

ALMOST g_α''' -CLOSED FUNCTIONS AND SEPARATION AXIOMS

¹O. RAVI*, ²S. JEYASHRI, ³J. ANTONY REX RODRIGO AND ⁴S. LEELAVATHI

^{1,2}Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai Dt., Tamil Nadu, India

E-mail: siingam@yahoo.com, jeyabala1@yahoo.com

³Department of Mathematics, V. O. Chidambaram College, Thoothukudi, Tamil Nadu, India

E-mail: antonyrexrodrigo@yahoo.co.in

⁴Department of Mathematics, Government Arts College, Mannarkudi, Thiruvarur Dt., Tamil Nadu, India

E-mail: s.leelavathy40@yahoo.com

(Received on: 15-02-11; Accepted on: 10-07-11)

ABSTRACT

We introduce a new class of functions called almost g_α''' -closed and use the functions to improve several preservation theorems of normality and regularity and also their generalizations. The main result of the paper is that normality and weak normality are preserved under almost g_α''' -closed continuous surjections.

2000 Mathematics Subject Classification: 54C10, 54C08, 54C05

Key words and Phrases: Topological space, g''' -closed set, g_α''' -closed set, α gs-closed set, α g-closed set.

1. INTRODUCTION

In topological spaces, it is well known that normality is preserved under closed continuous surjections. Many authors have tried to weaken the condition “closed” in this theorem. In 1978, Long and Herrington [12] used almost closedness due to Singal [31]. In 1982, Malghan [15] used g -closedness. In 1986, Greenwood and Reilly [8] used α -closedness due to Mashhour et al. [16]. In 1995, Yoshimura et al. [36] used almost g -closedness which is a generalization of both almost closedness and g -closedness. In 1999, Noiri [20] introduced almost αg -closedness using αg -closed sets [14]. Ravi et. al. [27] introduced almost α gs-closedness using α gs-closed sets [24]. Recently, Ravi et. al. [26] have introduced the notion of g_α''' -closed sets which are strictly weaker than both α -closed sets and g''' -closed sets [22].

We use g_α''' -closed sets to define a new class of functions called almost g_α''' -closed functions. The purpose of the present paper is to improve preservation theorems of separation axioms, that is, normality, weak normality, mild normality, almost normality, regularity, almost regularity, quasi-regularity and strong s -regularity. The following properties are main results of the present paper.

Theorem: A Normality and weak normality are preserved under almost g_α''' -closed continuous surjections.

Theorem: B Regularity and strong s -regularity are preserved under almost α -open almost g_α''' -closed continuous surjections.

2. PRELIMINARIES

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) , $\text{cl}(A)$, $\text{int}(A)$ and A^c denote the closure of A , the interior of A and the complement of A respectively.

Corresponding author: ¹O. RAVI, *E-mail: siingam@yahoo.com

We recall the following definitions which are useful in the sequel.

Definition: 2.1 A subset A of a space (X, τ) is called:

- (i) semi-open set [11] if $A \subseteq \text{cl}(\text{int}(A))$;
- (ii) α -open set [17] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$;
- (iii) regular open set [33] if $A = \text{int}(\text{cl}(A))$.

The complements of the above mentioned open sets are called their respective closed sets.

The family of regular open (resp. regular closed) sets of a space (X, τ) is denoted by $\text{RO}(X, \tau)$ (resp. $\text{RC}(X, \tau)$) or simply by $\text{RO}(X)$ (resp. $\text{RC}(X)$).

The family of α -open sets of a space (X, τ) is denoted by τ^α . It is known [17] that $\tau \subset \tau^\alpha$ and τ^α is a topology for X. The closure (resp. interior) of a subset A of X with respect to τ^α is denoted by $\alpha\text{cl}(A)$ (resp. $\alpha\text{int}(A)$). It is known in [1] that $\alpha\text{cl}(A) = A \cup \text{cl}(\text{int}(\text{cl}(A)))$ and $\alpha\text{int}(A) = A \cap \text{int}(\text{cl}(\text{int}(A)))$ for any subset A of a space (X, τ) .

The semi-closure [4] of a subset A of X, denoted by $\text{scl}(A)$, is defined to be the intersection of all semi-closed sets of (X, τ) containing A. It is known that $\text{scl}(A)$ is a semi-closed set.

