International Journal of Mathematical Archive-2(7), July - 2011, Page: 1194-1201
Available online through www.ijma.info ISSN 2229 – 5046

COEFFICIENT BOUNDS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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(Received on: 10-07-11; Accepted on: 22-07-11)

ABSTRACT

Let A be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *analytic in the unit disc* $E = \{z: |z| < 1\}$. *For starlike univalent function* $g(z) \in A$, *we denote by* C *and* C^* *the subclasses of functions* $f(z)$ *in A satisfying* $Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0$ *and* $Re\left\{\frac{f(z)}{g(z)}\right\} > 0$ *respectively. We wish to obtain the sharp upper bounds for the functional* $|a_2 a_4 - a_3^2|$.

Keywords: *Analytic univalent functions, Close-to-Star functions, Close-to-Convex functions, Carathéodory class.*

Mathematics Subject Classification: *30C45.*

1. INTRODUCTION

Let A denote the class of functions

(1.1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $E = \{z: |z| < 1\}$.

Carathéodory [1] introduced the class $\wp$ of functions of the form

(1.2) $$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

which are analytic in E and satisfy Re $\{p(z)\} > 0$, $z \in E$.

(1.3) $$S^* = \left\{f \in A\,;\, \frac{zf'(z)}{f(z)} = p(z)\right\}$$

is the class of starlike univalent functions.

(1.4) $$K = \left\{f \in A\,;\, \frac{\left(zf'(z)\right)'}{f'(z)} = p(z)\right\}$$

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is the class of convex univalent functions.

(1.5) $\mathrm{C} = \left\{ f \in \mathrm{A}\,;\ \frac{\mathrm{z}f'(\mathrm{z})}{\mathrm{g}(\mathrm{z})} = p(\mathrm{z}),\ \mathrm{g}(\mathrm{z}) \in \mathrm{S}^* \right\}$

is the class of close-to-convex functions introduced by Kaplan[5].

(1.6) $\mathrm{C}_1 = \left\{ f \in \mathrm{A}\,;\ \frac{\mathrm{z}f'(\mathrm{z})}{\mathrm{h}(\mathrm{z})} = p(\mathrm{z}),\ \mathrm{h}(\mathrm{z}) \in \mathrm{K} \right\}$

is the subclass of C.

(1.7) $\mathrm{C}^* = \left\{ f \in \mathrm{A}\,;\ \frac{f(\mathrm{z})}{\mathrm{g}(\mathrm{z})} = p(\mathrm{z}),\ \mathrm{g}(\mathrm{z}) \in \mathrm{S}^* \right\}$

is the class of close-to-star functions introduced by Reade [9].

(1.8) $\mathrm{C}_1^* = \left\{ f \in \mathrm{A}\,;\ \frac{f(\mathrm{z})}{\mathrm{h}(\mathrm{z})} = p(\mathrm{z}),\ \mathrm{h}(\mathrm{z}) \in \mathrm{K} \right\}$

is the subclass of C^*.

(1.9) $\mathrm{R}'(\alpha) = \{ f \in \mathrm{A}\,;\ f'(\mathrm{z}) + \alpha \mathrm{z} f''(\mathrm{z}) = p(\mathrm{z}),\ 0 \le \alpha \le 1 \}$

Janteng et al. [3, 4] obtained sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ for the classes S^*, K and $R'(0)$. Also Soh and Mohamad [1] obtained sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ for certain classes of close-to-convex functions.

2. PRELIMINARY LEMMAS

Lemma: 2.1 ([8]). If $p \in \wp$, then $|\mathrm{p}_\mathrm{k}| \le 2$ (k = 1, 2, 3 ...).

The result is sharp for $p(\mathrm{z}) = \frac{1+\mathrm{z}}{1-\mathrm{z}} = 1 + \sum_{\mathrm{k}=1}^{\infty} 2\mathrm{z}^\mathrm{k}$.

