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SOME NEARLY OPEN SETS IN A FUZZY SEQUENTIAL TOPOLOGICAL SPACE

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ABSTRACT

The present article gives a study of fs-semiopen sets, fs-regular open sets and fs-semicontinuous functions in a fuzzy sequential topological space. Other studied notions are fs-almost continuous functions, fs-weakly continuous functions and it has been shown that both of these functions and fs-semicontinuous functions are independent notions. Further, many results relating these functions together with fs-continuous functions have been obtained.

Keywords and Phrases: *Fuzzy sequential topological spaces, fs-semiopen sets, fs-semicontinuous functions, fs-regular open sets, fs-almost continuous functions, fs-weakly continuous functions.*

***AMS Subject Classification**: 54A40.*

1. PRELIMINARIES

The introduction of fuzzy sets in 1965, by L.A. Zadeh [12] leads to the foundation of a new area of research called fuzzy mathematics. Since then, many researchers have been working in this area and related areas. As a generalization of a topological space, C. L. Chang [3] introduced the concept of fuzzy topological space in 1968. Fuzzy semi-open sets and fuzzy semicontinuity were introduced and studied by K. K. Azad [1].

The purpose of this work is to study the concept of semi-open sets and semicontinuity in fuzzy sequential topological spaces.

Throughout the paper, X will denote a non empty set and I the unit interval [0, 1]. Sequences of fuzzy sets in X called fuzzy sequential sets (fs-sets) will be denoted by the symbols $A_f(s)$, $B_f(s)$, $C_f(s)$ etc. An fs-set $X_f^l(s)$ is a sequence of fuzzy sets $\{X_f^n\}_n$, where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$.

A family $\delta(s)$ of fuzzy sequential sets on a non-empty set X satisfying the properties:

i. $X_f^r(s) \in \delta(s)$ for all $r \in \{0, 1\}$,

ii. $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$

iii. for any family $\{A_{fj}(s);\ j \in J\} \subseteq \delta(s), \vee_{j \in J} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called a fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets. Complement of an open fuzzy sequential set is called closed fuzzy sequential set. In an FSTS $(X, \delta(s))$, the closure $\overline{A_f(s)}$ and interior $A_f^\circ(s)$ of any fs-set $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \le C_f(s),\ (C_f(s))^c \in \delta(s)\},$$

$$A_f^o(s) = \vee\{O_f(s);\ O_f(s) \le A_f(s)), O_f(s) \in \delta(s)\},$$

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[10] Let g be a mapping from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, then g is called

(i) fs-continuous if $g^{-1}(B_f(s))$ is open in $(X, \delta(s))$ for every open fs-set $B_f(s)$ in $(Y, \eta(s))$.

(ii) fs-open if $g(A_f(s))$ is fs-open in Y for every fs-open set $A_f(s)$ in X.

(iii) fs-closed if $g(A_f(s))$ is fs-closed in Y for every fs-closed set $A_f(s)$ in X.

Section **2** deals with the introduction and study of fs-semiopen sets as well as fs-semicontinuity. Section **3** deals with the introduction of fs-regular open sets and functions like fs-almost continuous and fs-weakly continuous functions. In this section, the interrelations among these functions together with fs-continuous and fs-semicontinuous functions have been investigated.

2. FS-SEMIOPEN SETS AND FS-SEMICONTINUITY

Definition 2.1: An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiopen set if $A_f(s) \leq \overline{A_f^o(s)}$. An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiclosed set if its complement is fs-semiopen.

Fundamental properties of fs-semiopen (fs-semiclosed) sets are:

- Any union (intersection) of fs-semiopen (fs-semiclosed) sets is fs-semiopen (fs-semiclosed).
- Every fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).
- Closure (interior) of an fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).

