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ALMOST g_α'''-CLOSED FUNCTIONS AND SEPARATION AXIOMS

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ABSTRACT

We introduce a new class of functions called almost g_α'''-closed and use the functions to improve several preservation theorems of normality and regularity and also their generalizations. The main result of the paper is that normality and weak normality are preserved under almost g_α'''-closed continuous surjections.

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1. INTRODUCTION

In topological spaces, it is well known that normality is preserved under closed continuous surjections. Many authors have tried to weaken the condition "closed" in this theorem. In 1978, Long and Herrington [12] used almost closedness due to Singal [31]. In 1982, Malghan [15] used g-closedness. In 1986, Greenwood and Reilly [8] used α-closedness due to Mashhour et al. [16]. In 1995, Yoshimura et al. [36] used almost g-closedness which is a generalization of both almost closedness and g-closedness. In 1999, Noiri [20] introduced almost αg-closedness using αg-closed sets [14]. Ravi et. al. [27] introduced almost αgs-closedness using αgs-closed sets [24]. Recently, Ravi et. al. [26] have introduced the notion of g_α'''-closed sets which are strictly weaker than both α-closed sets and g'''-closed sets [22]. We use g_α'''-closed sets to define a new class of functions called almost g_α'''-closed functions. The purpose of the present paper is to improve preservation theorems of separation axioms, that is, normality, weak normality, mild normality, almost normality, regularity, almost regularity, quasi-regularity and strong s-regularity. The following properties are main results of the present paper.

Theorem: A Normality and weak normality are preserved under almost g_α'''-closed continuous surjections.

Theorem: B Regularity and strong s-regularity are preserved under almost α-open almost g_α'''-closed continuous surjections.

2. PRELIMINARIES

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively.

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We recall the following definitions which are useful in the sequel.

Definition: 2.1 A subset A of a space (X,τ) is called:

(i) semi-open set [11] if $A \subseteq cl(int(A))$;
(ii) α-open set [17] if $A \subseteq int(cl(int(A)))$;
(iii) regular open set [33] if $A = int(cl(A))$.

The complements of the above mentioned open sets are called their respective closed sets.

The family of regular open (resp. regular closed) sets of a space (X, τ) is denoted by RO (X, τ) (resp. RC (X, τ)) or simply by RO (X) (resp. RC (X)).

The family of α-open sets of a space (X, τ) is denoted by τ^{α}. It is known [17] that $\tau \subset \tau^{\alpha}$ and τ^{α} is a topology for X. The closure (resp. interior) of a subset A of X with respect to τ^{α} is denoted by $\alpha cl(A)$ (resp. $\alpha int(A)$). It is known in [1] that $\alpha cl(A) = A \cup cl(int(cl(A)))$ and $\alpha int(A) = A \cap int(cl(int(A)))$ for any subset A of a space (X, τ).

The semi-closure [4] of a subset A of X, denoted by scl(A), is defined to be the intersection of all semi-closed sets of (X, τ) containing A. It is known that scl(A) is a semi-closed set.

Definition: 2.2 A subset A of a space (X, τ) is called:

(i) a generalized semi-closed (briefly gs-closed) set [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of gs-closed set is called gs-open set;

(ii) a g'''-closed set [22] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gs-open in (X, τ). The complement of g'''-closed set is called g'''-open set;

(iii) a g_{α}'''-closed set [26] if $\alpha\, cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gs-open in (X, τ). The complement of g_{α}'''-closed set is called g_{α}'''-open set;

(iv) a α-generalized semi-closed (briefly αgs-closed) set [24] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ). The complement of αgs-closed set is called αgs-open set;

(v) a ĝ-closed set [34] (= ω-closed set [28]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ). The complement of ĝ-closed set is called ĝ-open set;

(vi) a α-generalized closed (briefly αg-closed) set [14] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of αg-closed set is called αg-open set;

(vii) a rαg-closed [20] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ). The complement of rαg-closed set is called rαg-open set.

Remark: 2.3 From the Definition 2.1 and 2.2, we have the following implications.

$$\begin{tikzcd}
\text{closed} \arrow[r] \arrow[d] & g'''\text{-closed} \arrow[r] \arrow[d] & \hat{g}\text{-closed} \arrow[d] \\
\alpha\text{-closed} \arrow[r] & g_{\alpha}'''\text{-closed} \arrow[r] & \alpha gs\text{-closed}
\end{tikzcd}$$

Diagram I

In the above remark, none of implications is reversible in the related paper [22].

