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# COMPLEMENTARY TREE DOMINATION <br> IN BOOLEAN FUNCTION GRAPH B(K ${ }_{p}$, INC, $\left.\bar{K}_{q}\right)$ OF A GRAPH 

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#### Abstract

For any graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$ respectively. The Boolean function graph $B\left(K_{p}, I N C, \bar{K}_{q}\right)$ of $G$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B\left(K_{p}\right.$, INC, $\left.\bar{K}_{q}\right)$ are adjacent if and only if they correspond to two adjacent vertices of $G$, two nonadjacent vertices of $G$ or to $a$ vertex and an edge incident to it in $G$. For brevity, this graph is denoted by $B_{4}(G)$. In this paper, bounds of complementary tree domination number of Boolean function graph $B_{4}(G)$ are obtained and this number is found for Boolean function graphs of particular graphs. Also a characterization of graphs for which tree domination number is equal to 2 is obtained.


Key Words: Boolean Function Graph, Complementary tree dominating set, tree dominating set.

## 1. INTRODUCTION

Graphs discussed in this paper are undirected and simple graphs. For a graph G , let $\mathrm{V}(\mathrm{G})$ and $\mathrm{E}(\mathrm{G})$ denote its vertex set and edge set respectively. The graph $G$ with $p$ vertices and $q$ edges is denoted by $G(p, q)$. The points and edges of a graph are called its elements. Two elements of a graph are neighbors, if they are either incident or adjacent. For a connected graph $G$, the eccentricity $e(v)=\{d(u, v): u \in V(G)\}$, where $d(u, v)$ is the distance between $u$ and $v$ in $G$. The radius of $G$ is $\operatorname{rad}(G)=\min \{e(v) u \in V(G)\}$. A vertex $v$ is a central vertex if $e(v)=\operatorname{rad}(G)$. A Bistar whose central vertices have degree $m$ and $n$ is denoted by $S_{m, n}$.

The concept of domination in graphs was introduced by Ore [11]. A set $\mathrm{D} \subseteq \mathrm{V}(\mathrm{G})$ is said to be a dominating set of G , if every vertex in $V(G)$ - $D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set. We call a set of vertices a $\gamma$ - set, if it is a dominating set with cardinality $\gamma(\mathrm{G})$. Many domination parameters are obtained by combining domination with another graph theoretical property. Some domination parameters are defined by imposing additional constraint on the complement of a dominating set. Such parameters are called codomination parameters. Based on these, the concepts of split and nonsplit domination in graphs were introduced by Kulli and Janakiram [8, 9]. Chen et.al. [2] defined a tree dominating set D to be a set D whose induced subgraph $<\mathrm{D}>$ is a tree. The minimum cardinality of a tree dominating set of G is the tree domination number $\gamma_{\mathrm{tr}}(\mathrm{G})$. If there is no tree dominating set in $G$, then let $\gamma_{\mathrm{tr}}(\mathrm{G})=0$. A dominating set $\mathrm{D} \subseteq \mathrm{V}(\mathrm{G})$ is said to be tree dominating set if the induced subgraph $<\mathrm{D}>$ is a tree. Muthammai, Bhanumathi and Vidhya [10] introduced the concept of complement tree dominating set. A dominating set $\mathrm{D} \subseteq \mathrm{V}(\mathrm{G})$ is said to be complementary tree dominating set (ctd-set) if the induced subgraph $<\mathrm{V}(\mathrm{G})-\mathrm{D}>$ is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{\mathrm{ctd}}(\mathrm{G})$.

Whitney [12] introduced the concept of the line graph $L(G)$ of a given graph $G$ in 1932. The concept of total graphs was introduced by Behzad [1] in 1966. Janakiraman et al. introduced the concepts of Boolean and Boolean function graphs [4-7].

The Boolean function graph $B\left(K_{p}\right.$, NINC, $\left.\bar{K}_{q}\right)$ ) of $G$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B\left(K_{p}\right.$, NINC, $\left.\bar{K}_{q}\right)$ ) are adjacent if and only if they correspond to two adjacent vertices of $G$, two nonadjacent vertices of $G$ to a vertex and an edge incident to it in $G$. For brevity, this graph is denoted by $B_{4}(G)$. In this paper, bounds of complementary tree domination number of Boolean function graph $B_{4}(G)$ are obtained and this number is found for Boolean function graphs of particular graphs. Also a characterization of graphs for which tree domination number is equal to 2 is obtained. For graph theoretic terminology, Harary [3] is referred.