Example 2.1 shows that an fs-semiopen (fs-semiclosed) set may not be fs open (fs-closed), the intersection (union) of any two fs-semiopen (fs semiclosed) sets need not be an fs-semiopen (fs-semoclosed) set. Unlike in a general topological space, the intersection of an fs-semiopen set with an fs open set may fail to be an fs-semiopen set.

Example 2.1: Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \dots \dots\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{2}}, \bar{0}, \bar{0}, \dots \dots\right\}$$

$$C_f(s) = \left\{\overline{\frac{3}{8}}, \bar{1}, \bar{1}, \dots \dots\right\}$$

$$D_f(s) = \left\{\overline{\frac{3}{8}}, \bar{0}, \bar{0}, \dots \dots\right\}$$

Consider $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Now,

(i) $B_f(s)$ is fs-open, hence fs-semiopen and $C_f(s)$ is fs-semiopen but their intersection $D_f(s)$ is not fs-semiopen.

(ii) $C_f(s)$ is fs-semiopen but is not fs-open.

Theorem 2.1: Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-semiopen if and only if there exist an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$.

Proof: Straightforward.

Theorem 2.2: Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-semiclosed if and only if there exist an fs-closed set $C_f(s)$ in X such that $C_f^o(s) \leq A_f(s) \leq C_f(s)$.

Proof: Straightforward.

We will denote the set of all fs-semiopen sets in X by $FSSO(X)$.

Theorem 2.3: In an FSTS $(X, \delta(s))$, (i) $\delta(s) \subseteq FSSO(X)$. (ii) If $A_f(s) \in FSSO(X)$ and $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, then $B_f(s) \in FSSO(X)$.

Proof:

(i) Follows from definition.

(ii) Let $A_f(s) \in FSSO(X)$. Then there exists an fs-open set $O_f(s)$ such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. So,

$$O_f(s) \leq A_f(s) \leq B_f(s) \leq \overline{A_f(s)} \leq \overline{O_f(s)}$$
$$\Rightarrow O_f(s) \leq B_f(s) \leq \overline{O_f(s)}.$$

$O_f(s)$ being fs-open, $B_f(s)$ is fs-semiopen.

Theorem 2.4: If in a fuzzy sequential topological space, $C_f^o(s) \le B_f(s) \le C_f(s)$, where $C_f(s)$ is fs-semiclosed, then $B_f(s)$ is also fs-semiclosed.

Proof: Omitted.

Theorem 2.5: Let $\mho = \{A_{\alpha f}(s);\ \alpha \in \Lambda\}$ be a collection of fs-sets in an FSTS $(X, \delta(s))$ such that (i) $\delta(s) \subseteq \mho$ and (ii) if $A_f(s) \in \mho$ and $A_f(s) \le B_f(s) \le \overline{A_f(s)}$, then $B_f(s) \in \mho$. Then $FSSO(X) \subseteq \mho$. that is, $FSSO(X)$ is the smallest class of fs-sets in X satisfying (i) and (ii).

Proof: Let $A_f(s) \in FSSO(X)$. Then $O_f(s) \le A_f(s) \le \overline{O_f(s)}$ for some $O_f(s) \in \delta(s)$. By (i), $O_f(s) \in \mho$ and thus $A_f(s) \in \mho$ by (ii).

If $\mho = \{A_{\alpha f}(s);\ \alpha \in \Lambda\}$ be a collection of fs-sets in X, then $Int\mho$ denotes the set $\{A^o_{\alpha f}(s);\ \alpha \in \Lambda\}$.

Theorem 2.6: If $(X, \delta(s))$ be a fuzzy sequential topological space, then $\delta(s) = Int(FSSO(X))$.

Proof: Every fs-open set being fs-semiopen, $\delta(s) \subseteq Int(FSSO(X))$. Conversely, let $O_f(s) \in Int(FSSO(X))$. Then $O_f(s) = A_f^o(s)$ for some $A_f(s) \in FSSO(X)$ and hence $O_f(s) \in \delta(S)$.