Definition: 2.2 A subset A of a space (X, τ) is called:

- (i) a generalized semi-closed (briefly gs-closed) set [2] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) . The complement of gs-closed set is called gs-open set;
- (ii) a g''' -closed set [22] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is gs-open in (X, τ) . The complement of g''' -closed set is called g''' -open set;
- (iii) a g'''_α -closed set [26] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is gs-open in (X, τ) . The complement of g'''_α -closed set is called g'''_α -open set;
- (iv) a α -generalized semi-closed (briefly α gs-closed) set [24] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ) . The complement of α gs-closed set is called α gs-open set;
- (v) a \hat{g} -closed set [34] (= ω -closed set [28]) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ) . The complement of \hat{g} -closed set is called \hat{g} -open set;
- (vi) a α -generalized closed (briefly α g-closed) set [14] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) . The complement of α g-closed set is called α g-open set;
- (vii) a α g-closed [20] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ) . The complement of α g-closed set is called α g-open set.

Remark: 2.3 From the Definition 2.1 and 2.2, we have the following implications.

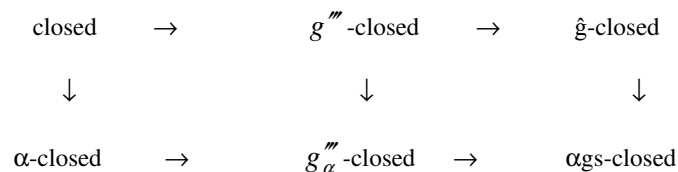


Diagram I

In the above remark, none of implications is reversible in the related paper [22].

3. ALMOST g'''_α -CLOSED FUNCTIONS

Definition: 3.1 A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be

(a) α -closed [16] (resp. g_α''' -closed [21], \hat{g} -closed [27], α gs-closed [27], g_α''' -closed) if for each closed set F of X, $f(F)$ is α -closed (resp. g_α''' -closed, \hat{g} -closed, α gs-closed, g_α''' -closed);

(b) almost closed [31] (resp. almost α -closed [20], almost \hat{g} -closed [27], almost α gs-closed [27], almost g_α''' -closed, almost g_α''' -closed) if for each $F \in RC(X, \tau)$, $f(F)$ is closed (resp. α -closed, \hat{g} -closed, α gs-closed, g_α''' -closed, g_α''' -closed).

Remark: 3.2 We have the following diagram for properties of functions:

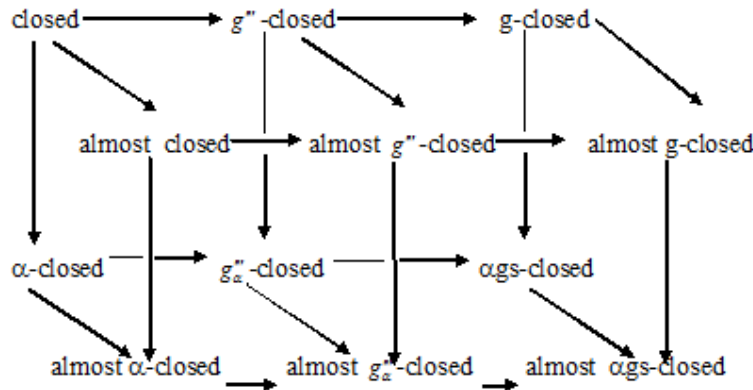


Diagram II

The following two examples show that almost g_α''' -closedness is strictly weaker than almost closedness and g_α''' -closedness.

Example: 3.3 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a, b\}, Y\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost g_α''' -closed. However, it is not almost closed since there exists $\{a, c\} \in RC(X, \tau)$ such that $f(\{a, c\}) = \{a, c\}$ is not closed in (Y, σ) .

Example: 3.4 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a, c\}, Y\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost g_α''' -closed. However, it is not g_α''' -closed since there exists a closed set $\{c\}$ of (X, τ) such that $f(\{c\}) = \{c\}$ is not g_α''' -closed in (Y, σ) .

The following two examples show that almost \hat{g} -closedness is strictly weaker than almost g_α''' -closedness and \hat{g} -closedness.

Example: 3.5 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a, c\}, Y\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost \hat{g} -closed. However, it is not almost g_α''' -closed since there exists $\{a\} \in RC(X, \tau)$ such that $f(\{a\}) = \{a\}$ is not g_α''' -closed in (Y, σ) .

Example: 3.6 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, Y\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost \hat{g} -closed. However, it is not \hat{g} -closed since there exists a closed set $\{c\}$ of (X, τ) such that $f(\{c\}) = \{c\}$ is not \hat{g} -closed in (Y, σ) .

The following three examples show that almost g_α''' -closedness is strictly weaker than almost α -closedness, g_α''' -closedness and almost g_α''' -closedness.

Example: 3.7 In Example 3.3, f is almost g_α''' -closed. However, it is not almost α -closed since there exists $\{a, c\} \in RC(X, \tau)$ such that $f(\{a, c\}) = \{a, c\}$ is not α -closed in (Y, σ) .

