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A COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACE UNDER WEAK** COMMUTATIVITY CONDITION FOR SIX SELF MAPS USING IMPLICIT RELATION

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ABSTRACT

*The aim of this paper is to present a common fixed point theorem in fuzzy metric space using weak** commuting property for six self maps satisfying an implicit relation which generalize and unify the existing results of [3], [6], [7], [8] ,[9] and [10].*

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Keywords: *Fuzzy metric space, Weak** commutative mapping, Common fixed point, Implicit relation.*

1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh in 1965. Later many authors used the concept of fuzziness in metric space. The idea of fuzzy metric space introduced by Kramosil and Michalek was modified by George and Veeramani [2].Recently in fuzzy metric space concept of R-weakly commuting map, compatible map, semi-compatible map, weak-compatible map etc are introduced and used by several authors. For instance Jungck and Rohades, R. Vasuki, Singh and Chauhan, Singh and Jain etc.

In recent years several authors have generalized commuting condition of mapping introduced by Jungck. Sessa initiated the tradition of improving commutative condition in fixed point theorems by introducing the notion of weakly commuting mapping. Pathak defines weak* commuting and weak** commuting mapping in metric space and proves some theorem. Popa [6] proved theorem for weakly compatible non-continuous mappings using implicit relation. It was extended by Imdad [3] using coincidence commuting property. Jain [9] also extend the result of Popa [7] and [8] in fuzzy metric space.

The main object of this paper is to obtain some common fixed point theorems in fuzzy metric space using “Implicit Relation”. Our result differs from all above authors in the following ways:

(1) We have taken six self maps.
(2) Weak** commuting property is used.
(3) Relaxing the continuity requirement completely.

2. PRELIMINARIES

For the terminologies and basic properties of fuzzy metric space readers refer to George and Veeramani [2]. Some other required definitions are as follows:

Weak Commuting:** Two self mappings A and T of fuzzy metric space (X, M, *) is called weak** commuting if $A(X) \subset T(X)$ and for any $x \in X$,

$$M(A^2T^2x, T^2A^2x, t) \geq M(A^2Tx, T^2Ax, t) \geq M(AT^2x, TA^2x, t) \geq M(ATx, TAx, t) \geq M(A^2x, T^2x, t)$$

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Remark: If A and T are idempotent maps i.e. $A^2 = A$ and $T^2 = T$ then weak**commutative reduces to weak commuting pair of (A, T).

2.1. A class of implicit relation: Let Φ be the set of all real continuous functions.

F: $(R+)^5 \rightarrow R$ non decreasing in the first argument satisfying the following conditions:

(a) For $u, v \geq 0$, $F(u, v, u, v, 1) \geq 0$ implies that $u \geq v$.

(b) $F\{u,1,1,u,1\} \geq 0$ or $F\{u,u,1,1,u\} \geq 0$ or $F\{u,1,u,1,u\} \geq 0$ implies that $u \geq 1$

2.2. Example

Define $F(t_1, t_2, t_3, t_4, t_5) \geq 20t_1 - 18t_2 + 10t_3 - 12t_4 - t_5 + 1$. Then $F \in \Phi$.

3. MAIN RESULTS

3.1. Theorem: Let P, A, B, Q, S and T be self mappings of a complete fuzzy metric space (X, M,*) satisfying:

(3.1.1). $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$

(3.1.2). The pairs (P, ST) and (Q, AB) are weak ** commutative,

(3.1.3). One of P(X), Q(X), AB(X) and ST(X) is a complete subspace of X

(3.1.4). For some $F \in \emptyset$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$F\{M(P^2x, Q^2y, kt), M((ST)^2x, (AB)^2y, t), M(P^2x, (ST)^2x, t), M(Q^2y, (AB)^2y, kt), M((AB)^2y, P^2x, t)\} \geq 0$.

Then P, Q, AB, ST has unique common fixed point in X.

If the pair (A, B), (S, T), (Q, B) and (T, P) are commuting mappings then A, B, S, T, P & Q have a unique Common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary point, as $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$ there exist $x_1, x_2 \in X$ such that $P^2x_0 = (AB)^2x_1$ and $Q^2x_1 = (ST)^2x_2$. Inductively construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that
$y_{2n+1} = P^2x_{2n} = (AB)^2x_{2n+1}$, $y_{2n+2} = Q^2x_{2n+1} = (ST)^2x_{2n+2}$, for n = 0,1,2,...

