International Journal of Mathematical Archive-7(2), 2016, 120-128
Available online through www.ijma.info ISSN 2229 – 5046

SOME NEARLY OPEN SETS IN A FUZZY SEQUENTIAL TOPOLOGICAL SPACE

N. TAMANG*[1], S. DE SARKAR[2]

[1]Assistant Professor, Department of Mathematics, University of North Bengal, India.

[2]Professor, Department of Mathematics, University of North Bengal, India.

(Received On: 19-01-16; Revised & Accepted On: 15-02-16)

ABSTRACT

The present article gives a study of fs-semiopen sets, fs-regular open sets and fs-semicontinuous functions in a fuzzy sequential topological space. Other studied notions are fs-almost continuous functions, fs-weakly continuous functions and it has been shown that both of these functions and fs-semicontinuous functions are independent notions. Further, many results relating these functions together with fs-continuous functions have been obtained.

Keywords and Phrases: *Fuzzy sequential topological spaces, fs-semiopen sets, fs-semicontinuous functions, fs-regular open sets, fs-almost continuous functions, fs-weakly continuous functions.*

***AMS Subject Classification**: 54A40.*

1. PRELIMINARIES

The introduction of fuzzy sets in 1965, by L.A. Zadeh [12] leads to the foundation of a new area of research called fuzzy mathematics. Since then, many researchers have been working in this area and related areas. As a generalization of a topological space, C. L. Chang [3] introduced the concept of fuzzy topological space in 1968. Fuzzy semi-open sets and fuzzy semicontinuity were introduced and studied by K. K. Azad [1].

The purpose of this work is to study the concept of semi-open sets and semicontinuity in fuzzy sequential topological spaces.

Throughout the paper, X will denote a non empty set and I the unit interval [0, 1]. Sequences of fuzzy sets in X called fuzzy sequential sets (fs-sets) will be denoted by the symbols $A_f(s)$, $B_f(s)$, $C_f(s)$ etc. An fs-set $X_f^l(s)$ is a sequence of fuzzy sets $\{X_f^n\}_n$, where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$.

A family $\delta(s)$ of fuzzy sequential sets on a non-empty set X satisfying the properties:

i. $X_f^r(s) \in \delta(s)$ for all $r \in \{0, 1\}$,

ii. $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$

iii. for any family $\{A_{fj}(s);\ j \in J\} \subseteq \delta(s), \vee_{j\in J} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called a fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets. Complement of an open fuzzy sequential set is called closed fuzzy sequential set. In an FSTS $(X, \delta(s))$, the closure $\overline{A_f(s)}$ and interior $A_f^{\circ}(s)$ of any fs-set $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \le C_f(s),\ (C_f(s))^c \in \delta(s)\},$$

$$A_f^o(s) = \vee\{O_f(s);\ O_f(s) \le A_f(s)), O_f(s) \in \delta(s)\},$$

Corresponding Author: N. Tamang[1]*
[1]Assistant Professor, Department of Mathematics, University of North Bengal, India.

[10] Let g be a mapping from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, then g is called
(i) fs-continuous if $g^{-1}(B_f(s))$ is open in $(X, \delta(s))$ for every open fs-set $B_f(s)$ in $(Y, \eta(s))$.
(ii) fs-open if $g(A_f(s))$ is fs-open in Y for every fs-open set $A_f(s)$ in X.
(iii) fs-closed if $g(A_f(s))$ is fs-closed in Y for every fs-closed set $A_f(s)$ in X.

Section **2** deals with the introduction and study of fs-semiopen sets as well as fs-semicontinuity. Section **3** deals with the introduction of fs-regular open sets and functions like fs-almost continuous and fs-weakly continuous functions. In this section, the interrelations among these functions together with fs-continuous and fs-semicontinuous functions have been investigated.

2. FS-SEMIOPEN SETS AND FS-SEMICONTINUITY

Definition 2.1: An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiopen set if $A_f(s) \leq \overline{A_f^o(s)}$. An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiclosed set if its complement is fs-semiopen.

Fundamental properties of fs-semiopen (fs-semiclosed) sets are:

- Any union (intersection) of fs-semiopen (fs-semiclosed) sets is fs-semiopen (fs-semiclosed).
- Every fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).
- Closure (interior) of an fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).