Example: 3.8 In Example 3.4, f is almost g_α''' -closed. However, it is not g_α''' -closed since there exists a closed set $\{b, c\}$ of (X, τ) such that $f(\{b, c\}) = \{b, c\}$ is not g_α''' -closed in (Y, σ) .

Example: 3.9 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, Y\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost g_α''' -closed. However, it is not almost g_α''' -closed since there exists $\{d\} \in RC(X, \tau)$ such that $f(\{d\}) = \{d\}$ is not g_α''' -closed in (Y, σ) .

The following three examples show that almost α gs-closedness is strictly weaker than almost \hat{g} -closedness, α gs-closedness and almost g_α''' -closedness.

Example: 3.10 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost α gs-closed. However, it is not almost g_α''' -closed since there exists $\{a, c\} \in RC(X, \tau)$ such that $f(\{a, c\}) = \{a, c\}$ is not g_α''' -closed in (Y, σ) .

Example: 3.11 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, Y\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost α gs-closed. However, it is not α gs-closed since there exists a closed set $\{c\}$ of (X, τ) such that $f(\{c\}) = \{c\}$ is not α gs-closed in (Y, σ) .

Example: 3.12 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, Y\}$. Then the identity function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost α gs-closed. However, it is not almost \hat{g} -closed since there exists $\{d\} \in RC(X, \tau)$ such that $f(\{d\}) = \{d\}$ is not \hat{g} -closed in (Y, σ) .

Theorem: 3.13 A surjection $f: X \rightarrow Y$ is almost g_α''' -closed if and only if for each subset S of Y and each $U \in RO(X)$ containing $f^{-1}(S)$ there exists an g_α''' -open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof: Necessity. Suppose that f is almost g_α''' -closed. Let S be a subset of Y and $U \in RO(X)$ containing $f^{-1}(S)$. Put $V = Y - f(X - U)$, then V is an g_α''' -open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X . Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in RO(X)$. There exists an g_α''' -open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset f^{-1}(Y - V)$. Hence, we obtain $f(F) = Y - V$ and $f(F)$ is g_α''' -closed in Y . This shows that f is almost g_α''' -closed.

Corollary: 3.14 If $f: X \rightarrow Y$ is an almost g_α''' -closed surjection, then for each g s-closed set F of Y and each $U \in RO(X)$ containing $f^{-1}(F)$ there exists an α -open set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$.

Proof: Let F be a g s-closed set of Y and $U \in RO(X)$ containing $f^{-1}(F)$. By Theorem 3.13, there exists an g_α''' -open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset U$. Since W is g_α''' -open, we have $F \subset \alpha\text{int}(W)$. Put $V = \alpha\text{int}(W)$, then V is α -open in Y and $f^{-1}(V) \subset U$.

4. NORMAL SPACES

In this section, we make use of g_α''' -closed sets to obtain further characterizations and preservation theorems of normal spaces.

Theorem: 4.1 The following are equivalent for a space X :

- (a) X is normal;
- (b) For any disjoint closed sets A and B , there exist disjoint g_α''' -open sets U, V such that $A \subset U$ and $B \subset V$;
- (c) For any closed set A and any open set V containing A , there exists an g_α''' -open set U of X such that $A \subset U \subset \alpha\text{cl}(U) \subset V$.

Proof:

(a) \Rightarrow (b). This is obvious since every open set is g_α''' -open.

(b) \Rightarrow (c). Let A be a closed set and V an open set containing A . Then A and $X - V$ are disjoint closed sets. There exist disjoint g_α''' -open sets U and W such that $A \subset U$ and $X - V \subset W$. Since $X - V$ is closed and hence g_α''' -closed, we have $X - V \subset \alpha\text{int}(W)$ and $U \cap \alpha\text{int}(W) = \phi$. Therefore, we obtain $\alpha\text{cl}(U) \cap \alpha\text{int}(W) = \phi$ and hence $A \subset U \subset \alpha\text{cl}(U) \subset X - \alpha\text{int}(W) \subset V$.

(c) \Rightarrow (a) Let A, B be disjoint closed sets of X . Then $A \subset X - B$ and $X - B$ is open. There exists an g_α''' -open set G of X such that $A \subset G \subset \alpha\text{cl}(G) \subset X - B$. Since A is closed, we have $A \subset \alpha\text{int}(G)$. Put $U = \text{int}(\text{cl}(\text{int}(\alpha\text{int}(G))))$ and $V = \text{int}(\text{cl}(\text{int}(X - \alpha\text{cl}(G))))$. Then U and V are disjoint open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is normal.

Theorem: 4.2 If $f: X \rightarrow Y$ is a continuous almost g_α''' -closed surjection and X is a normal space, then Y is normal.

