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COMPLEMENTARY TREE DOMINATION IN BOOLEAN FUNCTION GRAPH B(K_p, INC, $\overline{K}_q$) OF A GRAPH

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ABSTRACT

For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph B(K_p, INC, $\overline{K}_q$) of G is a graph with vertex set V(G)$\cup$E(G) and two vertices in B(K_p, INC, $\overline{K}_q$) are adjacent if and only if they correspond to two adjacent vertices of G, two nonadjacent vertices of G or to a vertex and an edge incident to it in G. For brevity, this graph is denoted by B_4(G). In this paper, bounds of complementary tree domination number of Boolean function graph B_4(G) are obtained and this number is found for Boolean function graphs of particular graphs. Also a characterization of graphs for which tree domination number is equal to 2 is obtained.

Key Words: *Boolean Function Graph, Complementary tree dominating set, tree dominating set.*

1. INTRODUCTION

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. The graph G with p vertices and q edges is denoted by G(p, q). The points and edges of a graph are called its elements. Two elements of a graph are neighbors, if they are either incident or adjacent. For a connected graph G, the *eccentricity* e(v) = {d(u, v) : u$\in$V(G)}, where d(u, v) is the distance between u and v in G. The radius of G is rad(G) = min{e(v) u$\in$V(G)}. A vertex v is a central vertex if e(v) = rad(G). A Bistar whose central vertices have degree m and n is denoted by $S_{m,n}$.

The concept of domination in graphs was introduced by Ore [11]. A set D$\subseteq$V(G) is said to be a dominating set of G, if every vertex in V(G) –D is adjacent to some vertex in D. The domination number γ(G) of G is the minimum cardinality of a dominating set. We call a set of vertices a γ - set, if it is a dominating set with cardinality γ(G). Many domination parameters are obtained by combining domination with another graph theoretical property. Some domination parameters are defined by imposing additional constraint on the complement of a dominating set. Such parameters are called codomination parameters. Based on these, the concepts of split and nonsplit domination in graphs were introduced by Kulli and Janakiram [8, 9]. Chen *et.al.* [2] defined a tree dominating set D to be a set D whose induced subgraph <D> is a tree. The minimum cardinality of a tree dominating set of G is the tree domination number γ_{tr}(G). If there is no tree dominating set in G, then let γ_{tr}(G) = 0. A dominating set D$\subseteq$V(G) is said to be tree dominating set if the induced subgraph <D> is a tree. Muthammai, Bhanumathi and Vidhya [10] introduced the concept of complement tree dominating set. A dominating set D$\subseteq$V(G) is said to be complementary tree dominating set (ctd-set) if the induced subgraph <V(G) - D> is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by γ_{ctd}(G).

Whitney [12] introduced the concept of the line graph L(G) of a given graph G in 1932. The concept of total graphs was introduced by Behzad [1] in 1966. Janakiraman *et al.* introduced the concepts of Boolean and Boolean function graphs [4 - 7].

The *Boolean function graph* B(K_p, NINC, $\overline{K}_q$)) of G is a graph with vertex set V(G)$\cup$E(G) and two vertices in B(K_p, NINC, $\overline{K}_q$)) are adjacent if and only if they correspond to two adjacent vertices of G, two nonadjacent vertices of G to a vertex and an edge incident to it in G. For brevity, this graph is denoted by B_4(G). In this paper, bounds of complementary tree domination number of Boolean function graph B_4(G) are obtained and this number is found for Boolean function graphs of particular graphs. Also a characterization of graphs for which tree domination number is equal to 2 is obtained. For graph theoretic terminology, Harary [3] is referred.

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2. PREVIOUS RESULTS

Observation 2.1: [6]

1. K_p is an induced subgraph of $B_4(G)$ and the subgraph of $B_4(G)$ induced by q vertices is totally disconnected.
2. Number of vertices in $B_4(G)$ is p + q, since $B_4(G)$ contains vertices of both G and the line graph L(G) of G.
3. Number of edges in $B_4(G)$ is $(p(p-1))/2 + 2q$
4. For every vertex $v \in V(G)$, $d_{B4(G)}(v) = p - 1 + d_G(v)$
   (a) If G is complete, then $d_{B4(G)}(v) = 2(p - 1)$
   (b) If G is totally disconnected, then $d_{B4(G)}(v) = p - 1$
   (c) If G has atleast one edge, then $2 \leq d_{B4(G)}(v) \leq 2(p - 1)$ and $d_{B4(G)}(v) = 1$ if and only if $G \cong 2K_1$.
5. $\gamma(B_4(G)) = 1$ if and only if $G \cong K_{1, n} \cup mK_1$, n, m ≥ 1 .
6. For an edge $e \in E(G)$, $d_{B4(G)}(e) = 2$
7. $B_4(G)$ is always connected.