Example 2.1 shows that an fs-semiopen (fs-semiclosed) set may not be fs open (fs-closed), the intersection (union) of any two fs-semiopen (fs semiclosed) sets need not be an fs-semiopen (fs-semoclosed) set. Unlike in a general topological space, the intersection of an fs-semiopen set with an fs open set may fail to be an fs-semiopen set.

Example 2.1: Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \bar{0}, \bar{0}, \dots\dots\right\}$$
$$C_f(s) = \left\{\overline{\frac{3}{8}}, \bar{1}, \bar{1}, \dots\dots\right\}$$
$$D_f(s) = \left\{\overline{\frac{3}{8}}, \bar{0}, \bar{0}, \dots\dots\right\}$$

Consider $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Now,
(i) $B_f(s)$ is fs-open, hence fs-semiopen and $C_f(s)$ is fs-semiopen but their intersection $D_f(s)$ is not fs-semiopen.
(ii) $C_f(s)$ is fs-semiopen but is not fs-open.

Theorem 2.1: Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-semiopen if and only if there exist an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$.

Proof: Straightforward.

Theorem 2.2: Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-semiclosed if and only if there exist an fs-closed set $C_f(s)$ in X such that $C_f^o(s) \leq A_f(s) \leq C_f(s)$.

Proof: Straightforward.

We will denote the set of all fs-semiopen sets in X by $FSSO(X)$.

Theorem 2.3: In an FSTS $(X, \delta(s))$, (i) $\delta(s) \subseteq FSSO(X)$. (ii) If $A_f(s) \in FSSO(X)$ and $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, then $B_f(s) \in FSSO(X)$.

Proof:
(i) Follows from definition.
(ii) Let $A_f(s) \in FSSO(X)$. Then there exists an fs-open set $O_f(s)$ such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. So,

$$O_f(s) \leq A_f(s) \leq B_f(s) \leq \overline{A_f(s)} \leq \overline{O_f(s)}$$
$$\Rightarrow O_f(s) \leq B_f(s) \leq \overline{O_f(s)}.$$

$O_f(s)$ being fs-open, $B_f(s)$ is fs-semiopen.

Theorem 2.4: If in a fuzzy sequential topological space, $C_f^o(s) \leq B_f(s) \leq C_f(s)$, where $C_f(s)$ is fs-semiclosed, then $B_f(s)$ is also fs-semiclosed.

Proof: Omitted.

Theorem 2.5: Let $\mho = \{A_{\alpha f}(s);\ \alpha \in \Lambda\}$ be a collection of fs-sets in an FSTS $(X, \delta(s))$ such that (i) $\delta(s) \subseteq \mho$ and (ii) if $A_f(s) \in \mho$ and $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, then $B_f(s) \in \mho$. Then $FSSO(X) \subseteq \mho$. that is, $FSSO(X)$ is the smallest class of fs-sets in X satisfying (i) and (ii).

Proof: Let $A_f(s) \in FSSO(X)$. Then $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$ for some $O_f(s) \in \delta(s)$. By (i), $O_f(s) \in \mho$ and thus $A_f(s) \in \mho$ by (ii).

If $\mho = \{A_{\alpha f}(s);\ \alpha \in \Lambda\}$ be a collection of fs-sets in X, then $Int\mho$ denotes the set $\{A_{\alpha f}^o(s);\ \alpha \in \Lambda\}$.

Theorem 2.6: If $(X, \delta(s))$ be a fuzzy sequential topological space, then $\delta(s) = Int(FSSO(X))$.

Proof: Every fs-open set being fs-semiopen, $\delta(s) \subseteq Int(FSSO(X))$. Conversely, let $O_f(s) \in Int(FSSO(X))$. Then $O_f(s) = A_f^o(s)$ for some $A_f(s) \in FSSO(X)$ and hence $O_f(s) \in \delta(S)$.