Now, using condition (3.1.4) with $x = x_{2n}$, $y = x_{2n+1}$, we get

$$F\{M(P^2x_{2n}, Q^2x_{2n+1}, kt), M((ST)^2x_{2n}, (AB)^2x_{2n+1}, t), M(P^2x_{2n}, (ST)^2x_{2n}, t), M(Q^2x_{2n+1}, (AB)^2x_{2n+1}, kt), M((AB)^2x_{2n+1}, P^2x_{2n}, t)\} \geq 0.$$

That is,

$$F\{M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, kt), M(y_{2n+1}, y_{2n+1}, t)\} \geq 0.$$

Using condition (2.1) (a), we have

$$M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n+1}, y_{2n}, t)$$

Thus for any n and t, we have

$$M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t)$$

We shall prove that $\{y_n\}$ is a Cauchy sequence.

$$M(y_{n+1}, y_n, t) \geq M(y_n, y_{n-1}, t/k) \geq M(y_{n-1}, y_{n-2}, t/k^2) \geq \ldots\ldots\geq, M(y_1, y_0, t/k^n) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Thus the result holds for m = 1.

By induction hypothesis suppose that the result holds for m = r.

Now,

$$M(y_n, y_{n+r+1}, t) \geq M(y_n, y_{n+r}, t/2) * M(y_{n+1}, y_{n+r+1}, t/2) \rightarrow 1*1 = 1$$

Thus the result holds for m = r + 1

Hence $\{y_n\}$ is a Cauchy sequence in X which is complete. Therefore $\{y_n\}$ converges to $z \in X$.

Hence its subsequences $\{P^2x_{2n}\}$, $\{(AB)^2x_{2n+1}\}$, $\{Q^2x_{2n+1}\}$ and $\{(ST)^2x_{2n+2}\}$ also Convergence to z

Case I: AB(X) is a subsequence of X.

In this case $z \in AB(X)$, hence there exist $u \in X$ such that $z = (AB)^2u$.

Step I: Put $x = x_{2n}$ and $y = u$ in (3.1.4), we get

$$F\{M(P^2x_{2n}, Q^2u, kt), M((ST)^2x_{2n}, (AB)^2u, t), M(P^2x_{2n}, (ST)^2x_{2n}, t), M(Q^2u, (AB)^2u, kt), M((AB)^2u, P^2x_{2n}, t)\} \geq 0.$$

Taking limit $n \to \infty$, we get

$$F\{M(z, Q^2u, kt), M(z, z, t), M(z, z, t), M(Q^2u, z, kt), M(z, z, t)\} \geq 0$$

That is,

$$F\{M(z, Q^2u, kt), 1, 1, M(Q^2u, z, kt), 1\} \geq 0$$

$M(z, Q^2u, kt) \geq 1$ by (2.1) (b)

Hence $z = Q^2u$ Therefore

(3.1.6) $z = Q^2u = (AB)^2u$

Now, (Q, AB) is weak** commutative, therefore

(3.1.7) $M((AB)^2 Q^2u, Q^2(AB)^2u, t) \geq M((AB)^2Qu, Q^2(AB)u, t) \geq M((AB)Q^2u, Q(AB)^2u, t) \geq M((AB)Qu, Q(AB)u, t) \geq M((ST)^2u, Q^2u, t)$

Hence by (3.1.6), $(AB)^2Q^2u = Q^2(AB)^2u$

(3.1.8) Therefore $(AB)^2z = Q^2z$ (8)

StepII: Put $x = x_{2n}$ and $y = z$ in (3.1.4), we get from (3.1.8)

$$F\{M(P^2x_{2n}, Q^2z, kt), M((ST)^2x_{2n}, (AB)^2z, t), M(P^2x_{2n}, (ST)^2x_{2n}, t), M(Q^2z, (AB)^2z, kt), M((AB)^2z, P^2x_{2n}, t)\} \geq 0$$

Taking lim $n \to \infty$

$$F\{M(z, Q^2z, kt), M(z, Q^2z, t), M(z, z, t), M(Q^2z, Q^2z, kt), M(Q^2z, z, t)\} \geq 0$$

That is

$$F\{M(z, Q^2z, kt), M(z, Q^2z, t), 1, 1, M(Q^2z, (AB)^2 z, t)\} \geq 0$$

As F is non decreasing in the first argument, we have

$$F\{M(z, Q^2z, t), M(z, Q^2z, t), 1, 1, M(z, Q^2z, t)\} \geq 0$$

That is

$M(z, Q^2z, t) \geq 1$ by (2.1) (b)

$z = Q^2z$

(3.1.9) Hence $z = Q^2z = (AB)^2z$

Step III: As $Q(X) \in ST(X)$, there exist $v \in X$ such that $z = Q^2z = (ST)^2v$.