Definition 2.2: Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X. We define semi-closure $sCl(A_f(s))$ and semi-interior $sInt(A_f(s))$ of $A_f(s)$ by

$$sCl\left(A_f(s)\right) = \wedge\{B_f(s);\ A_f(s) \le B_f(s)\ and\ A_f^c(s) \in FSSO(X)\}$$
$$sInt\left(B_f^c(s)\right) = \vee\{C_f(s);\ C_f(s) \le A_f(s)\ and\ C_f(s) \in FSSO(X)\}.$$

Obviously, $sCl\left(A_f(s)\right)$ is the smallest fs-semiclosed set containing $A_f(s)$ and $sInt\left(A_f(s)\right)$ is the largest fs-semiopen set contained in $A_f(s)$. Further,

(i) $A_f(s) \le sCl\left(A_f(s)\right) \le \overline{A_f(s)}$ and $A_f^o(s) \le sInt\left(A_f(s)\right) \le A_f(s)$.
(ii) $A_f(s)$ is fs-semiopen if and only if $A_f(s) = sInt\left(A_f(s)\right)$.
(iii) $A_f(s)$ is fs-semiclosed if and only if $A_f(s) = sCl\left(A_f(s)\right)$.
(iv) $A_f(s) \le B_f(s)$ implies $sInt\left(A_f(s)\right) \le sInt\left(B_f(s)\right)$ and $sCl\left(A_f(s)\right) \le sCl\left(B_f(s)\right)$.

Definition 2.3: A mapping $g: (X, \delta(s)) \to (Y, \delta'(s))$ is said to be
(i) fs-semicontinuous if $g^{-1}(B_f(s))$ is fs-semiopen in X for every $B_f(s) \in \delta'(s)$.
(ii) fs-semiopen if $g(A_f(s))$ is fs-semiopen in Y for every $A_f(s) \in \delta(s)$.
(iii) fs-semiclosed if $g(A_f(s))$ is fs-semiclosed in Y for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-semicontinuous (fs-semiopen, fs-semiclosed). That the converse may not be true, is shown by Example 2.2.

Example 2.2: Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \bar{0}, \bar{0}, \dots\dots\right\}$$
$$C_f(s) = \left\{\overline{\frac{3}{8}}, \bar{1}, \bar{1}, \dots\dots\right\}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Let $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$. Define $g: (X, \delta(s)) \to (X, \delta'(s))$ by $g(x) = x$ for all $x \in X$. The function g is fs-semicontinuous but not fs-continuous.

Again the map $h: (X, \delta'(s)) \to (X, \delta(s))$ defined by $h(x) = x$ for all $x \in X$, is both fs-semiopen and fs-semiclosed but is neither fs-open nor fs-closed.

Now consider the map $t: (X, \eta(s)) \to (X, \delta(s))$ defined by $t(x) = x$ for all $x \in X$, where $\eta(s) = \{C_f^c(s), X_f^0(s), X_f^1(s)\}$. Then t is fs-semiclosed but not fs-closed.

Theorem 2.7: Let $g: (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then the following conditions are equivalent:

(i) g is fs-semicontinuous.

(ii) the inverse image of an fs-closed set in Y under g is fs-semiclosed in X.

(iii) For any fs-set $A_f(s)$ in X, $g\left(sCl\left(A_f(s)\right)\right) \le \overline{g\left(A_f(s)\right)}$.