Theorem 2.1: [10] $\gamma_{ctd}(G) = 1$ if and only if $G \cong T + K_1$, where T is a tree.

Theorem 2.2: [10] For any connected graph G with p ≥ 2, $\gamma_{ctd}(G) \leq p - 1$.

Theorem 2.3: [10] Let G be a connected graph with p ≥ 2. $\gamma_{ctd}(G) = p - 1$ if and only if G is a star on p vertices.

Theorem 2.4: [10] Let G be a connected graph containing a cycle. Then $\gamma_{ctd}(G) = p - 2$ if and only if G is isomorphic to one of the following graphs C_p, K_p or G is the graph obtained by attaching pendant edges at atleast one of the vertices of a complete graph.

Theorem 2.5: [10] Let T be a tree with p vertices which is not a star. Then $\gamma_{ctd}(T) = p - 2$ if and only if T is a path or T is obtained by attaching pendant edges at atleast one of the end vertices.

3. MAIN RESULTS

In the following, an upper bound of $\gamma_{ctd}(B_4(G))$ is found.

Theorem 3.1: For any graph G with p vertices, $\gamma_{ctd}(B_4(G)) \leq p + q - \Delta(G) - \delta(G) - 2$.

Proof: Let G be a graph with p vertices. Let u be a vertex of G with deg(u) = Δ(G) and let v be a vertex of G with deg(v) = t, where $t = Max\{deg_G(v): v \notin N(u)\}$. Then |N(v)| = t.

If $D = N(u) \cup N(v) \cup \{u, v\}$, then $V(B_4(G)) - D$ is a complementary tree dominating set of $B_4(G)$ and hence
$\gamma_{ctd}(B_4(G)) \leq |V(B_4(G)) - D| = p + q - (\Delta(G) + t + 2) = p + q - \Delta(G) - t - 2$.

$\leq p + q - \Delta(G) - \delta(G) - 2$.

Equality holds, if $G \cong K_{m,m}$ and C_n (m, n ≥ 4).

Note 3.1: If G contains atleast one edge and three vertices, then $\gamma_{ctd}(B_4(G)) \geq 2$.

Theorem 3.2: Let G be a connected graph with p vertices and $\gamma_{ctd}(G) = 1$. Then $\gamma_{ctd}(B_4(G)) \leq 2p - 5$.

Proof: Let G be a connected graph with p vertices and $\gamma_{ctd}(G) = 1$. Then G is isomorphic to $T + K_1$, where T is a tree on p − 1 vertices and hence $B_4(G)$ has 3(p − 1) vertices. Let $v \in V(K_1)$ and u be a vertex of T with $deg_T(u) = \Delta(T)$ and e = (u, v) and let E′ be the set of edges in G incident with u, v or both. If D′ be the set of vertices in $B_4(G)$ corresponding to the edges in E′, let $D = D' \cup \{u, v\} - \{e\}$and $|D'| = deg_G(v) + deg_T(u) + 2 - 1 = p - 1 + \Delta(T) + 1 = p + \Delta(T)$ and $V(B_4(G)) - D$ is a complementary tree dominating set of $B_4(G)$ and hence $\gamma_{ctd}(B_4(G)) \leq |V(B_4(G)) - D'| = 3p - 3 - (p + \Delta(T)) \leq 2p - 5$, since Δ(T) ≥ 2.

Equality holds, if $G \cong P_n + K_1$ where P_n is a path on n vertices.

In the following complementary tree domination number of $B_4(G)$ is found when G is a path, cycle, complete graph, complete bipartite graph, star and wheel.

Theorem 3.3: If G is a Path P_n on n ($n \geq 5$) vertices, then $\gamma_{ctd}(B_4(P_n)) = 2n - 7$.