Put $x = v$, $y = z$ in (3.1.4), we have from (3.1.9)

$$F\{M(P^2v, Q^2z, kt), M((ST)^2v, (AB)^2z, t), M(P^2v, (ST)^2v, t), M(Q^2z, (AB)^2z, kt), M((AB)^2z, P^2v, t)\} \geq 0$$

That is

$$F\{M(P^2v, z, kt), 1, M(P^2v, z, t), 1, M(z, P^2v, t)\} \geq 0$$

As F is non decreasing in the first argument, we have

$$F\{M(P^2v, z, t), 1, M(P^2v, z, t), 1, M(P^2v, z, t)\} \geq 0$$

That is,

$M(P^2v, z, t) \geq 1$ by (2.1) (b)

$z = P^2v$

(3.1.10) Therefore $z = P^2v = (AB)^2 v$ (P, ST) is weak** commutative,therefore

$$M(P^2(ST)^2v, (ST)^2P^2v, kt) \geq M(P^2(ST)v, (ST)^2Pv, t) \geq M(P(ST)^2v, (ST)P^2v, t) \geq M(P(ST)v, (ST)Pv, t) \geq M(P^2v, (ST)^2v, t)$$

Hence by (3.1.10), $P^2(ST)^2v = (ST)^2P^2v$

Therefore $P^2z = (ST)^2z$

Combining all the results, we have

$$(ST)^2z = P^2z = Q^2z$$

Put $x = Pz$ and $y = z$ in (3.1.4),

$$F\{M(P^2Pz, Q^2z, kt), M((ST)^2Pz, (AB)^2z, t), M(P^2Pz, (ST)^2Pz, t), M(Q^2z, (AB)^2z, kt), M((AB)^2z, P^2Pz, t)\} \geq 0$$

As (P, ST) is weak ** commutative, therefore

$$P^2(ST)z = (ST)P^2z \text{ and } (ST)^2Pz = P(ST)^2z$$

Hence

$$F\{M(Pz, z, kt), M(Pz, z, t), M(Pz, Pz, t), M(z, z, kt), M(z, Pz, t)\} \geq 0$$

As F is non decreasing in the first argument, we have

$F\{M(Pz, z, t), M(Pz, z, t), 1, 1, M(Pz, z, t) \geq 0$

$M(Pz, z, t) \geq 1$ by (2.1) (b)

$Pz = z.$

Similarly we can show that $Qz = z$, $STz = z$ and $ABz = z$

Hence $z = Pz = Qz = STz = ABz$

Thus z is the common fixed point of P, Q, AB & ST

Uniqueness: Let w and z be two common fixed points of maps P, Q, ST and AB.

Put x = z and y = w in (3.1.4), we get

$F\{M(Pz, Q^2w, kt), M((ST)^2z, (AB)^2w, t), M(P^2z, (ST)^2z, t), M(Q^2w, (AB)^2w, kt), M((AB)^2w, P^2z, t)\} \geq 0$

$F\{M(z, w, kt), M(z, w, t), M(z, z, t), M(w, w, kt), M(w, z, t)\} \geq 0$

As F is non decreasing in the first argument, we have

$F\{M(z, w, t), M(z, w, t), 1, 1, M(z, w, t)\} \geq 0$

$M(z, w, t) \geq 1$ by (2.1) (b)

That is z = w

Thus z is the unique common fixed point of P, Q, AB and ST.

Now, we show that z=Tz by putting x=Tz & $y=x_{2n+1}$ in (3.1.4) and using the commutivity of the pairs (T, P) & (S, T)

$F\{M(P^2Tz, Q^2x_{2n+1}, kt),M((ST)^2Tz, (AB)^2x_{2n+1}, t), M(P^2Tz, (ST)^2Tz, t),(Q^2x_{2n+1}, (AB)^2x_{2n+1}, kt), M((AB)^2 x_{2n+1}, P^2Tz, t)\} \geq 0$

Letting $n \to \infty$, we get

$F\{M(Tz, z, kt), M(Tz, z, t), M(Tz, Tz, t), M(z, z, kt), M(z, Tz, t)\} \geq 0$

$F\{M(Tz, z, kt), M(Tz, z, t), 1, 1, M(Tz, z, t)\} \geq 0$

As F non decreasing in the first argument, we have

$F\{M(Tz, z, t), M(Tz, z, t), 1, 1, M(Tz, z, t)\} \geq 0$

$F\{M(Tz, z, t)\} \geq 1$ by (2.1) (b)

Therefore Tz=z.