3. ALMOST g_{α}'''-CLOSED FUNCTIONS

Definition: 3.1 A function $f: (X,\tau) \to (Y,\sigma)$ is said to be

(a) α-closed [16] (resp. g'''-closed [21], ĝ-closed [27], αgs-closed [27], g_{α}'''-closed) if for each closed set F of X, f(F) is α-closed (resp. g'''-closed, ĝ-closed, αgs-closed, g_{α}'''-closed);

(b) almost closed [31] (resp. almost α-closed [20], almost ĝ-closed [27], almost αgs-closed [27], almost g'''-closed, almost g_{α}'''-closed) if for each F∈ RC (X, τ), f(F) is closed (resp. α-closed, ĝ-closed, αgs-closed, g'''-closed, g_{α}'''-closed).

Remark: 3.2 We have the following diagram for properties of functions:

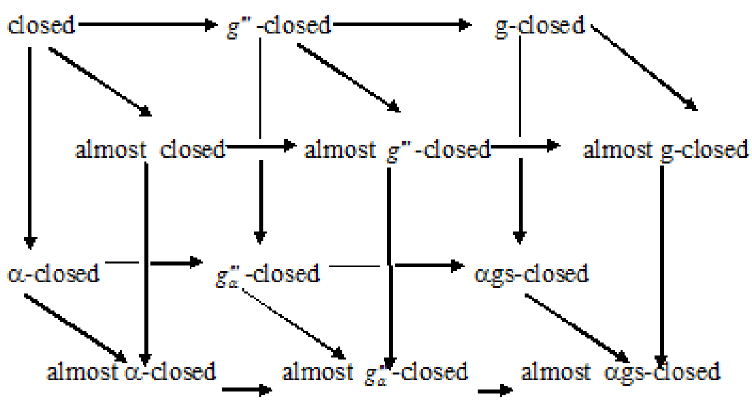

Diagram II

The following two examples show that almost g'''-closedness is strictly weaker than almost closedness and g'''-closedness.

Example: 3.3 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a, b}, Y}. Then the identity function f: (X,τ) → (Y,σ) is almost g'''-closed. However, it is not almost closed since there exists {a, c} ∈ RC (X,τ) such that f ({a, c}) = {a, c} is not closed in (Y, σ).

Example: 3.4 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X} and σ = {ϕ, {b}, {a, c}, Y}. Then the identity function f : (X, τ) → (Y, σ) is almost g'''-closed. However, it is not g'''-closed since there exists a closed set {c} of (X,τ) such that f({c}) = {c} is not g'''-closed in (Y, σ).

The following two examples show that almost ĝ-closedness is strictly weaker than almost g'''-closedness and ĝ-closedness.

Example: 3.5 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b, c}, X} and σ = {ϕ, {b}, {a, c}, Y}. Then the identity function f: (X,τ) → (Y,σ) is almost ĝ-closed. However, it is not almost g'''-closed since there exists {a} ∈ RC (X,τ) such that f({a}) = {a} is not g'''-closed in (Y, σ).

Example: 3.6 Let X = Y = {a, b, c, d}, τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X} and σ = {ϕ, {a}, {b, c}, {a, b, c}, Y}. Then the identity function f : (X, τ) → (Y, σ) is almost ĝ-closed. However, it is not ĝ-closed since there exists a closed set {c} of (X,τ) such that f({c}) = {c} is not ĝ-closed in (Y, σ).

The following three examples show that almost g_{α}'''-closedness is strictly weaker than almost α-closedness, g_{α}'''-closedness and almost g'''-closedness.

Example: 3.7 In Example 3.3, f is almost g_{α}'''-closed. However, it is not almost α-closed since there exists {a, c} ∈ RC (X,τ) such that f({a, c}) = {a, c} is not α-closed in (Y, σ).

Example: 3.8 In Example 3.4, f is almost $g_{\alpha}^{\prime\prime\prime}$-closed. However, it is not $g_{\alpha}^{\prime\prime\prime}$-closed since there exists a closed set {b, c} of (X,τ) such that f({b, c}) = {b, c} is not $g_{\alpha}^{\prime\prime\prime}$-closed in (Y, σ).