Lemma: 2.2 ([6, 7]). If $p \in \wp$, then

(2.1) $2\mathrm{p}_2 = \mathrm{p}_1{}^2 + (4 - \mathrm{p}_1{}^2)x,$

(2.2) $4\mathrm{p}_3 = \mathrm{p}_1{}^3 + 2\mathrm{p}_1(4 - \mathrm{p}_1{}^2)x - \mathrm{p}_1(4 - \mathrm{p}_1{}^2)x^2 + 2(4 - \mathrm{p}_1{}^2)(1 - |x|^2)\mathrm{z}$

for some x and z satisfying $|x| \le 1$ and $|\mathrm{z}| \le 1$.

Lemma: 2.3 ([4]) (i) If f ∈ S^*, then

(2.3) $|a_2a_4 - a_3^2| \le 1.$

(ii) If f ∈ K, then

(2.4) $|a_2a_4 - a_3^2| \le \frac{1}{8}.$

The results (2.3) and (2.4) are sharp.

Lemma: 2.4 ([1]). If f ∈ S^*, then

(2.5) $\left| \frac{a_2a_4}{8} - \frac{a_3^2}{9} \right| \le \frac{1}{8}.$

Lemma: 2.5 If f ∈ K, then

(2.6) $\left| \frac{a_2a_4}{8} - \frac{a_3^2}{9} \right| \le \frac{89}{4608}.$

The result is sharp.

Proof: Since f ∈ K, it follows from (1.4)

(2.7) $\left(\mathrm{z}f'(\mathrm{z})\right)' = f'(\mathrm{z})p(\mathrm{z}).$

Identifying terms in (2.7) yields

(2.8) $$\begin{cases} a_2 = \frac{p_1}{2} \\ a_3 = \frac{p_2}{6} + \frac{{p_1}^2}{6} \\ a_4 = \frac{p_3}{12} + \frac{p_1 p_2}{8} + \frac{{p_1}^3}{24} \end{cases}$$

From (2.1), (2.2) and (2.8), it can be established that

(2.9) $$\left|\frac{a_2a_4}{8} - \frac{a_3^2}{9}\right| = \frac{1}{10368}\left|54p_1p_3 + 17{p_1}^2p_2 - 32{p_2}^2 - 5{p_1}^4\right|$$
$$= \frac{1}{20736}\left|27p_1(4p_3) + 17{p_1}^2(2p_2) - 16(2p_2)^2 - 10{p_1}^4\right|$$
$$= \frac{1}{20736}\left|18{p_1}^4 + 39{p_1}^2(4-{p_1}^2)x - (64+11{p_1}^2)(4-{p_1}^2)x^2 + 54p_1(4-{p_1}^2)(1-|x|^2)z\right|$$

Let $p_1 = p$ and $p \in [0,2]$. Using Triangular Inequality and $|z| \le 1$, we get
$$\left|\frac{a_2a_4}{8} - \frac{a_3^2}{9}\right| \le \frac{1}{20736}\{18p^4 + 39p^2(4-p^2)|x| + (64+11p^2)(4-p^2)|x|^2 + 54p(4-p^2)(1-|x|^2)\}$$
$$= \frac{1}{20736}\{18p^4 + 54p(4-p^2) + 39p^2(4-p^2)|x| + (64-54p+11p^2)(4-p^2)|x|^2\}$$
$$= \frac{1}{20736}\{18p^4 + 54p(4-p^2) + 39p^2(4-p^2)\delta + (32-11p)(2-p)(4-p^2)\delta^2\}$$
$$= \frac{1}{20736}F(\delta), \qquad \text{where } \delta = |x| \le 1.$$

Since $F'(\delta) = 39p^2(4-p^2) + (32-11p)(2-p)(4-p^2)2\delta \ge 0$, $F(\delta)$ is an increasing function which implies Max $F(\delta) = F(1) = -32p^4 + 136p^2 + 256 = G(p)$, say. Consequently
$$\left|\frac{a_2a_4}{8} - \frac{a_3^2}{9}\right| \le \frac{1}{20736}G(p)$$

For $G(p) = -32p^4 + 136p^2 + 256$, $G'(p) = -128p^3 + 272p$ and $G''(p) = -384p + 272$.