Proof: Let A and B be any disjoint closed sets of Y . Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X . Since X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $G = \text{int}(\text{cl}(U))$ and $H = \text{int}(\text{cl}(V))$, then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 3.13, there exists g_α''' -open sets K and L of Y such that $A \subset K, B \subset L, f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L . It follows from Theorem 4.1 that Y is normal.

The following two corollaries are immediate consequences of Theorem 4.2.

Corollary: 4.3 [12] If $f: X \rightarrow Y$ is a continuous almost closed surjection and X is a normal space, then Y is normal.

Corollary: 4.4 [8] If $f: X \rightarrow Y$ is a continuous α -closed surjection and X is a normal space, then Y is normal.

Definition: 4.5 A space X is said to be

(a) weakly normal [37] if for each decreasing sequence $\{F_n\}$ of closed sets of X such that $\bigcap \{F_n : n \in \mathbb{N}\} = \phi$ and each closed set H of X with $H \cap F_1 = \phi$, there exist $n \in \mathbb{N}$ and an open set U of X such that $F_n \subset U$ and $\text{cl}(U) \cap H = \phi$;

(b) mildly normal [32] if for any disjoint regular closed sets A and B , there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$;

(c) almost normal [30] if for every pair of disjoint sets A and B , one of which is closed and the other is regular closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Lemma: 4.6 [20] If A is an α -open set of a space X , then the following hold:

$$\alpha\text{cl}(A) = \text{cl}(A) = \text{cl}(\text{int}(A)).$$

Lemma: 4.7 [18] A space X is weakly normal if and only if for each decreasing sequence $\{F_n\}$ of closed sets of X such that $\bigcap \{F_n : n \in \mathbb{N}\} = \phi$ and each open set U of X such that $F_1 \subset U$, there exist $n \in \mathbb{N}$ and an open set G of X such that $F_n \subset G \subset \text{cl}(G) \subset U$.

Theorem: 4.8 If $f: X \rightarrow Y$ is an almost g_α''' -closed continuous surjection and X is a weakly normal space, then Y is weakly normal.

Proof: Let $\{F_n\}$ be any decreasing sequence of closed sets of Y with no common point and any open set V of Y such that $F_1 \subset V$. Then $\{f^{-1}(F_n)\}$ is a decreasing sequence of closed sets of X with no common point and $f^{-1}(V)$ is an open set of X such that $f^{-1}(F_1) \subset f^{-1}(V)$. Since X is weakly normal, by Lemma 4.7, there exist $n \in \mathbb{N}$ and an open set U of X such that $f^{-1}(F_n) \subset U \subset \text{cl}(U) \subset f^{-1}(V)$. Therefore, $f^{-1}(F_n) \subset \text{int}(\text{cl}(U))$ and by Corollary 3.14, there exists an α -open set G of Y such that $F_n \subset G$ and $f^{-1}(G) \subset \text{int}(\text{cl}(U))$. Since $\text{cl}(U)$ is regular closed and f is almost g_α''' -closed, $f(\text{cl}(U))$ is g_α''' -closed in Y . Thus, we obtain $F_n \subset G \subset \alpha\text{cl}(G) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. Let $H = \text{int}(\text{cl}(\text{int}(G)))$, then by Lemma 4.6 we have $F_n \subset H \subset \text{cl}(H) = \alpha\text{cl}(G) \subset V$. It follows from Lemma 4.7 that Y is weakly normal.

Corollary: 4.9 [18]

Weak normality is preserved under almost closed continuous surjections.

Lemma: 4.10

- (i) A subset A of a space X is $r\alpha g$ -open if and only if $F \subset \alpha \text{int}(A)$ whenever $F \in RC(X)$ and $F \subset A$ [20].
- (ii) Every αg -closed set is αg -closed but not conversely [27].
- (iii) Every αg -closed set is $r\alpha g$ -closed but not conversely [20].

Theorem: 4.11

The following are equivalent for a space X:

- (a) X is mildly normal;
- (b) for any disjoint $H, K \in RC(X)$, there exist disjoint g_α''' -open sets U, V such that $H \subset U$ and $K \subset V$;
- (c) for any disjoint $H, K \in RC(X)$, there exist disjoint αg -open sets U, V such that $H \subset U$ and $K \subset V$;
- (d) for any disjoint $H, K \in RC(X)$, there exist disjoint $r\alpha g$ -open sets U, V such that $H \subset U$ and $K \subset V$;
- (e) for any $H \in RC(X)$ and any $V \in RO(X)$ containing H, there exists an $r\alpha g$ -open set U of X such that $H \subset U \subset \alpha \text{cl}(U) \subset V$;
- (f) for any $H \in RC(X)$ and any $V \in RO(X)$ containing H, there exists an α -open set U of X such that $H \subset U \subset \alpha \text{cl}(U) \subset V$;
- (g) for any disjoint $H, K \in RC(X)$, there exist disjoint α -open sets U, V such that $H \subset U$ and $K \subset V$.