Example: 3.9 Let X = Y = {a, b, c, d}, τ = {ϕ, {d}, {a, b, c}, X} and σ = {ϕ, {a}, {b}, {a, b}, {a, d}, {a, b, d}, Y}. Then the identity function f: (X,τ) → (Y, σ) is almost $g_{\alpha}^{\prime\prime\prime}$-closed. However, it is not almost $g^{\prime\prime\prime}$-closed since there exists {d} ∈ RC (X,τ) such that f ({d}) = {d} is not $g^{\prime\prime\prime}$-closed in (Y,σ).

The following three examples show that almost αgs-closedness is strictly weaker than almost ĝ-closedness, αgs-closedness and almost $g_{\alpha}^{\prime\prime\prime}$-closedness.

Example: 3.10 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X} and σ = {ϕ, {a}, {b, c}, Y}. Then the identity function f : (X, τ) → (Y, σ) is almost αgs-closed. However, it is not almost $g_{\alpha}^{\prime\prime\prime}$-closed since there exists {a, c} ∈ RC (X,τ) such that f({a, c}) = {a, c} is not $g_{\alpha}^{\prime\prime\prime}$-closed in (Y, σ).

Example: 3.11 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {b, c}, X} and σ = {ϕ, {a}, {c}, {a, c}, {b, c}, Y}. Then the identity function f : (X, τ) → (Y, σ) is almost αgs-closed. However, it is not αgs-closed since there exists a closed set {c} of (X, τ) such that f({c}) = {c} is not αgs-closed in (Y, σ).

Example: 3.12 Let X = Y = {a, b, c, d}, τ = {ϕ, {d}, {a, b, c}, X} and σ = {ϕ, {a}, {b}, {a, b}, {a, d}, {a, b, d}, Y}. Then the identity function f: (X,τ) → (Y,σ) is almost αgs-closed. However, it is not almost ĝ-closed since there exists {d} ∈ RC (X,τ) such that f ({d}) = {d} is not ĝ-closed in (Y,σ).

Theorem: 3.13 A surjection f : X → Y is almost $g_{\alpha}^{\prime\prime\prime}$-closed if and only if for each subset S of Y and each U ∈ RO (X) containing $f^{-1}(S)$ there exists an $g_{\alpha}^{\prime\prime\prime}$-open set V of Y such that S ⊂ V and $f^{-1}(V) \subset U$.

Proof: Necessity. Suppose that f is almost $g_{\alpha}^{\prime\prime\prime}$-closed. Let S be a subset of Y and U ∈ RO (X) containing $f^{-1}(S)$. Put V = Y – f (X – U), then V is an $g_{\alpha}^{\prime\prime\prime}$-open set of Y such that S ⊂ V and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and X – F ∈ RO (X). There exists an $g_{\alpha}^{\prime\prime\prime}$-open set V of Y such that Y – f (F) ⊂ V and $f^{-1}(V) \subset X - F$. Therefore, we have f (F) ⊃ Y – V and $F \subset f^{-1}(Y - V)$. Hence, we obtain f (F) = Y – V and f (F) is $g_{\alpha}^{\prime\prime\prime}$-closed in Y. This shows that f is almost $g_{\alpha}^{\prime\prime\prime}$-closed.

Corollary: 3.14 If f : X → Y is an almost $g_{\alpha}^{\prime\prime\prime}$-closed surjection, then for each gs-closed set F of Y and each U ∈ RO (X) containing $f^{-1}(F)$ there exists an α-open set V of Y such that F ⊂ V and $f^{-1}(V) \subset U$.

Proof: Let F be a gs-closed set of Y and U ∈ RO (X) containing $f^{-1}(F)$. By Theorem 3.13, there exists an $g_{\alpha}^{\prime\prime\prime}$-open set W of Y such that F ⊂ W and $f^{-1}(W) \subset U$. Since W is $g_{\alpha}^{\prime\prime\prime}$-open, we have F ⊂ αint(W). Put V = αint(W), then V is α-open in Y and $f^{-1}(V) \subset U$.

4. NORMAL SPACES

In this section, we make use of $g_{\alpha}^{\prime\prime\prime}$-closed sets to obtain further characterizations and preservation theorems of normal spaces.

Theorem: 4.1 The following are equivalent for a space X:

(a) X is normal;

(b) For any disjoint closed sets A and B, there exist disjoint $g_{\alpha}^{\prime\prime\prime}$-open sets U, V such that A ⊂ U and B ⊂ V;

(c) For any closed set A and any open set V containing A, there exists an $g_{\alpha}^{\prime\prime\prime}$-open set U of X such that A ⊂ U ⊂ αcl(U) ⊂ V.