Now $G'(p) = 0 \Rightarrow p = 0$ and $p = \sqrt{\frac{17}{8}}$.

Clearly $G''(0) > 0$ and $G''\left(\sqrt{\frac{17}{8}}\right) < 0$. Therefore Max.$G(p) = G\left(\sqrt{\frac{17}{8}}\right) = \frac{801}{2}$.

This proved the lemma.

The result is sharp for $p_1 = \sqrt{\frac{17}{8}}$, $p_2 = -1$ and $p_3 = -\frac{779}{96\sqrt{34}}$.]

3. Main Results

Theorem: 3.1 If $f \in C$, then $|a_2a_4 - a_3^2| \le \frac{73}{72}$.

Proof: Since $f \in C$, it implies from (1.5) that

(3.1) $$zf'(z) = g(z)p(z)$$

for some $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$.

Identifying terms in (3.1), we get

(3.2) $$\begin{cases} a_2 = \frac{b_2}{2} + \frac{p_1}{2} \\ a_3 = \frac{b_3}{3} + \frac{b_2p_1}{3} + \frac{p_2}{3} \\ a_4 = \frac{b_4}{4} + \frac{b_3p_1}{4} + \frac{b_2p_2}{4} + \frac{p_3}{4} \end{cases}$$

As $g \in S^*$, from (1.3)

(3.3) $$zg'(z) = g(z)p(z)$$

Equating coefficients in (3.3), we have

(3.4) $$\begin{cases} b_2 = p_1 \\ b_3 = \frac{p_2}{2} + \frac{{p_1}^2}{2} \\ b_4 = \frac{p_3}{3} + \frac{p_1p_2}{2} + \frac{{p_1}^3}{6} \end{cases}$$

From (3.2) and (3.4), we obtain

$$|a_2a_4 - a_3^2| = \left|\left(\frac{b_2}{2}+\frac{p_1}{2}\right)\left(\frac{b_4}{4}+\frac{b_3p_1}{4}+\frac{b_2p_2}{4}+\frac{p_3}{4}\right)-\left(\frac{b_3}{3}+\frac{b_2p_1}{3}+\frac{p_2}{3}\right)^2\right|$$

$$\Rightarrow \quad |a_2a_4 - a_3^2| = \left|\left(\frac{b_2b_4}{8}-\frac{b_3^2}{9}\right)+\left(\frac{7}{24}p_1p_3-\frac{1}{144}{p_1}^2p_2-\frac{2}{9}{p_2}^2-\frac{1}{144}{p_1}^4\right)\right|$$

$$\Rightarrow \quad |a_2a_4 - a_3^2| \le \left|\frac{b_2b_4}{8}-\frac{b_3^2}{9}\right|+\left|\frac{7}{24}p_1p_3-\frac{1}{144}{p_1}^2p_2-\frac{2}{9}{p_2}^2-\frac{1}{144}{p_1}^4\right|$$

By using Lemma 2.4, we get

(3.5) $$|a_2a_4 - a_3^2| \le \frac{1}{8} + \left|\frac{7}{24}p_1p_3-\frac{1}{144}{p_1}^2p_2-\frac{2}{9}{p_2}^2-\frac{1}{144}{p_1}^4\right|$$

Now using (2.1) and (2.2), we have

$$\left|\frac{7}{24}p_1p_3-\frac{1}{144}{p_1}^2p_2-\frac{2}{9}{p_2}^2-\frac{1}{144}{p_1}^4\right|$$
$$= \frac{1}{288}\left|-18{p_1}^4+9{p_1}^2(4-{p_1}^2)x-(64+5{p_1}^2)(4-{p_1}^2)x^2+42p(4-{p_1}^2)(1-|x|^2)z\right|$$