Proof:

$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous map and $B_f(s)$ be an fs-closed set in Y. Then

$B_f^c(s)$ is fs-open in Y

$$\Rightarrow \left(g^{-1}\left(B_f(s)\right)\right)^c = g^{-1}\left(B_f^c(s)\right) \text{ is fs-semiopen in } X$$
$$\Rightarrow g^{-1}\left(B_f(s)\right) \text{ is fs-semiclosed in } X.$$

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$: Suppose $A_f(s)$ be an fs-set in X. Then by (ii), $g^{-1}(\overline{g(A_f(s))})$ is fs-semiclosed in X and hence $g^{-1}\left(\overline{g\left(A_f(s)\right)}\right) = sCl(g^{-1}(\overline{g(A_f(s))}))$. Again

$$A_f(s) \le g^{-1}\left(g\left(A_f(s)\right)\right)$$
$$\Rightarrow sCl\left(A_f(s)\right) \le sCl\left(g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)\right) = g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)$$
$$\Rightarrow g\left(sCl\left(A_f(s)\right)\right) \le g(g^{-1}(\overline{g(A_f(s))})) \le \overline{g(A_f(s))}$$

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$: Let $B_f(s)$ be an fs-open set in Y. Then for the fs-closed set $B_f^c(s)$, we have

$$g\left(sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\right) \le \overline{g\left(g^{-1}\left(B_f^c(s)\right)\right)} \le \overline{B_f^c(s)} = B_f^c(s)$$

Thus $sCl\left(g^{-1}\left(B_f^c(s)\right)\right) \le g^{-1}\left(g\left(sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\right)\right) \le g^{-1}\left(B_f^c(s)\right)$.

Therefore $sCl\left(g^{-1}\left(B_f^c(s)\right)\right) = g^{-1}(B_f^c(s))$ and hence $(g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s))$ is fs-semiclosed in X.

Theorem 2.8: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiopen set in Y is fs-semiopen in X.

Proof: Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$O_f(s) \le B_f(s) \le \overline{O_f(s)}$$
$$\Rightarrow g^{-1}(O_f(s)) \le g^{-1}(B_f(s)) \le g^{-1}(\overline{O_f(s)})$$

We claim that $g^{-1}(\overline{O_f(s)}) \le \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in g^{-1}(\overline{O_f(s)})$. This implies $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $g(U_f(s))$ is a weak open Q-nbd of $g(P_f(s))$. Therefore

$$g(U_f(s))\ q_w\ O_f(s)$$
$$\Rightarrow U_f(s)\ q_w\ g^{-1}(O_f(s))$$
$$\Rightarrow P_f(s) \in \overline{g^{-1}(O_f(s))}.$$

Thus we have, $g^{-1}(O_f(s)) \le g^{-1}(B_f(s)) \le \overline{g^{-1}(O_f(s))}$. Hence, $g^{-1}(O_f(s))$ being fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen.

Corollary 2.1: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiclosed set in Y is fs-semiclosed in X.

Proof: Proof is omitted.

Corollary 2.2: If $g: (X, \delta(s)) \to (Y, \delta'(s))$ be an fs-semicontinuous open map and $h: (Y, \delta'(s)) \to (Z, \eta(s))$ be an fs-semicontinuous map, then $hog: (X, \delta(s)) \to (Z, \eta(s))$ is fs-semicontinuous.

Proof: Let $C_f(s)$ be an fs-open set in Z, then $h^{-1}(C_f(s))$ is fs-semiopen in Y and hence $(hog)^{-1}\left(C_f(s)\right) = g^{-1}(h^{-1}(C_f(s)))$ is fs-semiopen in X by Theorem 2.8.

Theorem 2.9: Let $g:(X,\delta(s)) \to (Y,\eta(s))$ be an fs-continuous open map. Then the g-image of an fs-semiopen set in X is fs-semiopen in Y.

Proof: Let $A_f(s)$ be an fs-semiopen set in X. Then there exists an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. This implies

$$g(O_f(s)) \leq g(A_f(s)) \leq g(\overline{O_f(s)}) \leq \overline{g(O_f(s))}.$$

Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-semiopen in Y.

Corollary 2.3: Semi-openness in an FSTS is a topological property.

Proof: Follows from Theorem 2.9.

Remark 2.1: Theorem 2.9 does not hold if g is not fs-open. This is shown by Example 2.3.