Proof:

(a) $\Rightarrow$ (b). This is obvious since every open set is g_α'''-open.

(b) $\Rightarrow$ (c). Let A be a closed set and V an open set containing A. Then A and X − V are disjoint closed sets. There exist disjoint g_α'''-open sets U and W such that $A \subset U$ and $X - V \subset W$. Since X − V is closed and hence gs-closed, we have $X - V \subset \alpha int(W)$ and $U \cap \alpha int(W) = \phi$. Therefore, we obtain $\alpha cl(U) \cap \alpha int(W) = \phi$ and hence $A \subset U \subset \alpha cl(U) \subset X - \alpha int(W) \subset V$.

(c) $\Rightarrow$ (a) Let A, B be disjoint closed sets of X. Then $A \subset X - B$ and X − B is open. There exists an g_α'''-open set G of X such that $A \subset G \subset \alpha cl(G) \subset X - B$. Since A is closed, we have $A \subset \alpha int(G)$. Put U = int (cl (int (αint (G)))) and V = int (cl (int(X − αcl(G)))). Then U and V are disjoint open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is normal.

Theorem: 4.2 If f: X → Y is a continuous almost g_α'''-closed surjection and X is a normal space, then Y is normal.

Proof: Let A and B be any disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X. Since X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let G = int(cl(U)) and H = int(cl(V)), then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 3.13, there exists g_α'''-open sets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. It follows from Theorem 4.1 that Y is normal.

The following two corollaries are immediate consequences of Theorem 4.2.

Corollary: 4.3 [12] If f: X → Y is a continuous almost closed surjection and X is a normal space, then Y is normal.

Corollary: 4.4 [8] If f: X → Y is a continuous α-closed surjection and X is a normal space, then Y is normal.

Definition: 4.5 A space X is said to be

(a) weakly normal [37] if for each decreasing sequence $\{F_n\}$ of closed sets of X such that $\cap \{F_n : n \in N\} = \phi$ and each closed set H of X with $H \cap F_1 = \phi$, there exist $n \in N$ and an open set U of X such that $F_n \subset U$ and $cl(U) \cap H = \phi$;

(b) mildly normal [32] if for any disjoint regular closed sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$;

(c) almost normal [30] if for every pair of disjoint sets A and B, one of which is closed and the other is regular closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Lemma: 4.6 [20] If A is an α-open set of a space X, then the following hold:
$\alpha cl(A) = cl(A) = cl(int(A))$.

Lemma: 4.7 [18] A space X is weakly normal if and only if for each decreasing sequence $\{F_n\}$ of closed sets of X such that $\cap \{F_n : n \in N\} = \phi$ and each open set U of X such that $F_1 \subset U$, there exist $n \in N$ and an open set G of X such that $F_n \subset G \subset cl(G) \subset U$.

Theorem: 4.8 If f : X → Y is an almost g_α'''-closed continuous surjection and X is a weakly normal space, then Y is weakly normal.

Proof: Let $\{F_n\}$ be any decreasing sequence of closed sets of Y with no common point and any open set V of Y such that $F_1 \subset V$. Then $\{ f^{-1}(F_n) \}$ is a decreasing sequence of closed sets of X with no common point and $f^{-1}(V)$ is an open set of X such that $f^{-1}(F_1) \subset f^{-1}(V)$. Since X is weakly normal, by Lemma 4.7, there exist $n \in N$ and an open set U of X such that $f^{-1}(F_n) \subset U \subset cl(U) \subset f^{-1}(V)$. Therefore, $f^{-1}(F_n) \subset int(cl(U))$ and by Corollary 3.14, there exists an α-open set G of Y such that $F_n \subset G$ and $f^{-1}(G) \subset int(cl(U))$. Since cl(U) is regular closed and f is almost g_α'''-closed, f(cl(U)) is g_α'''-closed in Y. Thus, we obtain $F_n \subset G \subset \alpha cl(G) \subset \alpha cl(f(cl(U))) \subset V$. Let H = int(cl(int(G))), then by Lemma 4.6 we have $F_n \subset H \subset cl(H) = \alpha cl(G) \subset V$. It follows from Lemma 4.7 that Y is weakly normal.

Corollary: 4.9 [18]

Weak normality is preserved under almost closed continuous surjections.