Proof: It is obvious that (a) \Rightarrow (b), (b) \Rightarrow (c) and (c) \Rightarrow (d).

(d) \Rightarrow (e). Let $H \in RC(X)$ and $V \in RO(X)$ containing H. There exist disjoint $r\alpha g$ -open sets U, W such that $H \subset U$ and $X - V \subset W$. By Lemma 4.10, we have $X - V \subset \alpha \text{int}(W)$ and $U \cap \alpha \text{int}(W) = \emptyset$. Therefore, we obtain $\alpha \text{cl}(U) \cap \alpha \text{int}(W) = \emptyset$ and hence $H \subset U \subset \alpha \text{cl}(U) \subset X - \alpha \text{int}(W) \subset V$.

(e) \Rightarrow (f). Let $H \in RC(X)$ and $V \in RO(X)$ containing H. There exists an $r\alpha g$ -open set G of X such that $H \subset G \subset \alpha \text{cl}(G) \subset V$. Since $H \in RC(X)$, by Lemma 4.10, we have $H \subset \alpha \text{int}(G)$. Put $U = \alpha \text{int}(G)$, then U is α -open in X and $H \subset U \subset \alpha \text{cl}(U) \subset V$.

(f) \Rightarrow (g). Let H and K be any disjoint regular closed sets of X. Then, since $H \subset X - K$ and $X - K \in RO(X)$, there exists an α -open set U of X such that $H \subset U \subset \alpha \text{cl}(U) \subset X - K$. Put $V = X - \alpha \text{cl}(U)$, then U and V are disjoint α -open sets of X such that $H \subset U$ and $K \subset V$.

(g) \Rightarrow (a). Let H and K be any disjoint regular closed sets of X. Then there exist disjoint α -open sets A and B of X such that $H \subset A$ and $K \subset B$. Since A and B are disjoint, we have $\text{int}(\text{cl}(\text{int}(A))) \cap \text{int}(\text{cl}(\text{int}(B))) = \emptyset$. Now, Put $U = \text{int}(\text{cl}(\text{int}(A)))$ and $V = \text{int}(\text{cl}(\text{int}(B)))$, then U and V are disjoint open sets of X such that $H \subset U$ and $K \subset V$. Therefore, X is mildly normal.

Definition: 4.12 A function $f : X \rightarrow Y$ is said to be

- (a) R-map [3] (resp. almost-continuous [31]) if $f^{-1}(V)$ is regular open (resp. open) in X for every $V \in RO(Y)$;
- (b) almost open [31] (resp. almost α -open [20]) if $f(U)$ is open (resp. α -open) in Y for every regular open set U of X;
- (c) α -open [16] if $f(U)$ is α -open in Y for every open set U of X;
- (d) almost αg -closed [20] if $f(U)$ is αg -closed in Y for every regular closed set U of X.

Remark: 4.13 [20] Both almost-openness and α -openness imply almost α -openness but not conversely as the following example shows.

Example: 4.14 [20] Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}$. Let $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}$. Then a function $f: (X, \tau) \rightarrow (Y, \sigma)$, defined as $f(a) = f(d) = a$, $f(b) = b$ and $f(c) = c$, is almost α -open. However, it is neither almost open nor α -open.

Theorem: 4.15 Let $f: X \rightarrow Y$ be an R-map and an almost α gs-closed surjection. If X is a mildly normal space, then Y is mildly normal.

Proof: Let A and B be any disjoint regular closed sets of Y . Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular closed sets of X . Since X is mildly normal, there exist disjoint open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put $G = \text{int}(\text{cl}(U))$ and $H = \text{int}(\text{cl}(V))$, then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 3.13 [27], there exist α gs-open sets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L . It follows from Theorem 4.11 that Y is mildly normal.

Corollary: 4.16 Let $f: X \rightarrow Y$ be an R-map and an almost g_α''' -closed surjection and X is mildly normal then Y is mildly normal.

Lemma: 4.17 [20] If a function $f: X \rightarrow Y$ is almost continuous almost α -open and V is regular open in Y , then $f^{-1}(V)$ is regular open in X .

Theorem: 4.18 If $f: X \rightarrow Y$ is an almost α -open almost α gs-closed continuous surjection and X is an almost normal space, then Y is almost normal.