Suppose $p_1 = p$ and $p \in [0, 2]$. Application of triangular inequality and $|z| \le 1$ gives

$$\left|\frac{7}{24}p_1p_3-\frac{1}{144}{p_1}^2p_2-\frac{2}{9}{p_2}^2-\frac{1}{144}{p_1}^4\right|$$
$$\le \frac{1}{288}\{18p^4+42p(4-p^2)+9p^2(4-p^2)\delta+(32-5p)(2-p)(4-p^2)\delta^2\}$$
$$= \frac{1}{288}F(\delta), \qquad \text{where } \delta = |x| \le 1.$$

Since $F'(\delta) = 9p^2(4-p^2)+2(32-5p)(2-p)(4-p^2)\delta \ge 0$, $F(\delta)$ is an increasing function. Therefore, Max $F(\delta) = F(1)$. Consequently

$$\left|\frac{7}{24}p_1p_3-\frac{1}{144}{p_1}^2p_2-\frac{2}{9}{p_2}^2-\frac{1}{144}{p_1}^4\right| \le \frac{1}{288}\{4p^4-8p^2+256\}$$

One can easily see that $G(p) = 4p^4 - 8p^2 + 256$ attains its maximum at $p = 0$ and maximum value of $G(p) = 256$. Therefore

(3.6) $$\left|\frac{7}{24}p_1p_3-\frac{1}{144}{p_1}^2p_2-\frac{2}{9}{p_2}^2-\frac{1}{144}{p_1}^4\right| \le \frac{8}{9}.$$

The result follows from (3.5) and (3.6).

Taking $p_1 = 0$, $p_2 = -2$, and $p_3 = -2$ in (3.5) shows the result is sharp.

Theorem: 3.2 If $f \in C_1$, then $|a_2a_4 - a_3^2| \le \frac{8459}{13824}$.

Proof: Since $f \in C_1$, it implies from (1.6) that

(3.7) $$zf'(z) = h(z)p(z)$$

for some $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in K$.

Identifying terms in (3.7), we get

(3.8) $$\begin{cases} a_2 = \frac{c_2}{2}+\frac{p_1}{2} \\ a_3 = \frac{c_3}{3}+\frac{c_2p_1}{3}+\frac{p_2}{3} \\ a_4 = \frac{c_4}{4}+\frac{c_3p_1}{4}+\frac{c_2p_2}{4}+\frac{p_3}{4} \end{cases}$$

As $h \in K$, from (1.4)

(3.9) $$(zh'(z))' = h'(z)p(z)$$

Equating coefficients in (3.9), we have

(3.10) $$\begin{cases} c_2 = \frac{p_1}{2} \\ c_3 = \frac{p_2}{6}+\frac{{p_1}^2}{6} \\ c_4 = \frac{p_3}{12}+\frac{p_1p_2}{8}+\frac{{p_1}^3}{24} \end{cases}$$

From (3.8) and (3.10), we obtain

$$|a_2a_4-a_3^2| = \left|\left(\frac{c_2}{2}+\frac{p_1}{2}\right)\left(\frac{c_4}{4}+\frac{c_3p_1}{4}+\frac{c_2p_2}{4}+\frac{p_3}{4}\right)-\left(\frac{c_3}{3}+\frac{c_2p_1}{3}+\frac{p_2}{3}\right)^2\right|$$

$$\Rightarrow \quad |a_2a_4-a_3^2| = \left|\left(\frac{c_2c_4}{8}-\frac{c_3^2}{9}\right)+\left(\frac{19}{96}p_1p_3-\frac{5}{192}{p_1}^2p_2-\frac{4}{27}{p_2}^2-\frac{17}{1728}{p_1}^4\right)\right|$$

$$\Rightarrow \quad |a_2a_4-a_3^2| \le \left|\frac{c_2c_4}{8}-\frac{c_3^2}{9}\right|+\left|\frac{19}{96}p_1p_3-\frac{5}{192}{p_1}^2p_2-\frac{4}{27}{p_2}^2-\frac{17}{1728}{p_1}^4\right|$$