Definition 2.2: Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X. We define semi-closure $sCl(A_f(s))$ and semi-interior $sInt(A_f(s))$ of $A_f(s)$ by

$$sCl\left(A_f(s)\right) = \wedge\{B_f(s);\ A_f(s) \leq B_f(s)\ and\ A_f^c(s) \in FSSO(X)\}$$
$$sInt\left(B_f^c(s)\right) = \vee\{C_f(s);\ C_f(s) \leq A_f(s)\ and\ C_f(s) \in FSSO(X)\}.$$

Obviously, $sCl\left(A_f(s)\right)$ is the smallest fs-semiclosed set containing $A_f(s)$ and $sInt\left(A_f(s)\right)$ is the largest fs-semiopen set contained in $A_f(s)$. Further,

(i) $A_f(s) \leq sCl\left(A_f(s)\right) \leq \overline{A_f(s)}$ and $A_f^o(s) \leq sInt\left(A_f(s)\right) \leq A_f(s)$.

(ii) $A_f(s)$ is fs-semiopen if and only if $A_f(s) = sInt\left(A_f(s)\right)$.

(iii) $A_f(s)$ is fs-semiclosed if and only if $A_f(s) = sCl\left(A_f(s)\right)$.

(iv) $A_f(s) \leq B_f(s)$ implies $sInt\left(A_f(s)\right) \leq sInt\left(B_f(s)\right)$ and $sCl\left(A_f(s)\right) \leq sCl\left(B_f(s)\right)$.

Definition 2.3: A mapping $g: (X, \delta(s)) \to (Y, \delta'(s))$ is said to be

(i) fs-semicontinuous if $g^{-1}(B_f(s))$ is fs-semiopen in X for every $B_f(s) \in \delta'(s)$.

(ii) fs-semiopen if $g(A_f(s))$ is fs-semiopen in Y for every $A_f(s) \in \delta(s)$.

(iii) fs-semiclosed if $g(A_f(s))$ is fs-semiclosed in Y for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-semicontinuous (fs-semiopen, fs-semiclosed). That the converse may not be true, is shown by Example 2.2.

Example 2.2: Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \bar{0}, \bar{0}, \dots\dots\right\}$$
$$C_f(s) = \left\{\overline{\frac{3}{8}}, \bar{1}, \bar{1}, \dots\dots\right\}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Let $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$. Define $g: (X, \delta(s)) \to (X, \delta'(s))$ by $g(x) = x$ for all $x \in X$. The function g is fs-semicontinuous but not fs-continuous.

Again the map $h: (X, \delta'(s)) \to (X, \delta(s))$ defined by $h(x) = x$ for all $x \in X$, is both fs-semiopen and fs-semiclosed but is neither fs-open nor fs-closed.

Now consider the map $t: (X, \eta(s)) \to (X, \delta(s))$ defined by $t(x) = x$ for all $x \in X$, where $\eta(s) = \{C_f^c(s), X_f^0(s), X_f^1(s)\}$. Then t is fs-semiclosed but not fs-closed.

Theorem 2.7: Let $g: (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then the following conditions are equivalent:
- (i) g is fs-semicontinuous.
- (ii) the inverse image of an fs-closed set in Yunder g is fs-semiclosed in X.
- (iii) For any fs-set $A_f(s)$ in X, $g\left(sCl\left(A_f(s)\right)\right) \leq \overline{g\left(A_f(s)\right)}$.

Proof:
$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous map and $B_f(s)$ be an fs-closed set in Y. Then

$B_f^c(s)$ is fs-open in Y

$$\Rightarrow \left(g^{-1}\left(B_f(s)\right)\right)^c = g^{-1}\left(B_f^c(s)\right) \text{ is fs-semiopen in } X$$
$$\Rightarrow g^{-1}\left(B_f(s)\right) \text{ is fs-semiclosed in } X.$$

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$: Suppose $A_f(s)$ be an fs-set in X. Then by (ii), $g^{-1}(\overline{g(A_f(s))})$ is fs-semiclosed in X and hence $g^{-1}\left(\overline{g\left(A_f(s)\right)}\right) = sCl(g^{-1}(\overline{g(A_f(s))}))$. Again

$$A_f(s) \leq g^{-1}\left(g\left(A_f(s)\right)\right)$$
$$\Rightarrow sCl\left(A_f(s)\right) \leq sCl\left(g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)\right) = g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)$$
$$\Rightarrow g\left(sCl\left(A_f(s)\right)\right) \leq g(g^{-1}(\overline{g(A_f(s))})) \leq \overline{g(A_f(s))}$$

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$: Let $B_f(s)$ be an fs-open set in Y. Then for the fs-closed set $B_f^c(s)$, we have

$$g\left(sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\right) \leq \overline{g\left(g^{-1}\left(B_f^c(s)\right)\right)} \leq \overline{B_f^c(s)} = B_f^c(s)$$

Thus $sCl\left(g^{-1}\left(B_f^c(s)\right)\right) \leq g^{-1}\left(g\left(sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\right)\right) \leq g^{-1}\left(B_f^c(s)\right)$.

