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GRAPHS ON COMMUTATIVE RINGS OF TYPE Z_n AND Z_n x Z_n

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ABSTRACT

The properties of the Zero divisor graph of a commutative ring are studied by Philip.S.Livingston and David F.Anderson. We recall several results of Zero divisor graphs, total graphs of commutative rings. In this paper we discuss such kind of graphs obtained on the direct product of rings of type Z_n with suitable examples.

Key words: *Commutative ring, Zero divisors, Zero divisor graph Triangle graph, Bipartite Graph, Complete Bipartite graph.*

I. INTRODUCTION

Istvan Beck first introduce the concept of relating a commutative ring to a graph. This paper forms a new bridge between graph theory and the algebraic concept of ring R. Here R denote a commutative ring with identity. David F. Anderson and Philip S.Livingston associated a graph Γ(R) to a commutative ring R with identity with vertices Z(R)*= Z(R)\{0} the set of non zero zero divisors of R. Let x, y∈ V(R) be two distinct vertices .Then the vertices x and y are adjacent if and only if xy=0.They also investigated the interplay between the ring theoretic properties of R and the graph theoretic properties of Γ(R).

II. PRELIMINARIES

Ring: A non-empty set R together with two binary operations + and · is called a ring if the following conditions are satisfied.

(i) $(R,+)$ is an abelian group
(ii) $(R,\cdot)$ is a semi ring, where $R^* = R - \{0\}$.
(iii) The operation · is distributive over +, that is, $a \cdot (b+c) = a.b + a.c$ and $(a+b).c = a.c + b.c$.

Commutative ring: The ring $(R, +, \cdot)$ is called a commutative ring if $(R,\cdot)$ is commutative. That is $a.b = b.a$ for all $a,b \in R$.

Zero – divisor graph: A zero-divisor graph $\Gamma(R)$ associated to R is the graph whose vertices are the elements of $Z(R)^*$ where. $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero-divisor of R and the edge set is that the vertices x and y are adjacent if and only if $xy = 0$.

Triangle graph: A Triangle graph G(R) associated to ring R is the graph whose vertex set consisting the elements of R and the edge set is such that the vertices x,y and z are adjacent if and only if x+y+z=0 where $x \neq y \neq z$.

Bipartite Graph: A bipartite graph is a graph G whose vertex set is partitioned into two disjoint subsets X and Y such that each edge in G has one end in X and the other end in Y. Such a partition (X, Y) is called a bipartition of the graph.

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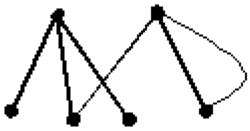
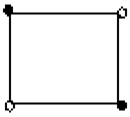
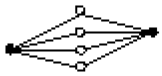
Example:



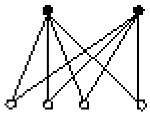
Complete Bipartite graph: A bipartite graph is complete if every vertex of X is joined to all other vertices of Y. We denote the complete bipartite graph by $K_{m,n}$ where m and n represent the number of vertices in the disjoint vertex set X and Y respectively.

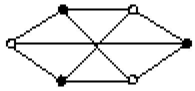
Example:





$K_{2,2}$ $K_{2,4}$





$K_{2,4}$ $K_{3,3}$

III. MAIN RESULTS

We consider (R, $+_n$, $\cdot_n$) be the commutative ring when R = Zn, the ring of integer modulo n where $+_n$ is addition modulo n and $\cdot_n$ is multiplication modulo n, and Zn={0,1,2--- n-1}. Let G(R) denote the graph of R, such that G(R) = (V(R), E(R)) where V(R) is the vertex set of G(R) and E(R) is the edge set of G(R). The elements of the ring R are consider as the vertices of the graph G(R) and x and y are adjacent then (x, y) $\in$E(R).

3.1.Theorem: Let R be a commutative ring (Z_n) of integers modulo n and let G(R) = (V(R), E(R) be the graph of R where V(R) is vertex set of G(R) and E(R) is edge set of G(R). If E(R) = {x, y, z $\in$ R/ x, y and z are adjacent to each other iff x+y+z=0, x≠y≠z and exactly one of x, y and z must be zero}. Then the graph G(R) is triangle graph when n≥3.

Proof: Let R be a commutative ring of integers modulo n.

Let G(R) = (V(R), E(R) denote the graph of R with vertex set V(R) and edge set E(R).

Define E(R) = {x, y, z $\in$R / x, y and z are adjacent to each other iff x + y + z = 0, x ≠ y ≠ z and exactly one of x, y and z must be zero}

Then we consider the following **cases:**

Case-(i): Let n=1 or 2 then the ring Z_n consists of either 1 or 2 elements and there does not exist any triangle graph with 1 or 2 vertices. Since in a triangle graph there are at least three vertices.