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## 2. PREVIOUS RESULTS

## Observation 2.1: [6]

1. $\quad K_{p}$ is an induced subgraph of $B_{4}(G)$ and the subgraph of $B_{4}(G)$ induced by $q$ vertices is totally disconnected.
2. Number of vertices in $B_{4}(G)$ is $p+q$, since $B_{4}(G)$ contains vertices of both $G$ and the line graph $L(G)$ of $G$.
3. Number of edges in $B_{4}(G)$ is $(p(p-1)) / 2+2 q$
4. For every vertex $v \in V(G), d_{B 4(G)}(v)=p-1+d_{G}(v)$
(a) If $G$ is complete, then $d_{B 4(G)}(v)=2(p-1)$
(b) If G is totally disconnected, then $\mathrm{d}_{\mathrm{B4}(\mathrm{G})}(\mathrm{v})=\mathrm{p}-1$
(c) If G has atleast one edge, then $2 \leq \mathrm{d}_{\mathrm{B} 4(\mathrm{G})}(\mathrm{v}) \leq 2(\mathrm{p}-1)$ and $\mathrm{d}_{\mathrm{B4}(\mathrm{G})}(\mathrm{v})=1$ if and only if $\mathrm{G} \cong 2 \mathrm{~K}_{1}$.
5. $\gamma\left(B_{4}(G)\right)=1$ if and only if $\mathrm{G} \cong \mathrm{K}_{1, \mathrm{n}} \cup \mathrm{mK}_{1}, \mathrm{n}, \mathrm{m} \geq 1$.
6. For an edge $\mathrm{e} \in \mathrm{E}(\mathrm{G}), \mathrm{d}_{\mathrm{B} 4(\mathrm{G})}(\mathrm{e})=2$
7. $\mathrm{B}_{4}(\mathrm{G})$ is always connected.

Theorem 2.1: [10] $\gamma_{c t d}(G)=1$ if and only if $G \cong T+K_{1}$, where $T$ is a tree.
Theorem 2.2: [10] For any connected graph $G$ with $p \geq 2, \gamma_{c t d}(G) \leq p-1$.
Theorem 2.3: [10] Let $G$ be a connected graph with $p \geq 2 . \gamma_{c t d}(G)=p-1$ if and only if $G$ is a star on $p$ vertices.
Theorem 2.4: [10] Let $G$ be a connected graph containing a cycle. Then $\gamma_{c t d}(G)=p-2$ if and only if $G$ is isomorphic to one of the following graphs $\mathrm{C}_{\mathrm{p}}, \mathrm{K}_{\mathrm{p}}$ or G is the graph obtained by attaching pendant edges at atleast one of the vertices of a complete graph.

Theorem 2.5: [10] Let $T$ be a tree with $p$ vertices which is not a star. Then $\gamma_{c t d}(T)=p-2$ if and only if $T$ is a path or $T$ is obtained by attaching pendant edges at atleast one of the end vertices.

## 3. MAIN RESULTS

In the following, an upper bound of $\gamma_{c t d}\left(B_{4}(G)\right)$ is found.
Theorem 3.1: For any graph $G$ with $p$ vertices, $\gamma_{c t d}\left(B_{4}(G)\right) \leq p+q-\Delta(G)-\delta(G)-2$.
Proof: Let $G$ be a graph with $p$ vertices. Let $u$ be a vertex of $G$ with $\operatorname{deg}(u)=\Delta(G)$ and let $v$ be a vertex of $G$ with $\operatorname{deg}(\mathrm{v})=\mathrm{t}$, where $\mathrm{t}=\operatorname{Max}\left\{\operatorname{deg}_{\mathrm{G}}(\mathrm{v}): \mathrm{v} \notin \mathrm{N}(\mathrm{u})\right\}$. Then $|\mathrm{N}(\mathrm{v})|=\mathrm{t}$.

If $\mathrm{D}=\mathrm{N}(\mathrm{u}) \cup \mathrm{N}(\mathrm{v}) \cup\{\mathrm{u}, \mathrm{v}\}$, then $\mathrm{V}\left(\mathrm{B}_{4}(\mathrm{G})\right)-\mathrm{D}$ is a complementary tree dominating set of $\mathrm{B}_{4}(\mathrm{G})$ and hence
$\gamma_{\mathrm{ctd}}\left(\mathrm{B}_{4}(\mathrm{G})\right) \leq\left|\mathrm{V}\left(\mathrm{B}_{4}(\mathrm{G})\right)-\mathrm{D}\right|=\mathrm{p}+\mathrm{q}-(\Delta(\mathrm{G})+\mathrm{t}+2)=\mathrm{p}+\mathrm{q}-\Delta(\mathrm{G})-\mathrm{t}-2$.

$$
\leq \mathrm{p}+\mathrm{q}-\Delta(\mathrm{G})-\delta(\mathrm{G})-2
$$

Equality holds, if $G \cong K_{m, m}$ and $C_{n}(m, n \geq 4)$.
Note 3.1: If $G$ contains atleast one edge and three vertices, then $\gamma_{\mathrm{ctd}}\left(B_{4}(G)\right) \geq 2$.
Theorem 3.2: Let $G$ be a connected graph with $p$ vertices and $\gamma_{c t d}(G)=1$. Then $\gamma_{c t d}\left(B_{4}(G)\right) \leq 2 p-5$.
Proof: Let $G$ be a connected graph with $p$ vertices and $\gamma_{c t d}(G)=1$. Then $G$ is isomorphic to $T+K_{1}$, where $T$ is a tree on $p-1$ vertices and hence $B_{4}(G)$ has $3(p-1)$ vertices. Let $v \in V\left(K_{1}\right)$ and $u$ be a vertex of $T$ with $\operatorname{deg}_{T}(u)=\Delta(T)$ and $e=(u, v)$ and let $E^{\prime}$ be the set of edges in $G$ incident with $u$, $v$ or both. If $D^{\prime}$ be the set of vertices in $B_{4}(G)$ corresponding to the edges in $\mathrm{E}^{\prime}$, let $\mathrm{D}=\mathrm{D}^{\prime} \cup\{\mathrm{u}, \mathrm{v}\}-\{\mathrm{e}\}$ and $\left|\mathrm{D}^{\prime}\right|=\operatorname{deg}_{\mathrm{G}}(\mathrm{v})+\operatorname{deg}_{\mathrm{T}}(\mathrm{u})+2-1=\mathrm{p}-1+\Delta(\mathrm{T})+1=\mathrm{p}+$ $\Delta(T)$ and $V\left(B_{4}(G)\right)-D$ is a complementary tree dominating set of $B_{4}(G)$ and hence $\gamma_{c t d}\left(B_{4}(G)\right) \leq\left|V\left(B_{4}(G)\right)-D^{\prime}\right|=3 p-$ $3-(p+\Delta(T)) \leq 2 p-5$, since $\Delta(T) \geq 2$.