By using Lemma 2.5, we get

$$(3.11) \qquad |a_2a_4-a_3^2| \le \frac{89}{4608}+\left|\frac{19}{96}p_1p_3-\frac{5}{192}{p_1}^2p_2-\frac{4}{27}{p_2}^2-\frac{17}{1728}{p_1}^4\right|$$

Now using (2.1) and (2.2), we have

$$\left|\frac{19}{96}p_1p_3-\frac{5}{192}{p_1}^2p_2-\frac{4}{27}{p_2}^2-\frac{17}{1728}{p_1}^4\right|$$
$$=\frac{1}{3456}\left|-36{p_1}^4+41{p_1}^2(4-{p_1}^2)x-(512+43{p_1}^2)(4-{p_1}^2)x^2+342p_1(4-{p_1}^2)(1-|x|^2)z\right|$$

Suppose $p_1 = p$ and $p \in [0,2]$. Using Triangular Inequality and $|z| \le 1$, we get

$$\left|\frac{19}{96}p_1p_3-\frac{5}{192}{p_1}^2p_2-\frac{4}{27}{p_2}^2-\frac{17}{1728}{p_1}^4\right|$$
$$\le \frac{1}{3456}\{36p^4+342p(4-p^2)+41p^2(4-p^2)\delta+(256-43p)(2-p)(4-p^2)\delta^2\}$$
$$=\frac{1}{3456}F(\delta), \text{ where } \delta=|x|\le 1.$$

Since $F'(\delta) = 41p^2(4-p^2)+2(256-43p)(2-p)(4-p^2)\delta \ge 0$, $F(\delta)$ is an increasing function which implies Max $F(\delta) = F(1) = 256-15p^4-22p^2 = G(p)$, say. Thus

$$\left|\frac{19}{96}p_1p_3-\frac{5}{192}{p_1}^2p_2-\frac{4}{27}{p_2}^2-\frac{17}{1728}{p_1}^4\right| \le \frac{1}{432}G(p)$$

where $G(p)$ attains its maximum at $p = 0$ and maximum value of $G(p) = 256$.
Therefore

$$(3.12) \qquad \left|\frac{19}{96}p_1p_3-\frac{5}{192}{p_1}^2p_2-\frac{4}{27}{p_2}^2-\frac{17}{1728}{p_1}^4\right| \le \frac{16}{27}.$$

Using (3.12) in (3.11), we get the required inequality.

Taking $p_1 = 0$, $p_2 = -2$, and $p_3 = -2$ in (3.10) shows the result is sharp.

Theorem: 3.3 If $f \in C^*$, then $|a_2a_4-a_3^2| \le 9$.

Proof: Since $f \in C^*$, it implies from (1.7) that

$$(3.13) \qquad f(z) = g(z)p(z)$$

for some $g(z) = z+\sum_{n=2}^{\infty} b_n z^n \in S^*$.

Identifying terms in (3.13), we get

$$(3.14) \qquad \begin{cases} a_2 = b_2+p_1 \\ a_3 = b_3+b_2p_1+p_2 \\ a_4 = b_4+b_3p_1+b_2p_2+p_3 \end{cases}$$

As $g \in S^*$, from (1.3)

$$(3.15) \qquad zg'(z) = g(z)p(z)$$

Equating coefficients in (3.15), we have

$$(3.16) \qquad \begin{cases} b_2 = p_1 \\ b_3 = \frac{p_2}{2}+\frac{{p_1}^2}{2} \\ b_4 = \frac{p_3}{3}+\frac{p_1p_2}{2}+\frac{{p_1}^3}{6} \end{cases}$$

