A_M(L)$, there exists $a \in M$ such that $a \circ m = m$. Now, for any $t \in L$, $a \circ \mathrm{t} = \mathrm{a} \circ (\mathrm{m} \circ \mathrm{t}) = (\mathrm{a} \circ \mathrm{m}) \circ \mathrm{t} = \mathrm{m} \circ \mathrm{t} = \mathrm{t}$. Therefore, a is a unimaximal element. Let $s \in M$. Then $s \sim a$ and hence $s \circ a = a \circ s = s$. Hence $s \le a$ for all $s \in M$. Therefore, a is a unielement of M.

Lemma 6.7: Let M be a maximal set with unielement u. If u is a unimaximal element of L, then M is amicable.

Proof: Suppose M is a maximal set with a unielement u and suppose u is a unimaximal element of L. Then for any $t \in L$, $u \circ t = t$. Therefore every element in L is M-amicable. Thus $A_M(L) = L$. Hence M is amicable.

Now, we have the following corollary in the view of theorem 5.25..

Corollary 6.8: If L has a unimaximal element, then every maximal set can be embedded in any maximal set with unielement.

Recall that an element $a \in L$ is said to be minimal element of L if $x \le a$, then $x = a$, for all $x \in L$.

Theorem 6.9: The following are equivalent in L:

1. a is a minimal element of L
2. $x \circ a = a$ for all $x \in L$

Proof:

(1) ⇒(2): Assume (1). Let $x \in L$. Then we have $x \circ a \le a$. It follows that $x \circ a = a$ since a is minimal.

(2) ⇒(1): Assume (2). Suppose $y \in L$ such that $y \le a$. Then $y = y \circ a = a$ and hence a is minimal.

Corollary 6.10: L is discrete if and only if every element of L is minimal.

Proof: Suppose L is a discrete ASL and suppose $a \in L$. Then $x \circ a = a$ for all $x \in L$. Therefore by theorem 5.35, a is a minimal element of L. Conversely, suppose every elements of L is minimal. Then we have $x \circ a = a$ for all $x \in L$ and for all $a \in L$. Thus L is discrete ASL.

Finally, we conclude this section with the following theorem which explains every element in L contains a minimal element.

Theorem 6.11: If L has minimal elements, then for each $x \in L$, there exists a minimal element $m \in L$ such that $m \leq x$.

Proof: Suppose L has a minimal element say n. Now, let $x \in L$. Put $m = n \circ x$. Then for any $t \in L$ with $t \leq m$, we get $t = t \circ m = t \circ (n \circ x) = (t \circ n) \circ x = n \circ x = m$. Hence m is a minimal element of L. Also, $n \circ x \leq x$, we get $m \leq x$.

7. ALMOST SEMILATTICE WITH ZERO

In this section we introduce the concept of zero element in an Almost Semilattice analogous to that of the least element in a semilattice. We establish the independency of axioms in the definition. Further, we give few examples of Almost Semilattice with 0 and prove several properties of ASL with 0.

Definition 7.1: Let L be an ASL. An element $0 \in L$ is called a *zero element* of L if $0 \circ a = 0$ for all $a \in L$.

Observe that an ASL can have at most one zero element and it will be the least element of the poset $(L, \leq)$. We always denote the zero element of L, if it exists, by '0'. If L has 0, then the algebra $(L, \circ, 0)$ is called an ASL with '0'. Now, we have the following theorem whose proof is straight forward.

Theorem 7.2: Let $L = (L, \circ)$ be an ASL and 0 be any external element of L. For any $x, y \in L \cup \{0\}$, define:

$$x \circ y = \begin{cases} x \circ y, \ (in \ L) & \text{if} x, y \in \text{L}. \\ 0, & \text{Otherwise.} \end{cases} \tag{3}$$

Thus $(L \cup \{0\}, \circ . 0)$ is an ASL with 0. We denote this ASL by L^o.

According to *definition* 7.1, an ASL with 0 is an algebra $(L, \circ, 0)$ of type $(2,0)$ satisfying the following axioms:

(AS_1) $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)

(AS_2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)

(AS_3) $x \circ x = x$ (Idempotent Law)

(AS_0) $0 \circ x = 0$ (for all $x \in L$)

For brevity, in future, we will refer to this Almost Semilattice with 0 as ASL with 0. Now, we give examples to exhibit the idempotency of the axioms in the above definition.

Throughout the remaining of this section, by L^0 we mean an ASL with 0 $(L, \circ, 0)$ unless otherwise specified.

Example 7.1: Let L be a nonempty set. Define a binary operation $\circ$ on L^0 by

$$x \circ y = x \text{ for all } x, y \in L^0.$$

Here, the algebra $(L, \circ, 0)$, satisfies the axiom of (AS_1), (AS_3) and (AS_0). But, it fails to satisfy the axiom (AS_2), since for any distinct three elements $x, y, z \in L^0, (x \circ y) \circ z = x \circ z = x$ and $(y \circ x) \circ z = y \circ z = y$. Therefore, $(x \circ y) \circ z \neq (y \circ x) \circ z$.

Example 7.2: Let $L^0 = \{0, 1, 2, ...\}$. Define a binary operation $\circ$ on L^o by

$$x \circ y = x.y \text{ for all } x, y \in L^o.$$

Here, L^o satisfies the axioms of (AS_1), (AS_2) and (AS_0). But, it is not satisfy the axiom (AS_3), since $x \circ x = x.x \neq x$, for all $x (\neq 0, 1) \in L^o$.