Example 2.3: Let $(X,\delta(s))$ and $(Y,\delta'(s))$ be two fuzzy sequential topological spaces, where $\delta(s)$ contains all the constant fs-sets in X, $Y = [0,1]$ and $\delta^{'}(s) = \{Y_f^0(s),\ Y_f^1(s)\}$. Define a map $g:(X,\delta(s)) \to (Y,\delta'(s))$ by $g(x) = \frac{1}{2}$ for all $x \in X$. Then g is fs-continuous but not fs-open. Here, for any fs-semiopen set $A_f(s)$ in X, $g(A_f(s)) = \left\{\overline{\frac{1}{2}}\right\}_{n=1}^{\infty}$ is not fs-semiopen in Y.

Remark 2.2: Converse of Theorem 2.9 holds if g is one-one.

Theorem 2.10: Let $g:(X,\delta(s)) \to (Y,\delta'(s))$ and $h:(Y,\delta'(s)) \to (Z,\eta(s))$ be two mappings and $hog:(X,\delta(s)) \to (Z,\eta(s))$ be an fs-semiclosed mapping. Then, g is fs-semiclosed if h is an injective fs-semicontinuous open mapping.

Proof: Let $A_f(s)$ be an fs-closed set in X. Then $hog(A_f(s))$ is fs-semiclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-semiclosed in Y.

Theorem 2.11: If $g:(X,\delta(s)) \to (Y,\delta'(s))$ is fs-semicontinuous and $h:(Y,\delta'(s)) \to (Z,\eta(s))$ is fs-continuous, then $hog:(X,\delta(s)) \to (Z,\eta(s))$ is fs-semicontinuous.

Proof: Omitted.

3. FS-REGULAR OPEN SETS

Definition 3.1 An fs-set $A_f(s)$ in an FSTS $(X,\delta(s))$, is said to be fs-regular open in X if $(\overline{A_f(s)})^o = A_f(s)$. An fs-set $A_f(s)$ is said to be fs-regular closed in X if its complement is fs-regular open.

It is obvious that every fs-regular open (closed) set is fs-open (closed). The converse need not be true, is shown by Example 3.1. Example 3.2 shows that the union (intersection) of any two fs-regular open (closed) sets need not be an fs-regular open (closed) set.

Example 3.1: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \ldots\ldots\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \ldots\ldots\right\}$$

Let $\delta(s) = \{A_f(s),\ B_f(s),\ A_f(s) \wedge B_f(s),\ A_f(s) \vee B_f(s),\ X_f^0(s),\ X_f^1(s)\}$. Then $(X,\delta(s))$ is an FSTS where $A_f(s)$ is fs-open but not fs-regular open.

Example 3.2: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \ldots\ldots\right\}$$

$$B_f(s) = \left\{\overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}} \ldots\ldots\right\}$$

Let $\delta(s) = \{A_f(s),\ B_f(s),\ A_f(s) \wedge B_f(s),\ A_f(s) \vee B_f(s),\ X_f^0(s),\ X_f^1(s)\}$. Then $(X,\delta(s))$ is an FSTS. Here $A_f(s)$ and $B_f(s)$ are fs-regular open sets but their union is not fs-regular open.

Theorem 3.1:
(a) The intersection of two fs-regular open sets is an fs-regular open set.
(b) The union of two fs-regular closed sets is an fs-regular closed set.

Proof: We prove only (a). Let $A_f(s)$ and $B_f(s)$ be two fs-regular open sets in X. Since $A_f(s) \wedge B_f(s)$ is fs-open, we have $A_f(s) \wedge B_f(s) \leq (\overline{A_f(s) \wedge B_f(s)})^o$.

Now, $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{A_f(s)})^o = A_f(s)$ and $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{B_f(s)})^o = B_f(s)$ implies $(\overline{A_f(s) \wedge B_f(s)})^o \leq$ $Af(s) \wedge Bf(s)$. Hence the result.

Theorem 3.2:
(a) The closure of an fs-open set is fs-regular closed.
(b) The interior of an fs-closed set is fs-regular open.