Therefore $sCl\left(g^{-1}\left(B_f^c(s)\right)\right) = g^{-1}(B_f^c(s))$ and hence $(g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s))$ is fs-semiclosed in X.

Theorem 2.8: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiopen set in Y is fs-semiopen in X.

Proof: Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Ysuch that

$$O_f(s) \leq B_f(s) \leq \overline{O_f(s)}$$
$$\Rightarrow g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq g^{-1}(\overline{O_f(s)})$$

We claim that $g^{-1}(\overline{O_f(s)}) \leq \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in g^{-1}(\overline{O_f(s)})$. This implies $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $g(U_f(s))$ is a weak open Q-nbd of $g(P_f(s))$. Therefore

$$g(U_f(s))\ q_w\ O_f(s)$$
$$\Rightarrow U_f(s)\ q_w\ g^{-1}(O_f(s))$$
$$\Rightarrow P_f(s) \in \overline{g^{-1}(O_f(s))}.$$

Thus we have, $g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq \overline{g^{-1}(O_f(s))}$. Hence, $g^{-1}(O_f(s))$ being fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen.

Corollary 2.1: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiclosed set in Y is fs-semiclosed in X.

Proof: Proof is omitted.

Corollary 2.2: If $g: (X, \delta(s)) \to (Y, \delta'(s))$ be an fs-semicontinuous open map and $h: (Y, \delta'(s)) \to (Z, \eta(s))$ be an fs-semicontinuous map, then $hog: (X, \delta(s)) \to (Z, \eta(s))$ is fs-semicontinuous.

Proof: Let $C_f(s)$ be an fs-open set in Z, then $h^{-1}(C_f(s))$ is fs-semiopen in $Y$and hence
$(hog)^{-1}\left(C_f(s)\right) = g^{-1}(h^{-1}(C_f(s)))$ is fs-semiopen in X by Theorem 2.8.

Theorem 2.9: Let $g:(X,\delta(s))\to(Y,\eta(s))$ be an fs-continuous open map. Then the g-image of an fs-semiopen set in X is fs-semiopen in Y.

Proof: Let $A_f(s)$ be an fs-semiopen set in X. Then there exists an fs-open set $O_f(s)$ in X such that $O_f(s)\le A_f(s)\le \overline{O_f(s)}$. This implies

$$g(O_f(s))\le g(A_f(s))\le g(\overline{O_f(s)})\le \overline{g(O_f(s))}.$$

Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-semiopen in Y.

Corollary 2.3: Semi-openness in an FSTS is a topological property.

Proof: Follows from Theorem 2.9.

Remark 2.1: Theorem 2.9 does not hold if g is not fs-open. This is shown by Example 2.3.

Example 2.3: Let $(X,\delta(s))$ and $(Y,\delta'(s))$ be two fuzzy sequential topological spaces, where $\delta(s)$ contains all the constant fs-sets in X, $Y=[0,1]$ and $\delta'(s)=\{Y_f^0(s),\ Y_f^1(s)\}$. Define a map $g:(X,\delta(s))\to(Y,\delta'(s))$ by $g(x)=\frac{1}{2}$ for all $x\in X$. Then g is fs-continuous but not fs-open. Here, for any fs-semiopen set $A_f(s)$ in X, $g(A_f(s))=\left\{\overline{\frac{1}{2}}\right\}_{n=1}^{\infty}$ is not fs-semiopen in Y.

Remark 2.2: Converse of Theorem 2.9 holds if g is one-one.

Theorem 2.10: Let $g:(X,\delta(s))\to(Y,\delta'(s))$ and $h:(Y,\delta'(s))\to(Z,\eta(s))$ be two mappings and $hog:(X,\delta(s))\to(Z,\eta(s))$ be an fs-semiclosed mapping. Then, g is fs-semiclosed if h is an injective fs-semicontinuous open mapping.