Lemma: 4.10

(i) A subset A of a space X is rαg-open if and only if $F \subset \alpha int(A)$ whenever $F \in RC(X)$ and $F \subset A$ [20].

(ii) Every αgs-closed set is αg-closed but not conversely [27].

(iii) Every αg-closed set is rαg-closed but not conversely [20].

Theorem: 4.11

The following are equivalent for a space X:

(a) X is mildly normal;

(b) for any disjoint $H, K \in RC(X)$, there exist disjoint g_{α}'''-open sets U, V such that $H \subset U$ and $K \subset V$;

(c) for any disjoint $H, K \in RC(X)$, there exist disjoint αgs-open sets U, V such that $H \subset U$ and $K \subset V$;

(d) for any disjoint $H, K \in RC(X)$, there exist disjoint rαg-open sets U, V such that $H \subset U$ and $K \subset V$;

(e) for any $H \in RC(X)$ and any $V \in RO(X)$ containing H, there exists an rαg-open set U of X such that $H \subset U \subset \alpha cl(U) \subset V$;

(f) for any $H \in RC(X)$ and any $V \in RO(X)$ containing H, there exists an α-open set U of X such that $H \subset U \subset \alpha cl(U) \subset V$;

(g) for any disjoint $H, K \in RC(X)$, there exist disjoint α-open sets U, V such that $H \subset U$ and $K \subset V$.

Proof: It is obvious that (a) ⇒ (b), (b) ⇒ (c) and (c) ⇒ (d).

(d) ⇒ (e). Let $H \in RC(X)$ and $V \in RO(X)$ containing H. There exist disjoint rαg-open sets U, W such that $H \subset U$ and $X - V \subset W$. By Lemma 4.10, we have $X - V \subset \alpha int(W)$ and $U \cap \alpha int(W) = \phi$. Therefore, we obtain $\alpha cl(U) \cap \alpha int(W) = \phi$ and hence $H \subset U \subset \alpha cl(U) \subset X - \alpha int(W) \subset V$.

(e) ⇒ (f). Let $H \in RC(X)$ and $V \in RO(X)$ containing H. There exists an rαg-open set G of X such that $H \subset G \subset \alpha cl(G) \subset V$. Since $H \in RC(X)$, by Lemma 4.10, we have $H \subset \alpha int(G)$. Put $U = \alpha int(G)$, then U is α-open in X and $H \subset U \subset \alpha cl(U) \subset V$.

(f) ⇒ (g). Let H and K be any disjoint regular closed sets of X. Then, since $H \subset X - K$ and $X - K \in RO(X)$, there exists an α-open set U of X such that $H \subset U \subset \alpha cl(U) \subset X - K$. Put $V = X - \alpha cl(U)$, then U and V are disjoint α-open sets of X such that $H \subset U$ and $K \subset V$.

(g) ⇒ (a). Let H and K be any disjoint regular closed sets of X. Then there exist disjoint α-open sets A and B of X such that $H \subset A$ and $K \subset B$. Since A and B are disjoint, we have $int(cl(int(A))) \cap int(cl(int(B))) = \phi$. Now, Put $U = int(cl(int(A)))$ and $V = int(cl(int(B)))$, then U and V are disjoint open sets of X such that $H \subset U$ and $K \subset V$. Therefore, X is mildly normal.

Definition: 4.12 A function $f : X \to Y$ is said to be

(a) R-map [3] (resp. almost-continuous [31]) if $f^{-1}(V)$ is regular open (resp. open) in X for every $V \in RO(Y)$;

(b) almost open [31] (resp. almost α-open [20]) if f(U) is open (resp. α-open) in Y for every regular open set U of X;

(c) α-open [16] if f(U) is α-open in Y for every open set U of X;

(d) almost αg-closed [20] if f(U) is αg-closed in Y for every regular closed set U of X.

Remark: 4.13 [20] Both almost-openness and α-openness imply almost α-openness but not conversely as the following example shows.

Example: 4.14 [20] Let X = {a, b, c, d} and τ = {ϕ, {c}, {d}, {a, c}, {c, d}, {a, c, d}, X}. Let Y = {a, b, c} and σ = {ϕ, Y, {a}, {a, b}}. Then a function f: (X,τ) → (Y,σ), defined as f(a) = f(d) = a, f(b) = b and f(c) =c, is almost α-open. However, it is neither almost open nor α-open.