So consider n ≥ 3 when n=3 then R = {0, 1, 2}. In this case E(R) = {01, 02, 12} and V(R) = {0, 1, 2}

Therefore G(R) is a 1-Triangle graph with V(R) = {0, 1, 2} and every two vertices of V(R).

Similarly when n=4, R={0,1,2,3} then E(R)={01,03,13} and V(R)= {0,1,3}
(which is same as in the case n=3) so G(R) is a 1- Triangle graph.

Case-(ii): Let n=5 then R={0,1,2,3,4} .In this case E(R)={01,02,03,04,14,23} and V(R) = {0,1,2,3,4}

Therefore G(R) is a 2- Triangle graph with V(R)={0,1,2,3} and every two vertices of V(R).

Similarly when n=6, R={0,1,2,3,4} then E(R)={01,02,04,05,15,24} and V(R) = {0,1,2,3,4} (which is same as in the case n=5) so G(R) is a 2- Triangle graph

Continuing like these, in general if either n=r and n=r+1 .We get the graph of R is a r-1/2-Triangle graph.

3.1. Example: When n=3 .Then G(R)= (V(R), E(R))where V(R)= {0,1,2}and with 0+1+2=0 such that 0,1,2 are adjacent to each other and the graph of R G(R) is a 1- Triangle graph given in the following figure

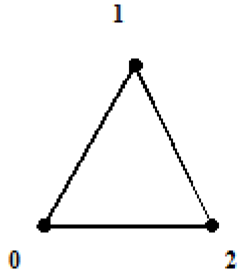


Fig. 3.1: 1- Triangle Graph

Similarly when n=4, we get the same graph with V(R) = {0, 1, 3} such that 0+1+3=0. which is same as in the case n=3 so G(R) is also a 1- Triangle graph.

Then we have the following graph

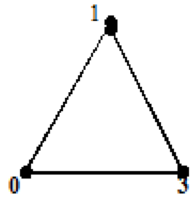


Fig. 3.2: 1 - Triangle graph

3.2. Example: When n=5.Then G(R)= (V(R), E(R)) where V(R)= {0,1,2,4,5}and with 0+1+4=0 and 0+2+3=0 such that 0,1,2,3,4 are adjacent to each other and the graph of R G(R) is a 2- Triangle graph given in the following figure

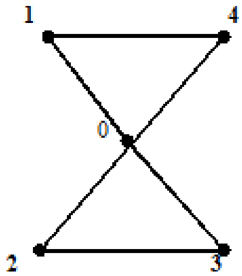


Fig. 3.3: 2 - Triangle Graph

Similarly when n=6, we get the same graph with V(R)= {0, 1, 2, 3, 4, 5} such that 0+1+5=0 and 0+2+4=0 which is same as in the case n=5 so G(R) is also a 2- Triangle graph.

Then we have the following graph

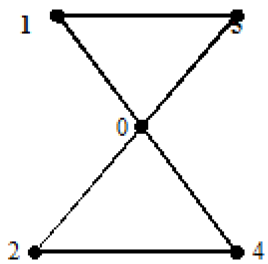


Fig. 3.4: 2- Triangle Graph

CONCLUSION

From the above examples we can observe that the same type of triangle n=r or r+1

When n=r or r+1 then G(R) is a r-1/2-Triangle graph the following table shows the number of triangle graphs for Z_n ($n\geq3$). We get the table

Number of vertices n	Number of Triangle graph G(R)
3&4	1
5&6	2
7&8	3
9&10	4
.	.
.	.
.	.
.	.
r,r+1	r-1/2

3.1.Corollary: Let R be a commutative ring (Z_n) of integers modulo n and let G(R)=(V(R), E(R)) be the graph of R where V(R) is vertex set of G(R) and E(R) is edge set of G(R). If E (R) = {x, y, z $\in$R / x, y and z are adjacent to each other if and only if x+y+z=0 and $x\neq y\neq z$}. Then the graph G(R) is triangle graph when $n\geq3$.

Proof: Let R= Z_n be a commutative ring of integers modulo n.

Let G(R) = (V(R), E(R)) denote the graph of R with vertex set V(R) and edge set E(R).

Define E (R) = {x, y, z $\in$R / x, y and z are adjacent to each other if and only if x+y+z=0 and $x\neq y\neq z$}

Then we consider the following cases:

Case-(i): Let n=1 or 2 then the ring R consists of either 1 or 2 elements and there does not exist any triangle graph with 1 or 2 vertices. Since in a triangle graph there are at least three vertices.