Proof: Let $A_f(s)$ be an fs-closed set in X. Then $hog(A_f(s))$ is fs-semiclosed in Z and hence $g(A_f(s))=h^{-1}(hog(A_f(s)))$ is fs-semiclosed in Y.

Theorem 2.11: If $g:(X,\delta(s))\to(Y,\delta'(s))$ is fs-semicontinuous and $h:(Y,\delta'(s))\to(Z,\eta(s))$ is fs-continuous, then $hog:(X,\delta(s))\to(Z,\eta(s))$ is fs-semicontinuous.

Proof: Omitted.

3. FS-REGULAR OPEN SETS

Definition 3.1 An fs-set $A_f(s)$ in an FSTS $(X,\delta(s))$, is said to be fs-regular open in X if $(\overline{A_f(s)})^o=A_f(s)$. An fs-set $A_f(s)$ is said to be fs-regular closed in X if its complement is fs-regular open.

It is obvious that every fs-regular open (closed) set is fs-open (closed). The converse need not be true, is shown by Example 3.1. Example 3.2 shows that the union (intersection) of any two fs-regular open (closed) sets need not be an fs-regular open (closed) set.

Example 3.1: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s)=\left\{\overline{\frac{1}{4}},\bar{1},\bar{1},\ldots\ldots\right\}$$

$$B_f(s)=\left\{\overline{\frac{1}{2}},\overline{\frac{1}{2}},\overline{\frac{1}{2}},\ldots\ldots\right\}$$

Let $\delta(s)=\{A_f(s),\ B_f(s),\ A_f(s)\wedge B_f(s),\ A_f(s)\vee B_f(s),\ X_f^0(s),\ X_f^1(s)\}$. Then $(X,\delta(s))$ is an FSTS where $A_f(s)$ is fs-open but not fs-regular open.

Example 3.2: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s)=\left\{\overline{\frac{1}{4}},\overline{\frac{3}{4}},\overline{\frac{1}{4}},\overline{\frac{3}{4}},\ldots\ldots\right\}$$

$$B_f(s)=\left\{\overline{\frac{3}{4}},\overline{\frac{1}{4}},\overline{\frac{3}{4}},\overline{\frac{1}{4}}\ldots\ldots\right\}$$

Let $\delta(s)=\{A_f(s),\ B_f(s),\ A_f(s)\wedge B_f(s),\ A_f(s)\vee B_f(s),\ X_f^0(s),\ X_f^1(s)\}$. Then $(X,\delta(s))$ is an FSTS. Here $A_f(s)$ and $B_f(s)$ are fs-regular open sets but their union is not fs-regular open.

Theorem 3.1:
(a) The intersection of two fs-regular open sets is an fs-regular open set.
(b) The union of two fs-regular closed sets is an fs-regular closed set.

Proof: We prove only (a). Let $A_f(s)$ and $B_f(s)$ be two fs-regular open sets in X. Since $A_f(s) \wedge B_f(s)$ is fs-open, we have $A_f(s) \wedge B_f(s) \leq (\overline{A_f(s) \wedge B_f(s)})^o$.

Now, $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{A_f(s)})^o = A_f(s)$ and $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{B_f(s)})^o = B_f(s)$ implies $(\overline{A_f(s) \wedge B_f(s)})^o \leq$ $Af(s) \wedge Bf(s)$. Hence the result.

Theorem 3.2:
(a) The closure of an fs-open set is fs-regular closed.
(b) The interior of an fs-closed set is fs-regular open.

Proof: We prove only (a). Let $A_f(s)$ be an fs-open set in X. Since $(\overline{A_f(s)})^o \leq \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \leq \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-open, $A_f(s) \leq (\overline{A_f(s)})^o$ and hence $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus $\overline{A_f(s)}$ is fs-regular closed.

Definition 3.2: A mapping $g: (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-almost continuous mapping if $g^{-1}(B_f(s)) \in \delta(s)$ for each fs-regular open set $B_f(s)$ in Y.