Proof: Let B be any closed set of Y and $V \in \text{RO}(Y)$ containing B . Since f is continuous and almost α -open, $f^{-1}(B)$ is closed and $f^{-1}(V) \in \text{RO}(X)$ by Lemma 4.17. Since X is almost normal and $f^{-1}(B) \subset f^{-1}(V)$, there exists $U \in \text{RO}(X)$ such that $f^{-1}(B) \subset U \subset \text{cl}(U) \subset f^{-1}(V)$ [30, Theorem 2.1]. Since f is almost α -open and almost α gs-closed, $f(U)$ is α -open and $f(\text{cl}(U))$ is α gs-closed in Y . Therefore, we obtain $B \subset f(U) \subset \alpha\text{cl}(f(U)) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. Put $G = \text{int}(\text{cl}(\text{int}(f(U))))$. Then G is open in Y and $\alpha\text{cl}(f(U)) = \text{cl}(\text{int}(f(U))) = \text{cl}(G)$ by Lemma 4.6. Therefore, we obtain $B \subset f(U) \subset G \subset \text{cl}(G) \subset V$. It follows from [30, Theorem 2.1] that Y is almost normal.

Corollary: 4.19 [27] Almost normality is preserved under almost α -open almost α g-closed continuous surjections.

5. REGULAR SPACES

In this section, we improve preservation theorems of regularity almost regularity, quasi-regularity.

Definition: 5.1 A space X is said to be

- (a) almost regular [29] if for each $F \in \text{RC}(X)$ and each $x \in X - F$, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$;
- (b) quasi-regular [23] if for every nonempty open set V of X , there exists a nonempty open set U in X such that $\text{cl}(U) \subset V$;
- (c) strongly s-regular [7] if for any closed set A of X and any point $x \in X - A$ there exists an $F \in \text{RC}(X)$ such that $x \in F$ and $F \cap A = \emptyset$.

It is shown in [7, Theorem 1] that a space X is strongly s-regular if and only if every open set of X is the union of regular closed sets. Strongly s-regular spaces are called $P\Sigma$ -spaces by Wang [35]. Ganster [7] showed that strong s-regularity is strictly weaker than regularity and is independent of almost regularity.

Theorem: 5.2 [20] The following are equivalent for a space (X, τ) :

- (a) (X, τ) is regular (resp. almost regular);
- (b) for each closed (resp. regular closed) set F and each $x \in X - F$, there exist disjoint $U, V \in \tau^\alpha$ such that $x \in U$ and $F \subset V$;
- (c) for each open (resp. regular open) set V and $x \in V$, there exists $U \in \tau^\alpha$ such that $x \in U \subset \alpha\text{cl}(U) \subset V$.

Theorem: 5.3 If $f: X \rightarrow Y$ is an almost α -open almost g_α''' -closed continuous surjection and X is a regular space, then Y is regular.

Proof: Let y be any point of Y and V any open neighbourhood of y . There exists a point $x \in X$ with $f(x) = y$. Since X is regular and f is continuous, there exists an open set U of X such that $x \in U \subset \text{cl}(U) \subset f^{-1}(V)$. Therefore, we have $y \in f(U) \subset f(\text{int}(\text{cl}(U))) \subset f(\text{cl}(U)) \subset V$ and $f(\text{int}(\text{cl}(U)))$ is α -open because $\text{int}(\text{cl}(U)) \in \text{RO}(X)$ and f is almost α -open. Since $\text{cl}(U) \in \text{RC}(X)$ and f is almost g_α''' -closed, $f(\text{cl}(U))$ is g_α''' -closed and hence $y \in f(\text{int}(\text{cl}(U))) \subset \alpha\text{cl}(f(\text{int}(\text{cl}(U)))) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. It follows from Theorem 5.2 that Y is regular.

Corollary: 5.4 [27] Regularity is preserved under almost α -open almost α gs-closed continuous surjections.

Theorem: 5.5 If $f: X \rightarrow Y$ is an almost α -open almost α gs-closed almost continuous surjection and X is an almost regular space, then Y is almost regular.

Proof: Let y be any point of Y and $V \in \text{RO}(Y)$ containing y . Since f is almost α -open almost continuous, $f^{-1}(V) \in \text{RO}(X)$ by Lemma 4.17. Take a point $x \in f^{-1}(y)$. Since X is almost regular, there exists $U \in \text{RO}(X)$ such that $x \in U \subset \text{cl}(U) \subset f^{-1}(V)$ [29, Theorem 2.2]. Hence $y \in f(U) \subset f(\text{cl}(U)) \subset V$. Since f is almost α -open almost α gs-closed, $f(U)$ is α -open in Y and $f(\text{cl}(U))$ is α gs-closed in Y and hence we have $y \in f(U) \subset \alpha\text{cl}(f(U)) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. It follows from Theorem 5.2 that Y is almost regular.