Equality holds, if $G \cong P_{n}+K_{1}$ where $P_{n}$ is a path on $n$ vertices.
In the following complementary tree domination number of $B_{4}(G)$ is found when $G$ is a path, cycle, complete graph, complete bipartite graph, star and wheel.

Theorem 3.3: If $G$ is a Path $P_{n}$ on $n(n \geq 5)$ vertices, then $\gamma_{c t d}\left(B_{4}\left(P_{n}\right)\right)=2 n-7$.
Proof: Let $G \cong P_{n}, n \geq 5 . B_{4}\left(P_{n}\right)$ has $2 n-1$ vertices. Let $v_{i}$, $v_{j}$ be two distinct vertices of degree 2 in $P_{n}$ such that $d_{G}\left(v_{i}, v_{j}\right) \geq 2$ and let $e_{i 1}, e_{i 2}$ be the edges incident with $v_{i}$ and $e_{j 1}, e_{j 2}$ be the edges incident with $v_{j}$. Then $v_{i}, v_{j}, e_{i 1}, e_{i 2}, e_{j 1}, e_{j 2} \in V\left(B_{4}\left(P_{n}\right)\right)$. If $D=\left\{v_{i}, v_{j}, e_{i 1}, e_{i 2}, e_{j 1}, e_{j 2}\right\}$, then $\langle D\rangle \cong S_{2,2}$ and $D$ is a ctd - set of $B_{4}\left(P_{n}\right)$ and hence $\gamma_{c t d}\left(P_{n}\right) \leq\left|V\left(B_{4}\left(P_{n}\right)\right)-D\right|=2 n-1-6=2 n-7$. Let $D^{\prime}$ be a ctd-set of $B_{4}\left(P_{n}\right)$. Since $K_{n}$ is an induced subgraph of $B_{4}\left(P_{n}\right), V\left(B_{4}\left(P_{n}\right)\right)-D$ contains atmost two vertices of $P_{n}$. Also each vertex of $L(G)$ is adjacent to two vertices of $G$ in $B_{4}(G)$. Therefore any tree of $B_{4}(G)$ contains atmost 6 vertices and hence $D^{\prime}$ contains atleast $2 n-1-6=2 n-7$ vertices. Therefore $\left|D^{\prime}\right| \geq 2 n-7$. Hence $\gamma_{c t d}\left(B_{4}\left(P_{n}\right)\right)=2 n-7$.

Remark 3.1: If $C_{n}$ is a cycle on $n(n \geq 4)$ vertices, then $\gamma_{\text {ctd }}\left(B_{4}\left(C_{n}\right)\right)=2 n-6$ and $\gamma_{c t d}\left(B_{4}\left(C_{3}\right)\right)=2$.
Theorem 3.4: If $K_{n}$ is a complete graph on $n$ vertices, then $\gamma_{c t d}\left(B_{4}\left(K_{n}\right)\right)=\left(n^{2}-3 n+4\right) / 2$, where $n \geq 4$.
Proof: $B_{4}\left(K_{n}\right)$ has $n+(n(n-1)) / 2$ vertices. Let $v_{1}, v_{2}, \ldots, v_{n}$ be the vertices of $K_{n}, e_{12}, e_{13}, \ldots, e_{1 n}$ be the edges incident with $v_{1}$ and let $e_{1 n}, e_{2 n}, \ldots, e_{n-1, n}$ be the edges incident with $v_{n}$.

Then $D=\left\{v_{1}, v_{n}, e_{12}, e_{13}, \ldots, e_{1 n-1}, e_{2 n}, \ldots, e_{n-1, n}\right\} \subseteq V\left(B_{4}\left(K_{n}\right)\right),\langle D\rangle \cong S_{n-2, n-2}$, and $|D|=2 n-2$, where $n \geq 4$. $V\left(B_{4}\left(K_{n}\right)\right)-D$ is a ctd-set of $\left.B_{4}\left(K_{n}\right)\right)$ and hence $\gamma_{c t d}\left(B_{4}\left(K_{n}\right)\right) \leq n+((n(n-1)) / 2)-(2 n-2)=\left(n^{2}-3 n+4\right) / 2$. Since $K_{n}$ is an induced subgraph of $B_{4}\left(K_{n}\right)$, any tree of $B_{4}\left(K_{n}\right)$ has atmost $2 n-2$ vertices. Therefore any ctd-set of $B_{4}\left(K_{n}\right)$ contains atleast $\left(n^{2}-3 n+4\right) / 2$ vertices. Hence $\gamma_{c t a}\left(B_{4}\left(K_{n}\right)\right) \geq\left(n^{2}-3 n+4\right) / 2$.