Example 7.3: Let L be a nonempty set. Define a binary operation $\circ$ on L by

$$x \circ y = y \text{ for all } x, y \in L.$$

Here, the algebra $(L,\circ,0)$ satisfies the axioms of (AS_1), (AS_2) and (AS_3). But, it fails to satisfy the axiom (AS_0), since $0\circ x = 0 \neq x$ for all $x \in L$.

Example 7.4: Let $L^o = \{0,a,b,c\}$. Define the binary operation $\circ$ on L as follows:

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	a	c
b	0	c	b	a
c	0	a	a	c

Here, L^o satisfies the axioms of $(AS_2),(AS_3)$ and (AS_0). But, it fails to satisfy the axiom (AS_1), since $(a\circ b)\circ c = a\circ c = c \neq a = a\circ a = a\circ(b\circ c)$.

Example 7.5: Let $L^o = \{0,a,b,c\}$. Define a binary operation $\circ$ as follows:

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c
c	0	a	b	c

It is easy to verify that $(L^o,\circ)$ is an ASL with 0.

Example 7.6: Let L be a nonempty set and fix $x_o \in L$. Define a binary operation $\circ$ on L by:

$$x\circ y = \begin{cases} y, & \text{if} \mathrm{x} \neq \mathrm{x_o}. \\ x_o, & \text{if} \mathrm{x} = \mathrm{x_o}. \end{cases} \tag{4}$$

Then L is an ASL with x_o as its zero.

In the rest of this section, we prove some results in the class of $ASLs$ with 0. Throughout the remaining of this section, by L we mean an ASL with 0 unless otherwise specified. Now, we prove the following.

Lemma 7.3: Let L be an ASL with 0. Then, for any $a \in L, a\circ 0 = 0$.

Proof: Suppose L is an ASL with 0. Then $a\circ 0 = a\circ(0\circ a) = (a\circ 0)\circ a = (0\circ a)\circ a = 0\circ a = 0$. Therefore $a\circ 0 = 0$.

Lemma 7.4: For any $a,b \in L, a\circ b = 0$ if and only if $b\circ a = 0$.

Proof: Suppose $a\circ b = 0$. Then $b\circ a = b\circ(a\circ a) = (b\circ a)\circ a = (a\circ b)\circ a = 0\circ a = 0$. Therefore $b\circ a = 0$. Similarly, the converse holds true.

Now, we prove the following corollary whose proof follows by leema 7.4.

Corollary 7.5: For any $a,b \in L, a\circ b = b\circ a$ whenever $a\circ b = 0$

Corollary 7.6: Let L be an ASL with 0. Then for any $a,b \in L, a \leq b$ implies that $a\circ x \leq b\circ x$ and $x\circ a \leq x\circ b$ for all $x \in L$.

Proof: Suppose L is an ASL with 0 and suppose $a \leq b$ for any $a,b \in L$. Then $a = a\circ b$. Now, for any $x \in L$, $(x\circ \mathrm{a})\circ(\mathrm{x}\circ\mathrm{b}) = (\mathrm{a}\circ\mathrm{x})\circ(\mathrm{x}\circ\mathrm{b}) = ((\mathrm{a}\circ\mathrm{x})\circ\mathrm{x})\circ\mathrm{b} = (\mathrm{a}\circ(\mathrm{x}\circ\mathrm{x}))\circ\mathrm{b} = (\mathrm{a}\circ\mathrm{x})\circ\mathrm{b} = (\mathrm{x}\circ\mathrm{a})\circ\mathrm{b} = \mathrm{x}\circ(\mathrm{a}\circ\mathrm{b}) = \mathrm{x}\circ\mathrm{a}$. Therefore $x\circ a \leq x\circ b$. Also, $(a\circ \mathrm{x})\circ(\mathrm{b}\circ\mathrm{x}) = (\mathrm{x}\circ\mathrm{a})\circ(\mathrm{b}\circ\mathrm{x}) = ((\mathrm{x}\circ\mathrm{a})\circ\mathrm{b})\circ\mathrm{x} = (\mathrm{x}\circ(\mathrm{a}\circ\mathrm{b}))\circ\mathrm{x} = (\mathrm{x}\circ\mathrm{a})\circ\mathrm{x} = (\mathrm{a}\circ\mathrm{x})\circ\mathrm{x} = \mathrm{a}\circ(\mathrm{x}\circ\mathrm{x}) = \mathrm{a}\circ\mathrm{x}$. Therefore $a\circ x \leq b\circ x$.

Since $a \circ 0 = 0$ and $0 \circ a = 0$ for all $a \in L$, $a \circ 0 = 0 \circ a$. Thus $0 \sim a$. Hence we have the following theorem.

Theorem 7.7: Let L be an ASL with 0. Then L is a semilattice with 0 if and only if $\sim$ is a transitive relation on L.

Proof: Suppose L is a semilattice with 0 and assume that $a \sim b$ and $b \sim c$ for $a, b, c \in L$. Then clearly $a \sim c$, since L is semilattice. Hence $\sim$ is transitive. Conversely, suppose $\sim$ is a transitive relation on L. Then for any $a, b \in L$, we have $a \sim 0$ and $0 \sim b$. Thus $a \sim b$, since $\sim$ is transitive. Therefore $a \circ b = b \circ a$ for any $a, b \in L$. Hence L is a semilattice.

But, *example* 3.6 (with L containing more than one element) shows that if L does not have 0, then the above theorem is not valid.

Definition 7.8: If L is a discrete ASL, then the $ASL\ (L^o, \circ)$ where $L^o = L \cup \{0\}$ is called a *simple* ASL.

Finally, we have the following theorem whose proof straightforward.

Theorem 7.9: let L be an ASL with 0. Then the following are equivalent.

(1) L is simple

(2) Every non zero element of L is unimaximal.

(3) $a \circ b = b$ for all $a \neq 0$

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