From (3.14) and (3.16), we obtain

$$|a_2a_4 - a_3^2| = |(b_2 + p_1)(b_4 + b_3p_1 + b_2p_2 + p_3) - (b_3 + b_2p_1 + p_2)^2|$$

$$\Rightarrow \quad |a_2a_4 - a_3^2| = \left|\left(b_2b_4 - {b_3}^2\right) + \left(\tfrac{7}{3}p_1p_3 - \tfrac{5}{3}{p_1}^2p_2 - 2{p_2}^2 + \tfrac{1}{3}{p_1}^4\right)\right|$$

$$\Rightarrow \quad |a_2a_4 - a_3^2| \le \left|b_2b_4 - {b_3}^2\right| + \left|\tfrac{7}{3}p_1p_3 - \tfrac{5}{3}{p_1}^2p_2 - 2{p_2}^2 + \tfrac{1}{3}{p_1}^4\right|$$

By using Lemma 2.3, we get

(3.17) $$|a_2a_4 - a_3^2| \le 1 + \left|\tfrac{7}{3}p_1p_3 - \tfrac{5}{3}{p_1}^2p_2 - 2{p_2}^2 + \tfrac{1}{3}{p_1}^4\right|$$

Now using (2.1) and (2.2), we have

$$\left|\frac{7}{3}p_1p_3 - \frac{5}{3}{p_1}^2p_2 - 2{p_2}^2 + \frac{1}{3}{p_1}^4\right|$$
$$= \frac{1}{12}\left|-5{p_1}^4 - 8{p_1}^2(4 - {p_1}^2)x - (24 + {p_1}^2)(4 - {p_1}^2)x^2 + 14p_1(4 - {p_1}^2)(1 - |x|^2)z\right|$$

Suppose $p_1 = p$ and $p \in [0, 2]$. Using Triangular Inequality and $|z| \le 1$, we get

$$\left|\tfrac{7}{3}p_1p_3 - \tfrac{5}{3}{p_1}^2p_2 - 2{p_2}^2 + \tfrac{1}{3}{p_1}^4\right| \le \tfrac{1}{12}\tfrac{1}{12}\{5p^4 + 14p(4 - p^2) + 8p^2(4 - p^2)\delta + (12 - p)(2 - p)(4 - p^2)\delta^2\}$$
$$\equiv \tfrac{1}{12}F(\delta), \qquad \text{where } \delta = |x| \le 1.$$

Since $F'(\delta) = 8p^2(4 - p^2) + 2(12 - p)(2 - p)(4 - p^2)\delta \ge 0$, $F(\delta)$ is an increasing function. Therefore, Max $F(\delta) = F(1)$. Consequently

$$\left|\frac{7}{3}p_1p_3 - \frac{5}{3}{p_1}^2p_2 - 2{p_2}^2 + \frac{1}{3}{p_1}^4\right| \le \frac{1}{3}\{24 - 3p^2 - p^4\}$$

It is easy to see that $G(p) = 24 - 3p^2 - p^4$ attains its maximum at $p = 0$ and maximum value of $G(p) = 24$. Therefore

(3.18) $$\left|\tfrac{7}{3}p_1p_3 - \tfrac{5}{3}{p_1}^2p_2 - 2{p_2}^2 + \tfrac{1}{3}{p_1}^4\right| \le 8.$$

(3.17) and (3.18) together gives the required inequality.

Putting $p_1 = 0$, $p_2 = -2$, and $p_3 = -2$ in (3.17) shows the equality holds.

Theorem: 3.4 If $f \in C_1^*$, then $|a_2a_4 - a_3^2| \le \frac{131}{24}$.

Proof: Since $f \in C_1^*$, it implies from (1.7) that

(3.19) $$f(z) = h(z)p(z)$$

for some $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in K$.