Theorem 3.5: If $K_{m, n}(m \geq n)$ is the complete bipartite graph, then $\gamma_{c t d}\left(B_{4}\left(K_{m, n}\right)\right)=m n-2, m, n \geq 2$.
Proof: Let $[A, B]$ be the bipartition of $K_{m, n}$ such that $|A|=m$ and $|B|=n$. Let $u, v \in B$. Then $\operatorname{deg}(u)=\operatorname{deg}(v)=m$. If $e_{1}, e_{2}, \ldots, e_{m}$ be the edges incident with $u$ and $f_{1}, f_{2}, \ldots, f_{m}$ be those edges incident with $v$, then $u, v, e_{1}, e_{2}, \ldots, e_{m}, f_{1}, f_{2}$, $\ldots, f_{m} \in V\left(B_{4}\left(K_{m, n}\right)\right)$.

Let $D=\left\{u, v, e_{1}, e_{2}, \ldots, e_{m}, f_{1}, f_{2}, \ldots, f_{m}\right\}$. Then $\langle D\rangle \cong S_{m, m}$ and $V\left(B_{4}\left(K_{m, n}\right)\right)-D$ is a minimum dominating set of $B_{4}\left(K_{m, n}\right)$. Therefore, $\gamma_{c t d}\left(B_{4}\left(K_{m, n}\right)\right)=\left|V\left(B_{4}\left(K_{m, n}\right)\right)-D\right|=m+n+m n-(m+n+2)=m n-2$.

Remark 3.2: If $G$ is a star on $n+1$ vertices, then $\gamma_{c t d}\left(B_{4}\left(K_{1, n}\right)\right)=n-2, n \geq 2$.
Theorem 3.6: If $W_{p}$ is the wheel on $p$ vertices, then $\gamma_{c t d}\left(B_{4}\left(W_{p}\right)\right)=3 p-9$, where $p \geq 5$.
Proof: Let $v, v_{1}, v_{2}, \ldots, v_{p-1}$ be vertices of $W_{p}$ and let $e_{i}=\left(v, v_{i}\right), i=1,2, \ldots, p-1$, $e_{i, i+1}=\left(v_{i}, v_{i+1}\right), i=1,2, \ldots, p-2$ and $e_{p-1,1}=\left(v_{p-1}, v_{1}\right) .\left|V\left(B_{4}\left(W_{p}\right)\right)\right|=3 p-1$.

Then $\mathrm{v}, \mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{p}-1}, \mathrm{e}_{\mathrm{i}, \mathrm{i}+1}, \mathrm{e}_{\mathrm{p}-1,1} \in \mathrm{~V}\left(\mathrm{~B}_{4}\left(\mathrm{~W}_{\mathrm{p}}\right)\right)$. Let vi, vj be two nonadjacent vertices in $\mathrm{W}_{\mathrm{p}}$. and let $\mathrm{e}_{\mathrm{i} 1}, \mathrm{e}_{\mathrm{i} 2}, \mathrm{e}_{\mathrm{i} 3}$ be the edges incident with $v_{i}$ and $e_{j 1}, e_{j 2}, e_{j 3}$ be the edges incident with $v_{j}$. Then $e_{i 1}, e_{i 2}, e_{i 3}, e_{j 1}, e_{j 2}, e_{j 3} \in V\left(B_{4}\left(W_{p}\right)\right)$. Let $D=\left\{e_{i 1}, e_{i 2}, e_{i 3}, e_{j 1}, e_{j 2}, e_{j 3}, v_{i}, v_{j}\right\} \subseteq V\left(B_{4}\left(W_{p}\right)\right)$. Then each vertex in $D$ is adjacent to atleast one vertex in $\mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{~W}_{\mathrm{p}}\right)\right)-\mathrm{D}$ and $<\mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{~W}_{\mathrm{p}}\right)\right)-\mathrm{D}>\cong \mathrm{S}_{3,3}$.

Therefore $V\left(B_{4}\left(W_{p}\right)\right)-D$ is a minimum ctd-set of $B_{4}\left(W_{p}\right)$ and hence $\gamma_{c t d}\left(B_{4}\left(W_{p}\right)\right)=\left|V\left(B_{4}\left(W_{p}\right)\right)-D\right|=3 p-1-8=3 p-9$.
Theorem 3.7: If $G$ is a graph obtained from $K_{1}+T$ with one pendant edge attached at the vertex of $K_{1}$, where $T$ is any tree on $p-2$ vertices, then $\gamma_{\text {ctd }}\left(B_{4}(G)\right) \leq 2 p-\Delta(T)-4$.