Proof: We prove only (a). Let $A_f(s)$ be an fs-open set in X. Since $(\overline{A_f(s)})^o \leq \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \leq \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-open, $A_f(s) \leq (\overline{A_f(s)})^o$ and hence $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus $\overline{A_f(s)}$ is fs-regular closed.

Definition 3.2: A mapping $g: (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-almost continuous mapping if $g^{-1}(B_f(s)) \in \delta(s)$ for each fs-regular open set $B_f(s)$ in Y.

Theorem 3.3: Let $g: (X, \delta(s)) \to (Y, \eta(s))$ be a mapping. Then the following are equivalent:
(i) g is fs-almost continuous.
(ii) $g^{-1}(B_f(s))$ is an fs-closed set for each fs-regular closed set $B_f(s)$ of Y .
(iii) $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$ for each fs-open set $B_f(s)$ of Y .
(iv) $\overline{g^{-1}(\overline{B_f^o(s)})} \leq g^{-1}(B_f(s))$ for each fs-closed set $B_f(s)$ of Y .

Proof: Here, we note that $g^{-1}\left(B_f^c(s)\right) = (g^{-1}(B_f(s)))^c$ for any fs-set $B_f(s)$ in Y.
$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$: Follows from the fact that an fs-set is fs-regular open if and only if its complement is fs-regular closed.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$: Let $B_f(s)$ be an fs-open set in Y. Then $B_f(s) \leq (\overline{B_f(s)})^o$ and hence $g^{-1}(B_f(s)) \leq g^{-1}((\overline{B_f(s)})^o)$. By Theorem 3.2 (b), $(\overline{B_f(s)})^o$ is an fs-regular open set in Y. Therefore, $g^{-1}((\overline{B_f(s)})^o)$ is fs-open in X and thus

$$g^{-1}\left(B_f(s)\right) \leq g^{-1}\left(\left(\overline{B_f(s)}\right)^o\right) = (g^{-1}((\overline{B_f(s)})^o))^o.$$

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$: Let $B_f(s)$ be an fs-regular open set in Y. Then by (iii), we have $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$. Hence $g^{-1}(B_f(s))$ is an fs-open set in X.

$(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iv})$: are easy to prove.

Clearly an fs-continuous map is an fs-almost continuous map but the converse may not be true, as is shown by Example 3.3.

Example 3.3: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \quad \overline{1}, \quad \overline{1}, \quad \overline{1}, \ldots \ldots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \ldots \ldots .\right\}$$

Let $\delta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), \quad A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-almost continuous but not fs-continuous. Again, since the inverse image of fs-open set $A_f(s)$ of $(X, \eta(s))$ is not fs-semiopen in $(X, \delta(s))$, g is not fs-semicontinuous.

Example 3.4: Example to show that an fs-semicontinuous map may not be fs-almost continuous. Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{0}, \quad \overline{0}, \quad \overline{0}, \ldots \ldots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \ldots \ldots .\right\}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-semicontinuous but not fs-almost continuous.

Remark 3.1: Example 3.3 and Example 3.4 shows that an fs-almost continuous mapping and an fs-semicontinuous mapping are independent notions.

Definition 3.3: An FSTS $(X, \delta(s))$ is called an fs-semiregular space if the collection of all fs-regular open sets in X forms a base for $\delta(s)$.

Theorem 3.4: Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping, where $(Y, \eta(s))$ is an fs-semiregular space. Then g is fs-almost continuous if and only if g is fs-continuous.