Definition: 5.6 A function $f: X \rightarrow Y$ is said to be

(a) feebly continuous [6] if $\text{int}(f^{-1}(V)) \neq \emptyset$ for every nonempty open set V of Y ;

(b) feebly open [6] if $\text{int}(f(U)) \neq \emptyset$ for every nonempty open set U of X ;

(c) almost feebly open [20] if $\text{int}(f(U)) \neq \emptyset$ for every nonempty $U \in \text{RO}(X)$.

Remark: 5.7 [20] It is obvious that every feebly open function is almost feebly open. However, the converse is not true in general as the following example shows.

Example: 5.8 [20] Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$.

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function defined as follows: $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then f is almost feebly open but it is not feebly open since we have $\text{RO}(X, \tau) = \{\emptyset, \{b\}, \{a, c\}, X\}$ and $\text{int}(f(\{a\})) = \emptyset$.

Theorem: 5.9 If $f: X \rightarrow Y$ is an almost feebly open feebly continuous almost g_α''' -closed surjection and X is a quasi-regular space, then Y is quasi-regular.

Proof: Let V be any nonempty open set of Y . Since f is feebly continuous, $\text{int}(f^{-1}(V)) \neq \emptyset$ and by the quasi-regularity of X there exists a nonempty open set U of X such that $U \subset \text{cl}(U) \subset \text{int}(f^{-1}(V))$. We have $f(\text{int}(\text{cl}(U))) \subset f(\text{cl}(U)) \subset V$. Since f is almost feebly open, $\text{int}(f(\text{int}(\text{cl}(U)))) \neq \emptyset$. Since f is almost g_α''' -closed, $f(\text{cl}(U))$ is g_α''' -closed and hence $\alpha\text{cl}(f(\text{cl}(U))) \subset V$. Now, put $G = \text{int}(f(\text{int}(\text{cl}(U))))$, then by Lemma 4.6 we obtain $\emptyset \neq G \subset \text{cl}(G) = \alpha\text{cl}(G) \subset \alpha\text{cl}(f(\text{cl}(U))) \subset V$. This shows that Y is quasi-regular.

Corollary: 5.10 [9] Quasi regularity is preserved under feebly open feebly continuous closed surjections.

We shall conclude the section with a preservation Theorem of strongly s -regular spaces.

Theorem: 5.11 If $f: X \rightarrow Y$ is an almost α -open almost g_α''' -closed continuous surjection and X is a strongly s -regular space, then Y is strongly s -regular.

Proof: Let V be any open set of Y and y any point of V . Since f is continuous, $f^{-1}(V)$ is open in X . For a point $x \in f^{-1}(y)$, there exists $F \in \text{RC}(X)$ such that $x \in F \subset f^{-1}(V)$; hence $y = f(x) \in f(F) \subset V$. Since f is continuous, we have $f(F) = f(\text{cl}(\text{int}(F))) \subset \text{cl}(f(\text{int}(F)))$. Since f is almost g_α''' -closed, $f(F)$ is g_α''' -closed and $\alpha\text{cl}(f(F)) \subset V$. Moreover, f is almost α -open, $f(\text{int}(F))$ is α -open in Y and by Lemma 4.6 we have $\text{cl}(f(\text{int}(F))) = \text{cl}(\text{int}(f(\text{int}(F)))) = \alpha\text{cl}(f(\text{int}(F))) \subset \alpha\text{cl}(f(F))$. Therefore, we obtain $\text{cl}(\text{int}(f(\text{int}(F)))) \in \text{RC}(Y)$ and $y \in f(F) \subset \text{cl}(f(\text{int}(F))) = \text{cl}(\text{int}(f(\text{int}(F)))) \subset \alpha\text{cl}(f(F)) \subset V$. It follows from [7, Theorem 1] that Y is strongly s -regular.