Theorem: 4.15 Let f : X → Y be an R-map and an almost αgs-closed surjection. If X is a mildly normal space, then Y is mildly normal.

Proof: Let A and B be any disjoint regular closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular closed sets of X. Since X is mildly normal, there exist disjoint open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put G = int(cl(U)) and H = int(cl(V)), then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 3.13 [27], there exist αgs-open sets K and L of Y such that A ⊂ K, B ⊂ L, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. It follows from Theorem 4.11 that Y is mildly normal.

Corollary: 4.16 Let f: X → Y be an R-map and an almost $g_{\alpha}^{\prime\prime\prime}$-closed surjection and X is mildly normal then Y is mildly normal.

Lemma: 4.17 [20] If a function f : X → Y is almost continuous almost α-open and V is regular open in Y, then $f^{-1}(V)$ is regular open in X.

Theorem: 4.18 If f: X → Y is an almost α-open almost αgs-closed continuous surjection and X is an almost normal space, then Y is almost normal.

Proof: Let B be any closed set of Y and V ∈ RO (Y) containing B. Since f is continuous and almost α-open, $f^{-1}(B)$ is closed and $f^{-1}(V) \in RO(X)$ by Lemma 4.17. Since X is almost normal and $f^{-1}(B) \subset f^{-1}(V)$, there exists U ∈ RO (X) such that $f^{-1}(B) \subset U \subset cl(U) \subset f^{-1}(V)$ [30, Theorem 2.1]. Since f is almost α-open and almost αgs-closed, f(U) is α-open and f(cl(U)) is αgs-closed in Y. Therefore, we obtain B ⊂ f(U) ⊂ αcl(f(U)) ⊂ αcl(f(cl(U)) ⊂ V. Put G = int(cl(int(f(U)))). Then G is open in Y and αcl(f(U)) = cl(int(f(U))) = cl(G) by Lemma 4.6. Therefore, we obtain B ⊂ f(U) ⊂ G ⊂ cl(G) ⊂ V. It follows from [30, Theorem 2.1] that Y is almost normal.

Corollary: 4.19 [27] Almost normality is preserved under almost α-open almost αg-closed continuous surjections.

5. REGULAR SPACES

In this section, we improve preservation theorems of regularity almost regularity, quasi-regularity.

Definition: 5.1 A space X is said to be

(a) almost regular [29] if for each F ∈ RC (X) and each x ∈ X – F, there exist disjoint open sets U and V of X such that x ∈ U and F ⊂ V;

(b) quasi-regular [23] if for every nonempty open set V of X, there exists a nonempty open set U in X such that cl(U) ⊂ V;

(c) strongly s-regular [7] if for any closed set A of X and any point x ∈ X – A there exists an F ∈ RC (X) such that x ∈ F and F ∩ A = ϕ.

It is shown in [7, Theorem 1] that a space X is strongly s-regular if and only if every open set of X is the union of regular closed sets. Strongly s-regular spaces are called $P\sum$-spaces by Wang [35]. Ganster [7] showed that strong s-regularity is strictly weaker than regularity and is independent of almost regularity.

Theorem: 5.2 [20] The following are equivalent for a space (X,τ):

(a) (X,τ) is regular (resp. almost regular);

(b) for each closed (resp. regular closed) set F and each x ∈ X – F, there exist disjoint U, V ∈ τ^{α} such that x ∈ U and F ⊂ V;

(c) for each open (resp. regular open) set V and x ∈ V, there exists U ∈ τ^{α} such that x ∈ U ⊂ αcl(U) ⊂ V.

Theorem: 5.3 If f: X → Y is an almost α-open almost g_α'''-closed continuous surjection and X is a regular space, then Y is regular.

Proof: Let y be any point of Y and V any open neighbourhood of y. There exists a point x ∈ X with f(x) = y. Since X is regular and f is continuous, there exists an open set U of X such that x ∈ U ⊂ cl(U) ⊂ f^{-1}(V). Therefore, we have y ∈ f(U) ⊂ f(int(cl(U))) ⊂ f(cl(U)) ⊂ V and f(int(cl(U)) is α-open because int(cl(U)) ∈ RO (X) and f is almost α-open. Since cl(U) ∈ RC (X) and f is almost g_α'''-closed, f(cl(U)) is g_α'''-closed and hence y ∈ f(int(cl(U))) ⊂ αcl(f(int(cl(U)))) ⊂ αcl(f(cl(U))) ⊂ V. It follows from Theorem 5.2 that Y is regular.