Proof: We need only to show that if g is fs-almost continuous, then it is fs-continuous. Suppose g is fs-almost continuous. Let $B_f(s) \in \eta(s)$, then $B_f(s) = \vee_{\lambda\in\Lambda} B_{\lambda f}(s)$, where $B_{\lambda f}(s)$'s are fs-regular open sets in Y . Then

$$\begin{aligned} g^{-1}(B_f(s)) &= \vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \vee_{\lambda\in\Lambda}(g^{-1}((\overline{B_{\lambda f}(s)})^o))^o \\ &= \vee_{\lambda\in\Lambda}(g^{-1}(B_{\lambda f}(s)))^o \\ &\leq (\vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

which shows $g^{-1}(B_f(s)) \in \delta(s)$.

Theorem 3.5: Let X, X_1 and X_2 be fuzzy sequential topological spaces and $\pi_i: X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g: X \to X_1 \times X_2$ is fs-almost continuous, then $\pi_i o g$ is also fs-almost continuous.

Proof: Let g be an fs-almost continuous map and let $B_f(s)$ be an fs-regular open set in X_i. Since π_i is fs-continuous, we have $\overline{\pi_i^{-1}(B_f(s))} \leq \pi_i^{-1}(\overline{B_f(s)})$ and since π_i is fs-open we have, $\pi_i^{-1}(B_f^o(s)) \leq (\pi_i^{-1}(B_f(s)))^o$. Also $B_f(s) \leq \pi_i^{-1}(\pi_i(B_f(s)))$ and $\pi_i(\pi_i^{-1}(B_f(s)) \leq B_f(s)$. Thus

$$\pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right) \leq \pi_i\left(\pi_i^{-1}\left(B_f(s)\right)\right) \leq B_f(s)$$

$$\Rightarrow \pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right) \leq B_f^o(s)$$

$$\Rightarrow \left(\pi_i^{-1}\left(B_f(s)\right)\right)^o \leq \pi_i^{-1}(\pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right)) \leq \pi_i^{-1}\left(B_f^o(s)\right) = \pi_i^{-1}\left(B_f(s)\right)$$

$$\Rightarrow \pi_i^{-1}\left(B_f(s)\right) = \left(\pi_i^{-1}\left(B_f(s)\right)\right)^o \leq \left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o \leq \left(\pi_i^{-1}(\overline{B_f(s)})\right)^o = \pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right) = \pi_i^{-1}\left(B_f(s)\right)$$

$$\Rightarrow \pi_i^{-1}\left(B_f(s)\right) = \left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o .$$

Therefore,

$$\begin{aligned} (\pi_i o g)^{-1}\left(B_f(s)\right) &= g^{-1}\left(\pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right)\right) \\ &= g^{-1}\left(\left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o\right) \\ &= \left(g^{-1}\left(\left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o\right)\right)^o \\ &\leq \left(g^{-1}\left(\left(\pi_i^{-1}(\overline{B_f(s)})\right)^o\right)\right)^o \\ &= \left(g^{-1}\left(\pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right)\right)\right)^o \\ &= \left(g^{-1}\left(\pi_i^{-1}(B_f(s))\right)\right)^o \\ &= \left((\pi_i o g)^{-1}\left(B_f(s)\right)\right)^o \end{aligned}$$

Hence the theorem.

Definition 3.4: A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-weakly continuous mapping if for each fs-open set $B_f(s)$ in Y, $g^{-1}(B_f(s)) \leq (g^{-1}(\overline{B_f(s)}))^o$.

Remark 3.2: It is clear that every fs-continuous mapping is fs-weakly continuous. The converse is not true, in general, which is shown by Example 3.5. The Example also shows that an fs-weakly continuous mapping may neither be fs-semicontinuous nor fs-almost continuous. However, it is clear that an fs-almost continuous mapping is also fs-weakly continuous.

Example 3.5: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots\dots\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{3}}, \quad \overline{\frac{1}{3}}, \quad \overline{\frac{1}{3}}, \dots\dots\right\}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-weakly continuous but not fs-continuous. Since the inverse image of fs-open set $B_f(s)$ of Y is not fs-semiopen in X, hence g is not fs-semicontinuous. Again, as the inverse image of fs-regular open set $B_f(s)$ of Y is not fs-open in X, g is not fs-almost continuous.