Proof: If $G$ is a graph as stated in the Theorem, then $\gamma_{c t d}(G)=2$. Number of vertices in $G$ is $p$ and the number of edges in $G$ is $|E(T)|+p-2+1=p-3+p-1=2 p-4$. Hence number of vertices in $B_{4}(G)$ is $p+2 p-4=3 p-4$. Let $\mathrm{V}\left(\mathrm{K}_{1}\right)=\{\mathrm{v}\}$ and $u$ be the pendant vertex adjacent to v in $\mathrm{G}, \mathrm{e}=(\mathrm{u}, \mathrm{v})$ and let $\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{p}-2}$ be the vertices of T with $\operatorname{deg}_{T}\left(v_{i}\right)=\Delta(T)$. Let $e_{i}=\left(v, v_{i}\right), i=1,2, \ldots, p-2$. Then $v_{1}, v_{2}, \ldots, v_{p-2}, e_{1}, e_{2}, \ldots, e_{p-2}, u, v, e \in V\left(B_{4}(G)\right)$. If $D=\left\{v, v_{i}, e, e_{1}, e_{2}, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{p-2}\right\} \cup N_{T}\left(v_{i}\right)\left(\subseteq V\left(B_{4}(G)\right)\right)$, then the subgraph of $B_{4}(G)$ induced by $D$ is isomorphic to $S_{p-2, \Delta(T)}$ and $|D|=p+\Delta(T)$. Since the subgraph of $B_{4}(G)$ induced by vertices of $G$ is complete, $v, v_{i}$ are adjacent atleast one vertex of $G$ in $V\left(B_{4}(G)\right)$ - D. Also e, $e_{1}, e_{2}, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{p-2}$ are adjacent to $u, v_{1}, v_{2}, \ldots, v_{i-1}$, $v_{i+1}, \ldots, v_{p-2}$ respectively in $V\left(B_{4}(G)\right)-D$. Therefore, $V\left(B_{4}(G)\right)-D$ is a dominating set of $B_{4}(G)$ and since $\mathrm{D} \cong S_{\mathrm{p}-2, \Delta(\mathrm{~T})}, \mathrm{V}\left(\mathrm{B}_{4}(\mathrm{G})\right)-\mathrm{D}$ is a ctd-set of $\mathrm{B} 4(\mathrm{G})$. Hence $\gamma_{\mathrm{ctd}}\left(\mathrm{B}_{4}(\mathrm{G})\right) \leq\left|\mathrm{V}\left(\mathrm{B}_{4}(\mathrm{G})\right)-\mathrm{D}\right|=3 \mathrm{p}-4-(\mathrm{p}+\Delta(\mathrm{T}))=2 \mathrm{p}-\Delta(\mathrm{T})$ -4 . Equality holds, if T is a path on $\mathrm{p}-2$ vertices.

Remark 3.3: If $T$ is a star, then $\Delta(T)=p-3$ and hence $\gamma_{c t d}\left(B_{4}(G)\right) \leq 2 p-(p-3)-4=p+7$.
Theorem 3.8: If $G$ is a graph obtained from a tree by joining each of the vertices of the tree to the vertices of $K_{2}$ such that $\operatorname{deg}(\mathrm{v}) \geq 2$, for all $\mathrm{v} \in \mathrm{V}\left(\mathrm{K}_{2}\right)$, then $\gamma_{\mathrm{ctd}}\left(\mathrm{B}_{4}(\mathrm{G})\right) \leq 2 \mathrm{p}-4$.

Proof: Let $V\left(K_{2}\right)=\{u, v\}$ and $\operatorname{deg}_{G}(u)=m$ and $\operatorname{deg}_{G}(v)=n$, where $m, n \geq 2$ and $m+n \leq p$ and let $T$ be a tree on $p-2$ vertices. Then $|\mathrm{E}(\mathrm{G})|=|\mathrm{E}(\mathrm{T})|+\mathrm{m}-1+\mathrm{n}-1+1=\mathrm{p}-3+\mathrm{m}+\mathrm{n}-1=\mathrm{p}+\mathrm{m}+\mathrm{n}-4$. Therefore $\mid \mathrm{V}\left(\mathrm{B}_{4}(\mathrm{G})|=|\mathrm{V}(\mathrm{G})|+\right.$ $|E(G)|=2 p+m+n-4$. Let $u_{1}, u_{2}, \ldots, u_{m-1}$ be the vertices of $T$ adjacent to $u$ and let $v_{1}, v_{2}, \ldots, v_{n-1}$ be the vertices of $T$ adjacent to v in G .

If $e_{i}=\left(u, u_{i}\right)$ and $f_{j}=\left(v, v_{j}\right)(i=1,2, \ldots, m-1, j=1,2, \ldots, n-1)$, then $D=\left\{u, v, e_{1}, e_{2}, \ldots, e_{m-1}, f_{1}, f_{2}, \ldots, f_{n-1}\right\} \subseteq V\left(B_{4}(G)\right)$ and $|\mathrm{D}|=\mathrm{m}+\mathrm{n}$. Also $<\mathrm{D}>\cong \mathrm{S}_{\mathrm{m}-1, \mathrm{n}-1}$ in $\mathrm{B}_{4}(\mathrm{G})$. Each vertex in D is adjacent to atleast one vertex in $\mathrm{V}\left(\mathrm{B}_{4}(\mathrm{G})\right)$ - D and $<D>$ is a tree in $B_{4}(G)$. Therefore $V\left(B_{4}(G)\right)-D$ is a ctd-set of $B_{4}(G)$ and hence $\gamma_{\text {ctd }}\left(B_{4}(G)\right) \leq\left|V\left(B_{4}(G)\right)-D\right|=2 p-4$.

Remark 3.4: If each vertex of the tree is adjacent to both $u$ and $v$ in $G$, then $\gamma_{c t d}\left(B_{4}(G)\right)=2 p-4$.
In the similar lines, the following theorem can be proved
Theorem 3.9: If $G$ is a graph obtained from a tree by joining each of the vertices of the tree to the vertices of $2 \mathrm{~K}_{1}$ such that $\operatorname{deg}(v) \geq 1$, for all $v \in V\left(2 K_{1}\right)$, then $\gamma_{c t d}\left(B_{4}(G)\right) \leq 2 p-5$.