Corollary: 5.4 [27] Regularity is preserved under almost α-open almost αgs-closed continuous surjections.

Theorem: 5.5 If f: X → Y is an almost α-open almost αgs-closed almost continuous surjection and X is an almost regular space, then Y is almost regular.

Proof: Let y be any point of Y and V ∈ RO (Y) containing y. Since f is almost α-open almost continuous, f^{-1}(V) ∈ RO (Y) by Lemma 4.17. Take a point x ∈ f^{-1}(y). Since X is almost regular, there exists U ∈ RO (X) such that x ∈ U ⊂ cl(U) ⊂ f^{-1}(V) [29, Theorem 2.2]. Hence y ∈ f(U) ⊂ f(cl(U)) ⊂ V. Since f is almost α-open almost αgs-closed, f(U) is α-open in Y and f(cl(U)) is αgs-closed in Y and hence we have y ∈ f(U) ⊂ αcl(f(U)) ⊂ αcl(f(cl(U))) ⊂ V. It follows from Theorem 5.2 that Y is almost regular.

Definition: 5.6 A function f : X → Y is said to be

(a) feebly continuous [6] if int(f^{-1}(V)) ≠ ϕ for every nonempty open set V of Y;

(b) feebly open [6] if int(f(U)) ≠ ϕ for every nonempty open set U of X;

(c) almost feebly open [20] if int(f(U)) ≠ ϕ for every nonempty U ∈ RO (X).

Remark: 5.7 [20] It is obvious that every feebly open function is almost feebly open. However, the converse is not true in general as the following example shows.

Example: 5.8 [20] Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X} and σ = {ϕ, {a}, {b}, {a, b}, Y}.
Let f: (X, τ) → (Y, σ) be a function defined as follows: f(a) = c, f(b) = a and f(c) = b. Then f is almost feebly open but it is not feebly open since we have RO (X, τ) = {ϕ, {b}, {a, c}, X} and int(f({a})) = ϕ.

Theorem: 5.9 If f: X → Y is an almost feebly open feebly continuous almost g_α'''-closed surjection and X is a quasi-regular space, then Y is quasi-regular.

Proof: Let V be any nonempty open set of Y. Since f is feebly continuous, int(f^{-1}(V)) ≠ ϕ and by the quasi-regularity of X there exists a nonempty open set U of X such that U ⊂ cl(U) ⊂ int(f^{-1}(V)). We have f(int(cl(U))) ⊂ f(cl(U)) ⊂ V. Since f is almost feebly open, int(f(int(cl(U)))) ≠ ϕ. Since f is almost g_α'''-closed, f(cl(U)) is g_α'''-closed and hence αcl(f(cl(U))) ⊂ V. Now, put G = int(f(int(cl(U)))), then by Lemma 4.6 we obtain ϕ ≠ G ⊂ cl(G) = αcl(G) ⊂ αcl(f(cl(U))) ⊂ V. This shows that Y is quasi-regular.

Corollary: 5.10 [9] Quasi regularity is preserved under feebly open feebly continuous closed surjections.

We shall conclude the section with a preservation Theorem of strongly s-regular spaces.

Theorem: 5.11 If f: X → Y is an almost α-open almost g_α'''-closed continuous surjection and X is a strongly s-regular space, then Y is strongly s-regular.

Proof: Let V be any open set of Y and y any point of V. Since f is continuous, f^{-1}(V) is open in X. For a point x ∈ f^{-1}(y), there exists F ∈ RC(X) such that x ∈ F ⊂ f^{-1}(V); hence y = f(x) ∈ f(F) ⊂ V. Since f is continuous, we have f(F) = f(cl(int(F))) ⊂ cl(f(int(F))). Since f is almost g_α'''-closed, f (F) is g_α'''-closed and αcl(f(F)) ⊂ V. Moreover, f is almost α-open, f(int(F)) is α-open in Y and by Lemma 4.6 we have cl(f(int(F))) = cl(int(f(int(F)))) = αcl(f(int(F))) ⊂ αcl(f(F)). Therefore, we obtain cl(int(f(int(F)))) ∈ RC (Y) and y ∈ f(F) ⊂ cl(f(int(F))) = cl(int(f(int(F)))) ⊂ αcl(f(F)) ⊂ V. It follows from [7, Theorem 1] that Y is strongly s-regular.

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