Theorem 3.3: Let $g: (X, \delta(s)) \to (Y, \eta(s))$ be a mapping. Then the following are equivalent:
- (i) g is fs-almost continuous.
- (ii) $g^{-1}(B_f(s))$ is an fs-closed set for each fs-regular closed set $B_f(s)$ of Y .
- (iii) $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$ for each fs-open set $B_f(s)$ of Y .
- (iv) $\overline{g^{-1}(\overline{B_f^o(s)})} \leq g^{-1}(B_f(s))$ for each fs-closed set $B_f(s)$ of Y .

Proof: Here, we note that $g^{-1}\left(B_f^c(s)\right) = (g^{-1}(B_f(s))\,)^c$ for any fs-set $B_f(s)$ in Y.
$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$: Follows from the fact that an fs-set is fs-regular open if and only if its complement is fs-regular closed.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$: Let $B_f(s)$ be an fs-open set in Y. Then $B_f(s) \leq (\overline{B_f(s)})^o$ and hence $g^{-1}(B_f(s)) \leq g^{-1}((\overline{B_f(s)})^o)$. By Theorem 3.2 (b), $(\overline{B_f(s)})^o$ is an fs-regular open set in Y. Therefore, $g^{-1}((\overline{B_f(s)})^o)$ is fs-open in X and thus

$$g^{-1}\left(B_f(s)\right) \leq g^{-1}\left(\left(\overline{B_f(s)}\right)^o\right) = (g^{-1}((\overline{B_f(s)})^o))^o.$$

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$: Let $B_f(s)$ be an fs-regular open set in Y. Then by (iii), we have $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$. Hence $g^{-1}(B_f(s))$ is an fs-open set in X.

$(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iv})$: are easy to prove.

Clearly an fs-continuous map is an fs-almost continuous map but the converse may not be true, as is shown by Example 3.3.

Example 3.3: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \quad \bar{1}, \quad \bar{1}, \quad \bar{1}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots\dots.\right\}$$

Let $\delta(s) = \left\{B_f(s), X_f^0(s), X_f^1(s)\right\}$ and $\eta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), \quad A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-almost continuous but not fs-continuous. Again, since the inverse image of fs-open set $A_f(s)$ of $(X, \eta(s))$ is not fs-semiopen in $(X, \delta(s))$, g is not fs-semicontinuous.

Example 3.4: Example to show that an fs-semicontinuous map may not be fs-almost continuous. Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{2}}, \quad \bar{0}, \quad \bar{0}, \quad \bar{0}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots\dots.\right\}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-semicontinuous but not fs-almost continuous.

Remark 3.1: Example 3.3 and Example 3.4 shows that an fs-almost continuous mapping and an fs-semicontinuous mapping are independent notions.

Definition 3.3: An FSTS $(X, \delta(s))$ is called an fs-semiregular space if the collection of all fs-regular open sets in X forms a base for $\delta(s)$.

Theorem 3.4: Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping, where $(Y, \eta(s))$ is an fs-semiregular space. Then g is fs-almost continuous if and only if g is fs-continuous.

Proof: We need only to show that if g is fs-almost continuous, then it is fs-continuous. Suppose g is fs-almost continuous. Let $B_f(s) \in \eta(s)$, then $B_f(s) = \vee_{\lambda\in\Lambda} B_{\lambda f}(s)$, where $B_{\lambda f}(s)$'s are fs-regular open sets in Y. Then

$$\begin{aligned} g^{-1}(B_f(s)) &= \vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\le \vee_{\lambda\in\Lambda} (g^{-1}((\overline{B_{\lambda f}(s)})^o))^o \\ &= \vee_{\lambda\in\Lambda} (g^{-1}(B_{\lambda f}(s)))^o \\ &\le (\vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

which shows $g^{-1}(B_f(s)) \in \delta(s)$.

Theorem 3.5: Let X, X_1 and X_2 be fuzzy sequential topological spaces and $\pi_i: X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g: X \to X_1 \times X_2$ is fs-almost continuous, then $\pi_i o g$ is also fs-almost continuous.