Identifying terms in (3.19), we get

(3.20) $$\begin{cases} a_2 = c_2 + p_1 \\ a_3 = c_3 + c_2p_1 + p_2 \\ a_4 = c_4 + c_3p_1 + c_2p_2 + p_3 \end{cases}$$

As $h \in K$, from (1.4)

(3.21) $$(zh'(z))' = h'(z)p(z)$$

Equating coefficients in (3.21), we have

(3.22) $$\begin{cases} c_2 = \frac{p_1}{2} \\ c_3 = \frac{p_2}{6} + \frac{{p_1}^2}{6} \\ c_4 = \frac{p_3}{12} + \frac{p_1p_2}{8} + \frac{{p_1}^3}{24} \end{cases}$$

From (3.20) and (3.22), we obtain

$$|a_2a_4 - a_3^2| = |(c_2 + p_1)(c_4 + c_3p_1 + c_2p_2 + p_3) - (c_3 + c_2p_1 + p_2)^2|$$

$$\Rightarrow \quad |a_2a_4 - a_3^2| \le |c_2c_4 - {c_3}^2| + \left|\tfrac{19}{12}p_1p_3 - \tfrac{3}{8}{p_1}^2p_2 - \tfrac{4}{3}{p_2}^2 - \tfrac{3}{24}{p_1}^4\right|$$

By using Lemma 2.4, we get

(3.23) $$|a_2a_4 - a_3^2| \leq \frac{1}{8} + \left|\frac{19}{12}p_1p_3 - \frac{3}{8}{p_1}^2p_2 - \frac{4}{3}{p_2}^2 - \frac{3}{24}{p_1}^4\right|$$

Now using (2.1) and (2.2), we have

$$\left|\frac{19}{12}p_1p_3 - \frac{3}{8}{p_1}^2p_2 - \frac{4}{3}{p_2}^2 - \frac{3}{24}{p_1}^4\right| = \frac{1}{48}|-6{p_1}^4 - 3{p_1}^2(4-{p_1}^2)x - (64+3{p_1}^2)(4-{p_1}^2)x^2 + 38p_1(4-{p_1}^2)(1-|x|^2)z|$$

Suppose $p_1 = p$ and $p \in [0,2]$. Using Triangular Inequality and $|z| \leq 1$, we get

$$\left|\frac{19}{12}p_1p_3 - \frac{3}{8}{p_1}^2p_2 - \frac{4}{3}{p_2}^2 - \frac{3}{24}{p_1}^4\right| \leq \frac{1}{48}\{6p^4 + 38p(4-p^2) + 3p^2(4-p^2)\delta + (64-38p+3p^2)(4-p^2)\delta^2\}$$
$$= \frac{1}{48}F(\delta), \qquad \text{where } \delta = |x| \leq 1.$$

Since $F'(\delta) = 3p^2(4-p^2) + 2(32-3p)(2-p)(4-p^2)\delta \geq 0$, $F(\delta)$ is an increasing function. Therefore, Max $F(\delta) = F(1)$. Consequently

$$\left|\frac{19}{12}p_1p_3 - \frac{3}{8}{p_1}^2p_2 - \frac{4}{3}{p_2}^2 - \frac{3}{24}{p_1}^4\right| \leq \frac{1}{6}\{32 - 5p^2\}$$

Obviously $G(p) = 32 - 5p^2$ has its maximum at $p = 0$ and maximum $G(p) = 32$. Therefore

(3.24) $$\left|\frac{19}{12}p_1p_3 - \frac{3}{8}{p_1}^2p_2 - \frac{4}{3}{p_2}^2 - \frac{3}{24}{p_1}^4\right| \leq \frac{16}{3}.$$

Combining (3.24) with (3.23), we get the required inequality.

Taking $p_1 = 0$, $p_2 = -2$, and $|p_3| \leq 2$ in (3.23) shows the result is sharp.

Theorem: 3.5 If $f \in R'(\alpha)$, then $|a_2a_4 - a_3^2| \leq \frac{4}{9(1+2\alpha)^2}$.