Remark 3.3: The map g defined in Example 3.4, is fs-semicontinuous but not fs-weakly continuous.

Remark 3.4: Example 3.5 and Remark 3.3 shows that fs-semicontinuity and fs-weakly continuity are independent notions.

Definition 3.5: An FSTS $(X, \delta(s))$ is called an Ωfs-semiregular space if each fs-open set $A_f(s)$ of X is the union of fs-open sets $A_{\lambda f}(s)$ $(\lambda \in \Lambda)$ of X such that $\overline{A_{\lambda f}(s)} \le A_f(s)$ for all $\lambda \in \Lambda$.

Theorem 3.6: An Ωfs-semiregular space is fs-semiregular.

Proof: Let $(X, \delta(s))$ be an Ωfs-semiregular space and $A_f(s)$ be an fs-open set in X. Then $A_f(s) = \vee_{\lambda \in \Lambda} A_{\lambda f}(s)$, where $A_{\lambda f}(s)$ are fs-open sets of X such that $\overline{A_{\lambda f}(s)} \le A_f(s)$ for all $\lambda \in \Lambda$. Since $A_{\lambda f}(s) \le (\overline{A_{\lambda f}(s)})^o \le A_f(s)$, we have $A_f(s) = \vee_{\lambda \in \Lambda} (\overline{A_{\lambda f}(s)})^o$. Now, for each $\lambda \in \Lambda$, $(\overline{A_{\lambda f}(s)})^o$ is fs-regular open in X and thus $(X, \delta(s))$ is a fs-semiregular space.

Remark 3.5: Example 3.6 shows that the converse of Theorem 3.6 may not be true.

Example 3.6: Consider the fuzzy sequential topological space $(X, \delta(s))$, where $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee$ Bfs, Afs∧Bfs, Xf0s, Xf1s and where the fs-sets *Afs* and *Bfs* in *X*, are defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \quad \bar{1}, \quad \bar{1}, \quad \bar{1}, \dots\dots\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \bar{0}, \quad \bar{0}, \quad \bar{0}, \dots\dots\right\}$$

Then $(X, \delta(s))$ is an fs-semiregular space. Now, the only way of writing $A_f(s)$ as the union of fs-open sets is the union of itself and $\overline{A_f(s)}$ is not contained in $A_f(s)$. Hence $(X, \delta(s))$ is not an Ωfs-semiregular space.

Theorem 3.7: Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping where $(X, \delta(s))$ is any FSTS and $(Y, \eta(s))$ is an Ωfs-semiregular space. Then g is fs-weakly continuous if and only if g is fs-continuous.

Proof: It suffices to show that if g is fs-weakly continuous, then it is fs-continuous. For this, let $B_f(s) \in \eta(s)$. Then $B_f(s) = \vee_{\lambda \in \Lambda} B_{\lambda f}(s)$, where for all $\lambda \in \Lambda$, $B_{\lambda f}(s) \in \eta(s)$ and $\overline{B_{\lambda f}(s)} \le B_f(s)$. Since g is fs- weakly continuous, we have

$$\begin{aligned} g^{-1}\left(B_f(s)\right) &= g^{-1}\left(\vee_{\lambda \in \Lambda} B_{\lambda f}(s)\right) = \vee_{\lambda \in \Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\le \vee_{\lambda \in \Lambda} (g^{-1}(\overline{B_{\lambda f}(s)}))^o \\ &\le \vee_{\lambda \in \Lambda} (g^{-1}(B_f(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

and hence $g^{-1}(B_f(s))$ is fs-open in X. Thus g is fs-continuous.

Theorem 3.8: Let X, X_1 and X_2 be FSTS's and $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g : X \to X_1 \times X_2$ is fs-weakly continuous, then $\pi_i o g$ is also fs-weakly continuous.

Proof: The proof is analogous to the proof of Theorem 3.5.

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