Theorem 3.10: For $n \geq 3, \gamma_{c t d}\left(B_{4}\left(P_{n}+K_{1}\right)\right)=2 n-3$.
Proof: Let $v_{1}, v_{2}, \ldots, v_{n} \in V\left(C_{n}\right), v \in V\left(K_{1}\right)$ and let $e_{i}=\left(v, v_{i}\right), i=1,2, \ldots, n ; e_{i, i+1}=\left(v_{i}, v_{i+1}\right), i=1,2, \ldots, n-1$. $\left.\left.\left|V\left(B_{4}\left(P_{n}+K_{1}\right)\right)\right|=\left|V\left(P_{n}\right)\right|+\mid V\left(K_{1}\right)\right)+\mid E\left(P_{n}\right)\right) \mid+n=n+1+n-1+n=3 n$. Let $D=\left\{v, v_{n-1}, e_{1}, e_{2}, \ldots, e_{n-2}, e_{n}, e_{n-2, n-1}\right.$, $\left.\mathrm{e}_{\mathrm{n}-1, \mathrm{n}}\right\}$. Then D is a subset of $\mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{P}_{\mathrm{n}}+\mathrm{K}_{1}\right)\right.$ and $|\mathrm{D}|=\mathrm{n}+3$. Each vertex in D is adjacent to atleast one vertex in $V\left(B_{4}\left(P_{n}+K_{1}\right)\right)-D$. Also $<D>\cong S_{n-1,2}$. Therefore $V\left(B_{4}\left(P_{n}+K_{1}\right)\right)-D$ is a minimum ctd-set of $B_{4}\left(P_{n}+K_{1}\right)$ and hence $\gamma_{c t d}\left(B_{4}\left(P_{n}+K_{1}\right)\right)=\left|V\left(B_{4}\left(P_{n}+K_{1}\right)\right)-D\right|=3 n-(n+3)=2 n-3$.

Theorem 3.11: For $n \geq 5, \gamma_{c t d}\left(B_{4}\left(C_{n}+K_{1}\right)\right)=2 n-2$.
Proof: Let $v_{1}, v_{2}, \ldots, v_{n} \in V\left(C_{n}\right), v \in V\left(K_{1}\right)$ and let $e_{i}=\left(v, v_{i}\right), i=1,2, \ldots, n ; e_{i, i+1}=\left(v_{i}, v_{i+1}\right), i=1,2, \ldots, n-1 ; e_{n},{ }_{1}=$ $\left.\left.\left(\mathrm{v}_{\mathrm{n}}, \mathrm{v}_{1}\right) .\left|\mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{C}_{\mathrm{n}}+\mathrm{K}_{1}\right)\right)\right|=\left|\mathrm{V}\left(\mathrm{C}_{\mathrm{n}}\right)\right|+\mid \mathrm{V}\left(\mathrm{K}_{1}\right)\right)+\mid \mathrm{E}\left(\mathrm{C}_{\mathrm{n}}\right)\right) \mid+\mathrm{n}=3 \mathrm{n}+1$.

Let $D=\left\{v, v_{n}, e_{1}, e_{2}, \ldots, e_{n-1}, e_{n-1, n}, e_{n 1}\right\}$. Then $D \subseteq V\left(B_{4}\left(C_{n}+K_{1}\right)\right.$, and $|D|=n+3 . v, v_{n}$ are adjacent to $v_{i}, i=2,3, \ldots$, $n-1$ and $e_{i}$ is adjacent to $v_{i}, i=1,2, \ldots, n-1, e_{n-1, n}, e_{n 1}$ are adjacent to $v_{n}$ in $V\left(B_{4}\left(C_{n}+K_{1}\right)\right)-D$. Therefore $V\left(B_{4}\left(C_{n}+K_{1}\right)\right)-D$ is a dominating set of $V\left(B_{4}\left(C_{n}+K_{1}\right)\right)$ and since $\left\langle D>\cong S_{n-1,2}, V\left(B_{4}\left(C_{n}+K_{1}\right)\right)-D\right.$ is a ctd-set of $B_{4}\left(C_{n}+K_{1}\right)$. Also there is no other ctd-set containing less than $\left|V\left(B_{4}\left(C_{n}+K_{1}\right)\right)-D\right|$ vertices. Therefore $\gamma_{\mathrm{ctd}}\left(\mathrm{B}_{4}\left(\mathrm{C}_{\mathrm{n}}+\mathrm{K}_{1}\right)\right) \leq\left|\mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{C}_{\mathrm{n}}+\mathrm{K}_{1}\right)\right)-\mathrm{D}\right|=3 \mathrm{n}+1-(\mathrm{n}+3)=2 \mathrm{n}-2$, where $\mathrm{n} \geq 5$.