Proof: As $f \in R'(\alpha)$, from (1.10)

(3.25) $$f'(z) + \alpha z f''(z) = p(z)$$

Equating corresponding coefficient in (3.25), we have

(3.26) $$\begin{cases} a_2 = \frac{p_1}{2(1+\alpha)} \\ a_3 = \frac{p_2}{3(1+2\alpha)} \\ a_4 = \frac{p_3}{4(1+3\alpha)} \end{cases}$$

(3.27) implies

$$|a_2a_4 - a_3^2| = \left|\left(\frac{p_1}{2(1+\alpha)}\right)\left(\frac{p_3}{4(1+3\alpha)}\right) - \left(\frac{p_2}{3(1+2\alpha)}\right)^2\right|$$
$$= \frac{1}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}|9(1+2\alpha)^2p_1(4p_3) - 8(1+\alpha)(1+3\alpha)(2p_2)^2|$$

Using (2.1) and (2.2), we have

$$|a_2a_4 - a_3^2| = \frac{1}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}|(1+4\alpha+12\alpha^2){p_1}^4 + 2(1+4\alpha+12\alpha^2){p_1}^2(4-{p_1}^2)x$$
$$- \{(32+128\alpha+96\alpha^2) + (1+4\alpha+12\alpha^2){p_1}^2\}(4-{p_1}^2)x^2 + 18(1+4\alpha+4\alpha^2)p_1(4-{p_1}^2)(1$$
$$- |x|^2)z|$$

Suppose $p_1 = p$ and $p \in [0,2]$. Using Triangular Inequality , $|z| \leq 1$ and $|x| = \delta$, we get

$$|a_2a_4 - a_3^2| \leq \frac{1}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\{(1+4\alpha+12\alpha^2)p^4 + 18(1+4\alpha+4\alpha^2)p(4-p^2)$$
$$+ 2(1+4\alpha+12\alpha^2)p^2(4-p^2)|x|$$
$$+ \{32(1+4\alpha+3\alpha^2) - 18(1+4\alpha+4\alpha^2)p + (1+4\alpha+12\alpha^2)p^2\}(4-p^2)|x|^2\}$$

$$= \frac{1}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\left\{\begin{matrix}(1+4\alpha+12\alpha^2)p^4 + 18(1+4\alpha+4\alpha^2)p(4-p^2) + 2(1+4\alpha+12\alpha^2)p^2(4-p^2)\delta \\ +\{16(1+4\alpha+3\alpha^2) - (1+4\alpha+12\alpha^2)p\}(2-p)(4-p^2)\delta^2\end{matrix}\right\}$$

which imply

(3.28) $$|a_2a_4 - a_3^2| \leq \frac{1}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}F(\delta),$$

$$F(\delta) = (1 + 4\alpha + 12\alpha^2)p^4 + 18(1 + 4\alpha + 4\alpha^2)p(4 - p^2) + 2(1 + 4\alpha + 12\alpha^2)p^2(4 - p^2)\delta + \{16(1 + 4\alpha + 3\alpha^2) - (1 + 4\alpha + 12\alpha^2)p\}(2 - p)(4 - p^2)\delta^2$$

Since $F'(\delta) \geq 0$, $F(\delta)$ is increasing function and, therefore, max. $F(\delta) = F(1)$.

Let $G(p) = F(1) = 128(1 + 4\alpha + 3\alpha^2) - 4(5 + 8\alpha + 12\alpha(1 - \alpha))p^2 - 2(1 + 4\alpha + 12\alpha^2)p^4$.

Since $G'(p) \leq 0$, $G(p)$ is decreasing function in [0, 2].

Therefore max. $G(p) = G(0) = 128(1 + 4\alpha + 3\alpha^2)$.

From (3.28), we have

$$|a_2 a_4 - a_3^2| \leq \frac{4}{9(1+2\alpha)^2}.$$

Corollary: 3.1 If $f \in R'(0)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{4}{9}.$$

This result is proved by Janteng et al.[3].

Corollary: 3.2 If $f \in R'(1)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{4}{81}.$$

The result is sharp for $p(z) = \frac{1+z^2}{1-z^2}$.

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