Proof: Let g be an fs-almost continuous map and let $B_f(s)$ be an fs-regular open set in X_i. Since π_i is fs-continuous, we have $\overline{\pi_i^{-1}(B_f(s))} \le \pi_i^{-1}(\overline{B_f(s)})$ and since π_i is fs-open we have, $\pi_i^{-1}(B_f^o(s)) \le (\pi_i^{-1}(B_f(s)))^o$. Also $B_f(s) \le \pi_i^{-1}(\pi_i(B_f(s)))$ and $\pi_i(\pi_i^{-1}(B_f(s)) \le B_f(s)$. Thus

$$\pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right) \le \pi_i\left(\pi_i^{-1}\left(B_f(s)\right)\right) \le B_f(s)$$

$$\Rightarrow \pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right) \le B_f^o(s)$$

$$\Rightarrow \left(\pi_i^{-1}\left(B_f(s)\right)\right)^o \le \pi_i^{-1}(\pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right)) \le \pi_i^{-1}\left(B_f^o(s)\right) = \pi_i^{-1}\left(B_f(s)\right)$$

$$\Rightarrow \pi_i^{-1}\left(B_f(s)\right) = \left(\pi_i^{-1}\left(B_f(s)\right)\right)^o \le \left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o \le \left(\pi_i^{-1}(\overline{B_f(s)})\right)^o = \pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right) = \pi_i^{-1}\left(B_f(s)\right)$$

$$\Rightarrow \pi_i^{-1}\left(B_f(s)\right) = \left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o .$$

Therefore,

$$\begin{aligned} (\pi_i o g)^{-1}\left(B_f(s)\right) &= g^{-1}\left(\pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right)\right) \\ &= g^{-1}\left(\left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o\right) \\ &= \left(g^{-1}\left(\left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o\right)\right)^o \\ &\le \left(g^{-1}\left(\left(\pi_i^{-1}(\overline{B_f(s)})\right)^o\right)\right)^o \\ &= \left(g^{-1}\left(\pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right)\right)\right)^o \\ &= \left(g^{-1}\left(\pi_i^{-1}(B_f(s))\right)\right)^o \\ &= \left((\pi_i o g)^{-1}\left(B_f(s)\right)\right)^o \end{aligned}$$

Hence the theorem.

Definition 3.4: A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-weakly continuous mapping if for each fs-open set $B_f(s)$ in Y, $g^{-1}(B_f(s)) \le (g^{-1}(\overline{B_f(s)}))^o$.

Remark 3.2: It is clear that every fs-continuous mapping is fs-weakly continuous. The converse is not true, in general, which is shown by Example 3.5. The Example also shows that an fs-weakly continuous mapping may neither be fs-semicontinuous nor fs-almost continuous. However, it is clear that an fs-almost continuous mapping is also fs-weakly continuous.

Example 3.5: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots\dots\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{3}}, \quad \overline{\frac{1}{3}}, \quad \overline{\frac{1}{3}}, \dots\dots\right\}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-weakly continuous but not fs-continuous. Since the inverse image of fs-open set $B_f(s)$ of Y is not fs-semiopen in X, hence g is not fs-semicontinuous. Again, as the inverse image of fs-regular open set $B_f(s)$ of Y is not fs-open in X, g is not fs-almost continuous.

Remark 3.3: The map g defined in Example 3.4, is fs-semicontinuous but not fs-weakly continuous.

Remark 3.4: Example 3.5 and Remark 3.3 shows that fs-semicontinuity and fs-weakly continuity are independent notions.

Definition 3.5: An FSTS $(X, \delta(s))$ is called an Ωfs-semiregular space if each fs-open set $A_f(s)$ of X is the union of fs-open sets $A_{\lambda f}(s)$ $(\lambda \in \Lambda)$ of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$.

Theorem 3.6: An Ωfs-semiregular space is fs-semiregular.

Proof: Let $(X, \delta(s))$ be an Ωfs-semiregular space and $A_f(s)$ be an fs-open set in X. Then $A_f(s) = \vee_{\lambda \in \Lambda} A_{\lambda f}(s)$, where $A_{\lambda f}(s)$ are fs-open sets of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$. Since $A_{\lambda f}(s) \leq (\overline{A_{\lambda f}(s)})^o \leq A_f(s)$, we have $A_f(s) = \vee_{\lambda \in \Lambda} (\overline{A_{\lambda f}(s)})^o$. Now, for each $\lambda \in \Lambda$, $(\overline{A_{\lambda f}(s)})^o$ is fs-regular open in X and thus $(X, \delta(s))$ is a fs-semiregular space.

Remark 3.5: Example 3.6 shows that the converse of Theorem 3.6 may not be true.