Remark 3.5: $\gamma_{c t d}\left(B_{4}\left(\mathrm{C}_{4}+\mathrm{K}_{1}\right)\right)=5$.
Theorem 3.12: For $m, n \geq 2, \gamma_{c t d}\left(B_{4}\left(K_{m, n}+K_{1}\right)\right)=n(m+1)$
Proof: Let [A, B] be the bipartition of $K_{m, n}$. Assume $A=\left\{v_{1}, v_{2}, \ldots, v_{m}\right\}, B=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $V\left(K_{1}\right)=\{v\}$. Let $e_{i j}=\left(u_{i}, v_{j}\right), e_{i}=\left(v, v_{i}\right)$ and $f_{j}=\left(v, u_{j}\right), i=1,2, \ldots, m$ and $j=1,2, \ldots, n .\left|V\left(B_{4}\left(K_{m, n}+K_{1}\right)\right)\right|=\left|V\left(K_{m, n}\right)\right|+\left|V\left(K_{1}\right)\right|+$ $\left|E\left(K_{m, n}\right)\right|+\left|\left\{e_{i}, i=1,2, \ldots, m\right\}\right|+\left|\left\{f_{j}, j=1,2, \ldots, n\right\}\right|=m+n+1+m n+m+n=m n+2 m+2 n+1$. If $D=\left\{v, u_{1}, e_{1}\right.$, $\left.e_{2}, \ldots, e_{m}, f_{2}, \ldots, f_{n}, e_{11}, e_{21}, \ldots, e_{m 1}\right\}$, then $D \subseteq V\left(B_{4}\left(K_{m, n}+K_{1}\right)\right),|D|=2 m+n+1$ and $<D>\cong S_{m+n-1, m}$. As in Theorem, $\mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{~K}_{\mathrm{m}, \mathrm{n}}+\mathrm{K}_{1}\right)\right)-\mathrm{D}$ is a minimum ctd - set of $\mathrm{B}_{4}\left(\mathrm{~K}_{\mathrm{m}, \mathrm{n}}+\mathrm{K}_{1}\right)$ and hence $\gamma_{\mathrm{ctd}}\left(\mathrm{B}_{4}\left(\mathrm{~K}_{\mathrm{m}, \mathrm{n}}+\mathrm{K}_{1}\right)\right)=\left|\mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{~K}_{\mathrm{m}, \mathrm{n}}+\mathrm{K}_{1}\right)\right)-\mathrm{D}\right|=$ $(m n+2 m+2 n+1)-(2 m+n+1)=n(m+1)$.

Theorem 3.13: For $m, n \geq 3, \gamma_{c t d}\left(B_{4}\left(K_{1, n}+K_{1}\right)\right)=n+3$.
Proof: Let $v_{1}, v_{2}, \ldots, v_{n}, v_{n+1} \in V\left(K_{1, n}\right)$, where $v_{1}$ is the central vertex and $V\left(K_{1}\right)=\{v\}$. Let $e_{i}=\left(v, v_{i}\right), i=1,2, \ldots, n+1$, $e_{1 j}=\left(v_{1}, v_{j}\right), j=2,3, \ldots, n+1$. Then $\left|V\left(B_{4}\left(K_{1, n}+K_{1}\right)\right)\right|=\left|V\left(K_{1, n}\right)\right|+\left|V\left(K_{1}\right)\right|+\left|\left\{e_{i}, i=1,2, \ldots, n+1\right\}\right|+\mid\left\{e_{1, j}, j=2,3, \ldots\right.$, $n+1\} \mid=n+1+1+n+1+n=3 n+3$.

If $D=\left\{v, v_{1}, e_{2}, e_{3}, \ldots, e_{n}, e_{12}, e_{13}, \ldots, e_{1 n}\right\} \subseteq V\left(B_{4}\left(K_{1, n}+K_{1}\right)\right)$, then $|D|=2 n$ and $<D>\cong S_{n-1, n-1}$. Also $V\left(B_{4}\left(K_{1, n}+K_{1}\right)\right.$ $-D$ is a minimum ctd-set of $B_{4}\left(K_{1, n}+K_{1}\right)$ and hence $\gamma_{c t d}\left(B_{4}\left(K_{1, n}+K_{1}\right)\right)=3 n+3-2 n=n+3$.

Theorem 3.14: For $p \geq 5, \gamma_{c t d}\left(B_{4}\left(W_{p}+K_{1}\right)\right)=2 p-1$.
Proof: Let $v_{1}, v_{2}, \ldots, v_{p}$ be vertices of $W_{p}$, where $v_{1}$ is the central vertex and $v \in V\left(K_{1}\right)$. Let the edges of $B_{4}\left(W_{p}+K_{1}\right)$ be denoted by $e_{i}=\left(v, v_{i}\right), i=1,2, \ldots, p ; e_{1 j}=\left(v_{1}, v_{j}\right), j=2,3, \ldots, p ;\left(e_{i}, e_{i+1}\right)=\left(v i, v_{i+1}\right), i=2,3, \ldots, p-1$ and $\mathrm{e}_{\mathrm{p} 2}=\left(\mathrm{v}_{\mathrm{p}}, \mathrm{v}_{2}\right) . \mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{~W}_{\mathrm{p}}+\mathrm{K}_{1}\right)\right)=\left(\mathrm{V}\left(\mathrm{W}_{\mathrm{p}}+\mathrm{K}_{1}\right)\right) \cup\left(\mathrm{E}\left(\mathrm{W}_{\mathrm{p}}+\mathrm{K}_{1}\right)\right)$ and $\left|\mathrm{V}\left(\mathrm{B}_{4}\left(\mathrm{~W}_{\mathrm{p}}+\mathrm{K}_{1}\right)\right)\right|=\left|\mathrm{V}\left(\mathrm{W}_{\mathrm{p}}+\mathrm{K}_{1}\right)\right|+\left|\mathrm{E}\left(\mathrm{W}_{\mathrm{p}}+\mathrm{K}_{1}\right)\right|=$ $(p+1)+p+(p-1)+(p-2)+1=4 p-1$. If $D=\{v, v 1, e 2, e 3, \ldots, e p, e 12, e 13, \ldots, e 1 p\} \subseteq V\left(B_{4}\left(W_{p}+K_{1}\right)\right)$, then $V\left(B_{4}\left(W_{p}+K_{1}\right)\right)-D$ is a dominating set of $W_{p}+K_{1}$. Also $<D>\cong S_{p-1, p-1}$. Therefore $V\left(B_{4}\left(W_{p}+K_{1}\right)\right)-D$ is ctd-set of $B_{4}\left(W_{p}+K_{1}\right)$ and hence $\gamma_{c t d}\left(B_{4}\left(W_{p}+K_{1}\right)\right) \leq\left|V\left(B_{4}\left(W_{p}+K_{1}\right)\right)-D\right|=4 p-1-2 p=2 p-1$. Also there exists no ctd-set in $B_{4}\left(W_{p}+K_{1}\right)$ having $2 p-1$ vertices. Hence $\gamma_{c t}\left(B_{4}\left(W_{p}+K_{1}\right)\right)=2 p-1$.