So consider $n \geq 3$ when n=3 then R={0,1,2}. In this case E(R) = {01, 02, 12} and V(R) = {0, 1, 2}

Therefore G(R) is a **1-Triangle graph** with V(R) = {0, 1, 2} and every two vertices of V(R).

Similarly when n = 4, R={0, 1, 2, 3} then E(R)={01, 03, 13} and V(R)= {0, 1, 3}. (which is same as in the case n=3) so G(R) is a 1- Triangle graph.

Case-(ii): Let n=5 then R={0, 1, 2, 3, 4} .In this case E(R)={01, 02, 03, 04, 14, 23} and V(R) = {0, 1, 2,3, 4}

Therefore G(R) is a 2- Triangle graph with V(R) = {0, 1, 2, 3} and every two vertices of V(R).

Case-(iii): Let =6 then R={0,1,2,3,4 E(R)={ 01,02,04,05,15,24} and V(R)= {0,1,2,3,4} so G(R) is a 4- Triangle graph Continuing like these, which forms different type of triangle graphs.

3.1. Example: When n=3 .Then G(R)= (V(R), E(R)) where V(R) = {0, 1, 2} and with 0+1+2=0 such that 0, 1, 2 are adjacent to each other and the graph of R G(R) is a 1- Triangle graph given in the following figure

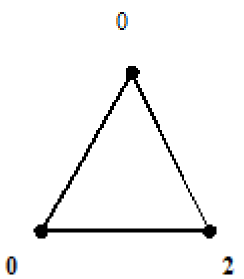


Fig. 3.5: 1- Triangle Graph

Similarly when n=4, we get the same graph with V(R) = {0, 1, 3} such that 0+1+3=0. which is same as in the case n=3 so G(R) is also a 1- Triangle graph.

Then we have the following graph

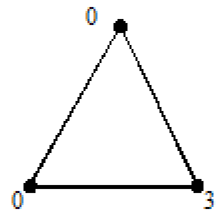


Fig. 3.6: 1 - Triangle graph

3.2. Example: When n=5.Then G(R)= (V(R), E(R)) where V(R)= {0,1,2,4,5}and with 0+1+4=0 and 0+2+3=0 such that 0,1,2,3,4 are adjacent to each other and the graph of R G(R) is a 2- Triangle graph given in the following figure

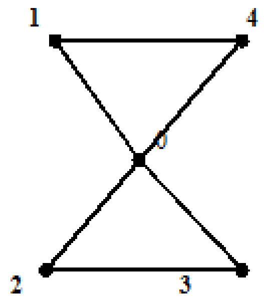


Fig 3.7: 2 - Triangle Graph

3.2. Example: when n=6, we get the same graph with V(R) = {0,1,2,3,4,5} such that 0+1+5=0 ,0+2+4=0, 1+2+3=0 and 3+4+5=0. So G(R) is a 4-Triangle graph.

Then we have the following graph

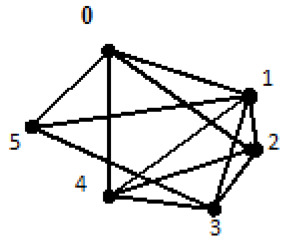


Fig. 3.8: 4-Triangle Graph

CONCLUSION

From the above examples we can observe that the different type of triangles are form.

Next we consider the commutative ring R of type $Z_n \times Z_n$.

Theorem: Let R be a commutative ring such that R= $Z_n \times Z_n$, where Z_n is a ring of integer modulo n and let G(R) = (V(R), E(R)) be the graph of R where V(R) is vertex set of G(R) and E(R) is edge set of G(R). If E(R)={x, y$\in$R /x and y are adjacent to each other iff x+y where x and y distant, $x=(x_1,y_1)$, $y=(x_2,y_2)$ and x,y$\neq$0}. Then the graph G(R) is a bipartite graph when n$\geq$3.

Proof: Let R be a commutative ring of integers modulo n.

Let G(R)=(V(R), E(R)) denote the graph of R with vertex set V(R) and edge set E(R).

Define by E(R)= {x, y$\in$R / x and y are adjacent to each other iff x+y where x and y distant, $x=(x_1,y_1)$, $y=(x_2,y_2)$ and x,y$\neq$0}.

Then we consider the following **cases:**

Case-(i): Let R = $Z_n \times Z_n$ when n=1 and 2k (k=1,2,3---). Then there does not exist any. *Bipartite Graph*.