Similarly, we can show that z=Bz by putting $x=x_{2n}$ & y=Bz in (3.1.4) and using the Commutativity of the pairs (A, B) and (Q, B)

$F\{M(P^2x_{2n},Q^2(Bz),kt),M((ST)^2x_{2n}, (AB)(Bz), t), M(P^2x_{2n}, (ST)^2x_{2n}, t),M(Q^2(Bz) ,(AB)^2(Bz), kt), M((AB)^2(Bz),P^2x_{2n}, t)\} \geq 0$

Let $n \to \infty$ we get,

$F\{M(z, Bz, kt), M(z, Bz, t), M(z, z, t), M(Bz, Bz, kt), M(Bz, z, t)\} \geq 0$

$F\{M(z, Bz, kt), M(z, Bz, t), 1,1, M(Bz, z, t)\} \geq 0$

As F non decreasing in the first argument, we have

$F\{M(z, Bz, t), M(z, Bz, t), 1, 1, M(z, Bz, t)\} \geq 0$

$M(z, Bz, t) \geq 1$ by (2.1) (b)

ABz = z, z=Bz implies A (Bz) = z which gives Az = z STz = z implies S (Tz) = z which gives Sz = z.

Hence z =Az =Bz = Sz = Tz =Pz = Qz is a Unique common fixed point

If we take B = T=I in theorem 3.1, we get following result.

Corollary 3.2: Let P, Q, A and S be self mappings of a complete fuzzy metric space (X, M,*) satisfying:

(3.2.1) $P(X) \subset A(X)$, $Q(X) \subset S(X)$

(3.2.2) The pairs (P, A),(Q,S) are weak** commutative,

(3.2.3) One of P(X),Q(X),A(X),B(X) complete subsequence of X

(3.2.4) for some $F \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$F\{M(P^2x, Q^2y, kt),M(S^2x,A^2y, t),M(P^2x, S^2x, t),M(Q^2y,A^2y, kt),M(A^2y, P^2x, t)\} \geq 0$

Then P, Q, S, and A have a unique common fixed point in X.

If we take S=T=A=B=I in theorem 3.1 the conditions (3.1.1), (3.1.2) and (3.1.3) are satisfied trivially and we get the following corollary.

Corollary 3.3: Let A and B be self mappings of a complete fuzzy metric space (X, M, *) satisfying:
(3.3.1) for some $F \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$F\{M(P^2x, Q^2y, kt), M(x, y, t), M(P^2x, x, t), M(Q^2y, y, kt), M(y, P^2x, t)\} \geq 0$$

Then P and Q have a unique common fixed point in X.

If we take P = I=A=B=T, the identity map on X then we have the following result for two self maps.

Corollary 3.4: Let Q and S be self mappings of a complete fuzzy metric space (X, M, *) satisfying:
(3.4.1) $Q(X) \subset S(X)$,
(3.4.2) the pair (Q, S) is weak ** commutative,
(3.4.3) S(X) is complete sub space of X,
(3.4.4) For some $F \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$F\{M(x, Qy, kt), M(Sx, Sy, t), M(x, Sx, t), M(Qy, Sy, kt), M(Sy, x, t)\} \geq 0$$

Then Q and S have a unique common fixed point is X

Theorem 3.5: Let U, V, W, N, L, M be self mapping of a complete fuzzy metric space (X, M,*) satisfying:
(3.5.1) $UV(X) \subset L(X)$, $WN(X) \subset M(X)$,
(3.5.2) the pairs (UV, M) and (WN, L) are weak** Commutative,
(3.5.3) one of L(X), M(X) is a complete sub space of X,
(3.5.4) for some $F \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$F\{M(U^2V^2x, W^2N^2y, kt), M(M^2x, L^2y, t), M(U^2V^2x, M^2x, t), M(W^2N^2x, L^2y, kt), M(L^2y, U^2V^2x, t)\} \geq 0$$

Then UV,L,WN and M have a Unique Common Fixed point more over (U,V),(W,N),(L,N),(M,V) are commuting mappings then U,V,W,L,M,N have a Unique fixed point in X.

CONCLUSION

In this paper we proved Common fixed point theorem in fuzzy metric space using weak** commuting property for six self maps satisfying an implicit relation.

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