In the following, tree domination number of $B_{4}(G)$ is found.
Observation 3.1: $\gamma\left(B_{4}(G)\right)=\gamma_{t r}\left(B_{4}(G)\right)=1$ if and only if $G \cong K_{1, n} \cup \mathrm{mK}_{1}, n, m \geq 1$.
Theorem 3.15: Let $G$ be not a star and $\delta(G) \geq 1$. Then $\gamma_{\mathrm{tr}}\left(\mathrm{B}_{4}(\mathrm{G})\right)=2$ if and only if there exists a minimum point cover of $G$ containing two adjacent vertices.

Proof: Let $D$ be a minimum point cover of $G$ containing two adjacent vertices say, $u$, $v$. Then each vertex of $G$ in $B_{4}(G)$ is adjacent to both $u$ and $v$. Since $D$ is a point cover, each edge in $G$ is incident with atleast one of $u$ and $v$. Therefore vertices in $B_{4}(G)$ corresponding to the edges of $G$ are adjacent to atleast one of $u$ and $v$ and hence $D$ is a dominating set of $B_{4}(G)$. Also $<D>\cong K_{2}$. Therefore $D$ is a tree dominating set of $B_{4}(G)$. Hence $\gamma_{r r}\left(B_{4}(G)\right) \leq|D|=2$. Since $G$ is not a star, $\gamma_{\mathrm{tr}}\left(\mathrm{B}_{4}(\mathrm{G})\right) \geq 2$. Therefore $\gamma_{\mathrm{tr}}\left(\mathrm{B}_{4}(\mathrm{G})\right)=2$.

Conversely assume $\gamma_{t r}\left(B_{4}(G)\right)=2$. Then there exists a tree dominating set $D$ of $B_{4}(G)$ containing two vertices and $D$ contains atleast one vertex of $G$, since vertices of line graph $L(G)$ of $G$ in $B_{4}(G)$ are independent. Let $D=\{u, v\}$. Then $<\mathrm{D}>\cong \mathrm{K}_{2}$ in $\mathrm{B}_{4}(\mathrm{G})$.

Case 1: $u \in V(G)$ and $v \in V(L(G))$
Then $v \in E(G)$ and $u$ is incident $v$ in $G$. Since $D$ is a dominating set of $B_{4}(G)$, each edge in $G$ is incident with $u$. That is, $\alpha_{0}(\mathrm{G})=1$ and $\mathrm{G} \cong \mathrm{K}_{1, \mathrm{n}}, \mathrm{n} \geq 2$. But $\gamma_{\mathrm{tr}}\left(\mathrm{B}_{4}\left(\mathrm{~K}_{1, \mathrm{n}}\right)\right)=1$.

Case 2: $u, v \in V(G)$
Then $D$ is minimum point cover of $G$ containing two adjacent vertices and $\alpha_{0}(G)=2$.
Theorem 3.16: For any graph $G$ with $\delta(G) \geq 1, \gamma_{t r}\left(B_{4}(G)\right)=0$ if and only if either $\alpha_{0}(G) \geq 3$ or all the point covers of $G$ containing two vertices are independent sets of $G$.

Proof: Assume $\alpha_{0}(G) \geq 3$. Then there exists a minimum point cover of $G$ containing three vertices. Then $D$ is a dominating set of $B_{4}(G)$ and but $\langle D\rangle \cong C_{3}$ in $B_{4}(G)$ and $D$ will not be tree dominating set of $B_{4}(G)$. Therefore $\gamma_{\mathrm{tr}}\left(\mathrm{B}_{4}(\mathrm{G})\right)=0$

Conversely assume $\gamma_{t r}\left(B_{4}(G)\right)=0$. If $\alpha_{0}(G)=1$, then $\gamma_{t r}\left(B_{4}(G)\right)=1$. If $\alpha_{0}(G)=2$ and if there exists a point cover $D$ of $G$ with $|D|=2$ and $\langle D\rangle \cong K_{2}$, then $\gamma_{\mathrm{tr}}\left(B_{4}(G)\right)=2$. Therefore either $\alpha_{0}(G) \geq 3$ or all the point covers of $G$ containing two vertices are independent sets of $G$.

## CONCLUSION

In this paper, bounds of complementary tree domination number of Boolean function graph $B_{4}(G)$ are obtained and this number is found for Boolean function graphs of particular graphs. Also a characterization of graphs for which tree domination number is equal to 2 is obtained.

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