REFERENCES

- [1] D. Andrijevic, Some properties of the topology of α -sets, *Mat. Vesnik*, 36(1984), 1-10.
- [2] S. P. Arya and T. Nour, Characterization of s-normal spaces, *Indian J. Pure. Appl. Math.*, 21(8) (1990), 717-719.
- [3] D. Carnahan, Some properties related to compactness in topological spaces, Ph.D. Thesis, Univ. of Arkansas, 1973.
- [4] S. G. Crossley and S. K. Hildebrand, Semi-closure, *Texas J. Sci.*, 22(1971), 99-112.
- [5] R. Devi, K. Balachandran and H. Maki, On generalized α -continuous maps and α -generalized continuous maps, *Far East J. Math. Sci.*, Special Volume (1997), Part I, 1-15.
- [6] Z. Frolik, Remarks concerning the invariance of Baire spaces under mapping, *Czechoslovak Math. J.*, 11(86)(1961), 381-385.
- [7] M. Ganster, On strongly s-regular spaces, *Glasnik Mat.*, 25(45)(1990), 195-201.
- [8] S. Greenwood and I. L. Reilly, On feebly closed mappings, *Indian J. Pure Appl. Math.*, 17(1986), 1101-1105.
- [9] D. S. Jankovic and Ch. Konstadilaki-Savvopoulou, On α -continuous functions, *Math., Bohemica*, 117(1992), 259-270.
- [10] N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo (2)*, 19 (1970), 89-96.
- [11] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.
- [12] P. E. Long and L. L. Herrington, Basic properties of regular-closed functions, *Rend. Circ. Mat. Palermo (2)*, 27 (1978), 20-28.
- [13] H. Maki, R. Devi and K. Balachandran, Generalized α -closed sets in topology, *Bull. Fukuoka Univ. Ed. III*, 42(1993), 13-21.
- [14] H. Maki, R. Devi and K. Balachandran, Associated topologies of generalized α -closed sets and α -generalized closed sets, *Mem. Fac. Sci. Kochi Univ. Math.*, 15(1994), 51-63.
- [15] S. R. Malghan, Generalized closed maps, *J. Karnatak Univ. Sci.*, 27(1982), 82-88.
- [16] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, α -continuous and α -open mappings, *Acta Math. Hungar.*, 41(1983), 213-218.
- [17] O. Njastad, On some classes of nearly open sets, *Pacific J. Math.*, 15(1965), 961-970.
- [18] T. Noiri, Almost-closed images of countably paracompact spaces, *Prace Mat.*, 20(1978), 423-426.
- [19] T. Noiri, Mildly normal spaces and some functions, *Kyungpook Math. J.*, 36(1996), 183-190.
- [20] T. Noiri, Almost α g-closed functions and separation axioms, *Acta Math. Hungar*, 82(3)(1999), 193-205.
- [21] S. Pious Missier, O. Ravi, S. Jeyashri and P. Herin Wise Bell, g_α''' -closed and g_α''' -open maps in topology (submitted).
- [22] S. Pious Missier, O. Ravi, S. Jeyashri, and P. Herin Wise Bell, g_α''' -closed sets in topology (submitted).
- [23] J. Porter and R. G. Woods, *Extensions and Absolute of Hausdorff spaces*, Springer Verlag, 1988.
- [24] M. Rajamani and K. Viswanathan, On α gs-closed sets in topological spaces, *Acta Ciencia Indica*, XXXM (3)(2004), 21-25.
- [25] M. Rajamani and K. Viswanathan, On α gs-continuous maps in topological spaces, *Acta Ciencia Indica*, XXXM (1)(2005), 293-303.

- [26] O. Ravi, S. Jeyashri, J. Antony Rex Rodrigo and S. Leelavathi, g_α''' -closed sets in topology (submitted).
- [27] O. Ravi, S. Ganesan and S. Chandrasekar, Almost α gs-closed functions and separation axioms, Bulletin of Mathematical Analysis and Applications, 3(1)(2011), 165-177.
- [28] M. Sheik John, A study on generalizations of closed sets and continuous maps in topological and bitopological spaces, Ph. D Thesis, Bharathiar University, Coimbatore, September 2002.
- [29] M. K. Singal and S. P. Arya, On almost-regular spaces, Glasnik Mat., 4(24) (1969), 89-99.
- [30] M. K. Singal and S. P. Arya, Almost normal and almost completely regular spaces, Glasnik Mat., 5(25)(1970), 141-152.
- [31] M. K. Singal and A. R. Singal, Almost-continuous mappings, Yokohama Math. J., 16(1968), 63-73.
- [32] M. K. Singal and A. R. Singal, Mildly normal spaces, Kyungpook Math. J., 13(1973), 27-31.
- [33] M. Stone, Application of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41(1937), 374-481.
- [34] M. K. R. S. Veera Kumar, \hat{g} -closed sets in Topological spaces, Bull. Allahabad Math. Soc., 18(2003), 99-112.
- [35] Guo Jun Wang, On S-closed spaces, Acta Math. Sinica, 24(1981), 55-63.
- [36] M. Yoshimura, T. Miwa and T. Noiri, A generalization of regular closed and g-closed functions, Stud. Cerc. Mat., 47(1995), 353-358.
- [37] P. Zenor, On countable paracompactness and normality, Prace Mat., 13(1969), 23-32.