Case-(ii): Let R = $Z_n \times Z_n$ when n=3. Then 2k+1, where k=1, 2,- - - to form a *Bipartite Graph.*

Let x, y$\in$R. Then x=(x_1, x_2) y = (y_1, y_2) where x_1, x_2, y_1, $y_2 \in$R, x is adjacent to y
<=>x+y=0
< = > (x_1,x_2) +(y_1,y_2) =(0,0)
< = > (x_1+y_1,x_2+y_2) = (0,0)
<= > $x_1+y_1 = 0$, $x_2+y_2 = 0$
<=> Since n is odd number there are exactly 2K+1 pair of elements adjacent in G(R) and the resulantant graph is a ***Bipartite Graph.***

If |G(R)|=2K(K+1) where n=2K+1, K= 1,2,3,... ...

3.6. Example: when n=3.Then R which consists of 3^2 elements, R={(0,0) (0,1) (0,2) (1,0) (1,1) (1,2) (2,0) (2,1) (2,2)}
Let E(R)={x, y$\in$R / x and y are adjacent to each other iff x+y where x and y distant, x=(x_1,y_1), y=(x_2,y_2) and x,y$\neq$0}
Such that (0,1)+(0,2)=0, (2,0)+ (1,1)=0, (1,1)+(2,2)=0, (1,2)+(2,1)=0

E(R)={(0,1)(0,2),(1,0)(2,0),(1,1)(2,2),(1,2)(2,1)}and V(R)={(0,1),(0,2),(1,0),(2,0),(1,1),(2,2),(1,2) (2,1)}

Then the graph of R is a **(4,4)- *Bipartite Graph*.**

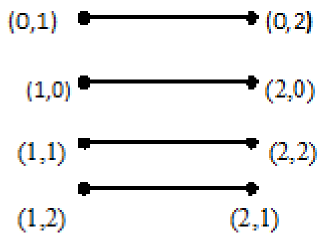


Fig 3.9: (4,4)- *Bipartite Graph*

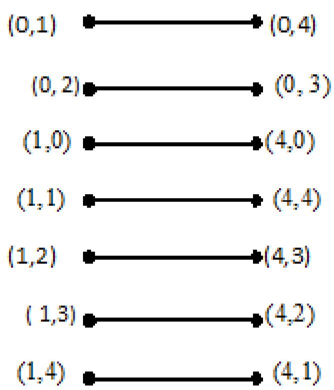
Example: When n=5.Then R which consists of 5^2 elements
R={(0,0)(0,1)(0,2)(0,3)(0,4)(1,0)(1,2)(1,3)(1,4)(2,0)(2,1)(2,2)(2,3)(2,4)(3,0)(3,1)(3,2)(3,3)(3,4)(4,0)(4,1)(4,2)(4,3)(4,4}

Let E(R)={x, y$\in$R / x and y are adjacent to each other iff x+y

where x and y distant, x=(x_1,y_1), y=(x_2,y_2) and x,y$\neq$0}. Such that (0,1)+(0,4)=0, (0,2)+(0,3)=0, (1,0)+(4,0)=0, (1,1)+(4,4)=0, (1,2)+(4,3)=0, (1,3)+(4,2)=0, (1,4)+(4,1)=0, (2,0)+(3,0)=0, (2,1)+(3,4)=0, (2,2)+(3,3)=0, (2,3)+(3,2)=0, (2,4)+(3,1)=0 in G(R)

E(R)={(0,1)(0,4),(0,2)(0,3),(1,0)(4,0),(1,1)(4,4),(1,2)(4,3),(1,3)(4,2),(1,4)(4,1),(2,0)(3,0),(2,1)(3,4),(2,2)(3,3),(2,3)(3,2), (2,4) (3,1)}

V(R)={(0,1),(0,2),(1,1),(1,2),(1,3),(1,4),(2,0)(2,1)(2,3)(2,4) (0,4),(0,3),(4,0),(4,4),(4,3),(4,1),(3,0) (3,4)(3,3) (3,2) (3,1) }
Then the graph of R is a **(12,12)- *Bipartite Graph*.**



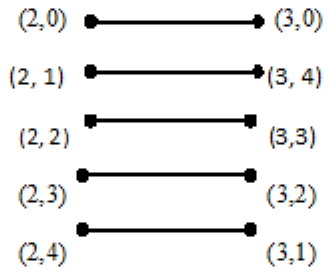


Fig 3.10: (4,4)- *Bipartite Graph*

In the same way z^7xz^7 and $z^{11}xz^{11}$ are also from a *Bipartite Graph*

CONCLUSION

From the above examples we can observe that the all odd numbers forms a *Bipartite Graph.*

In general if n=2K+1, where K=1, 2, 3,---- we get the graph of R is 2K(K+1) - *Bipartite Graph*

Let resulent graphs are obtained in the following table

Number of vertices n	Number of Triangle graph G(R)
3	4
5	12
7	24
9	40
11	60
.	.
.	.
.	.
.	.

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