Example 3.6: Consider the fuzzy sequential topological space $(X, \delta(s))$, where $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee$ Bfs, Afs∧Bfs, Xf0s, Xf1s and where the fs-sets Afs and Bfs in X, are defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \quad \bar{1}, \quad \bar{1}, \quad \bar{1}, \dots\dots\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \bar{0}, \quad \bar{0}, \quad \bar{0}, \dots\dots\right\}$$

Then $(X, \delta(s))$ is an fs-semiregular space. Now, the only way of writing $A_f(s)$ as the union of fs-open sets is the union of itself and $\overline{A_f(s)}$ is not contained in $A_f(s)$. Hence $(X, \delta(s))$ is not an Ωfs-semiregular space.

Theorem 3.7: Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping where $(X, \delta(s))$ is any FSTS and $(Y, \eta(s))$ is an Ωfs-semiregular space. Then g is fs-weakly continuous if and only if g is fs-continuous.

Proof: It suffices to show that if g is fs-weakly continuous, then it is fs-continuous. For this, let $B_f(s) \in \eta(s)$. Then $B_f(s) = \vee_{\lambda \in \Lambda} B_{\lambda f}(s)$, where for all $\lambda \in \Lambda$, $B_{\lambda f}(s) \in \eta(s)$ and $\overline{B_{\lambda f}(s)} \leq B_f(s)$. Since g is fs- weakly continuous, we have

$$\begin{aligned} g^{-1}\left(B_f(s)\right) &= g^{-1}\left(\vee_{\lambda \in \Lambda} B_{\lambda f}(s)\right) = \vee_{\lambda \in \Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \vee_{\lambda \in \Lambda} (g^{-1}(\overline{B_{\lambda f}(s)}))^o \\ &\leq \vee_{\lambda \in \Lambda} (g^{-1}(B_f(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

and hence $g^{-1}(B_f(s))$ is fs-open in X. Thus g is fs-continuous.

Theorem 3.8: Let X, X_1 and X_2 be FSTS's and $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g : X \to X_1 \times X_2$ is fs-weakly continuous, then $\pi_i o g$ is also fs-weakly continuous.

Proof: The proof is analogous to the proof of Theorem 3.5.

REFERENCES

1. Azad K. K., On Fuzzy Semicontinuity, Fuzzy Amost Continuity and Fuzzy Weakly Continuity, J. Math. Anal. Appl., 82 (1981) 14-32.
2. Bose M.K. and Lahiri I., "Sequential Topological Spaces and Separation Axioms", Bull. Allahabad Math. Soc., 17: 23-37, 2002.
3. Chang C. L., "Fuzzy Topological Spaces", J. Math. Anal. Appl., 24: 182-190, 1968.
4. Ganster M. and Reilly I., A Decomposition of Continuity, Acta Math. Hungarica 56(34) (1990), 299301.
5. Hamlett T. R., A Correction to the paper "Semi-open sets and Semi-continuity in Topological Spaces" by Norman Levine, Proceedings of American Mathematical Society, Vol. 49, No. 2 (June 1975), 458-460.
6. Levine N., Semi-Open Sets and Semi-Continuity in Topological Spaces, The American Mathematical Monthly, Vol. 70, No. 1, (Jan., 1963), 36-41.
7. Mashour A.S., Hasanein I. A. and EI-Deeb S. N., A Note on Semi-Continuity and Precontinuity, Indian J. pure appl. Math., 13 (10)(October 1982), 1119-1123.
8. Palaniappan N.,"Fuzzy Topology", Narosa Publishing House, New Delhi, 2006.
9. Singha M., Tamang N. and Sarkar S. De, Fuzzy Sequential Topological Spaces, International Journal of Computer and Mathematical Sciences, Vol. 3 (4) (June 2014), 2347-8527.
10. Tamang N., Singha M. and Sarkar S. De, On the notions of continuity and compactness in fuzzy sequential topological spaces, communicated.
11. Yalvac Tuna Hatice, Semi-interior and Semi-closure of a fuzzy set, J. Math. Anal. Appl., 132 (1988), 356-364.
12. Zadeh L.A., "Fuzzy Sets", Information And Control, 8: 338-353, 1965.

Source of support: Nil, Conflict of interest: None Declared

.