Proof: Let $G \cong P_n$, $n \geq 5$. $B_4(P_n)$ has $2n - 1$ vertices. Let v_i, v_j be two distinct vertices of degree 2 in P_n such that $d_G(v_i, v_j) \geq 2$ and let e_{i1}, e_{i2} be the edges incident with v_i and e_{j1}, e_{j2} be the edges incident with v_j. Then $v_i, v_j, e_{i1}, e_{i2}, e_{j1}, e_{j2} \in V(B_4(P_n))$. If $D = \{ v_i, v_j, e_{i1}, e_{i2}, e_{j1}, e_{j2}\}$, then $\langle D\rangle \cong S_{2,2}$ and D is a ctd – set of $B_4(P_n)$ and hence $\gamma_{ctd}(P_n) \leq | V(B_4(P_n)) - D| = 2n - 1 - 6 = 2n - 7$. Let D′ be a ctd-set of $B_4(P_n)$. Since K_n is an induced subgraph of $B_4(P_n)$, $V(B_4(P_n)) - D$ contains atmost two vertices of P_n. Also each vertex of L(G) is adjacent to two vertices of G in $B_4(G)$. Therefore any tree of $B_4(G)$ contains atmost 6 vertices and hence D′ contains atleast $2n - 1 - 6 = 2n - 7$ vertices. Therefore $|D'| \geq 2n - 7$. Hence $\gamma_{ctd}(B_4(P_n)) = 2n - 7$.

Remark 3.1: If C_n is a cycle on $n(n \geq 4)$ vertices, then $\gamma_{ctd}(B_4(C_n)) = 2n - 6$ and $\gamma_{ctd}(B_4(C_3)) = 2$.

Theorem 3.4: If K_n is a complete graph on n vertices, then $\gamma_{ctd}(B_4(K_n)) = (n^2 - 3n + 4)/2$, where $n \geq 4$.

Proof: $B_4(K_n)$ has $n + (n(n - 1))/2$ vertices. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n, $e_{12}, e_{13}, \ldots, e_{1n}$ be the edges incident with v_1 and let $e_{1n}, e_{2n}, \ldots, e_{n-1, n}$ be the edges incident with v_n.

Then $D = \{v_1, v_n, e_{12}, e_{13}, \ldots, e_{1n-1}, e_{2n}, \ldots, e_{n-1,n}\} \subseteq V(B_4(K_n))$, $\langle D\rangle \cong S_{n-2, n-2}$, and $|D| = 2n - 2$, where $n \geq 4$. $V(B_4(K_n)) - D$ is a ctd-set of $B_4(K_n))$ and hence $\gamma_{ctd}(B_4(K_n)) \leq n + ((n(n - 1))/2) - (2n - 2) = (n^2 - 3n + 4)/2$. Since K_n is an induced subgraph of $B_4(K_n)$, any tree of $B_4(K_n)$ has atmost $2n - 2$ vertices. Therefore any ctd-set of $B_4(K_n)$ contains atleast $(n^2 - 3n + 4)/2$ vertices. Hence $\gamma_{ctd}(B_4(K_n)) \geq (n^2 - 3n + 4)/2$.

Theorem 3.5: If $K_{m,n}$ ($m \geq n$) is the complete bipartite graph, then $\gamma_{ctd}(B_4(K_{m,n})) = mn - 2$, $m, n \geq 2$.

Proof: Let [A, B] be the bipartition of $K_{m,n}$ such that $|A| = m$ and $|B| = n$. Let $u, v \in B$. Then $deg(u) = deg(v) = m$. If $e_1, e_2, \ldots, e_m$ be the edges incident with u and $f_1, f_2, \ldots, f_m$ be those edges incident with v, then $u, v, e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m \in V(B_4(K_{m,n}))$.

Let $D = \{u, v, e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m\}$. Then $\langle D\rangle \cong S_{m, m}$ and $V(B_4(K_{m,n})) - D$ is a minimum dominating set of $B_4(K_{m,n})$. Therefore, $\gamma_{ctd}(B_4(K_{m,n})) = | V(B_4(K_{m,n})) - D| = m + n + mn - (m + n + 2) = mn - 2$.

Remark 3.2: If G is a star on n + 1 vertices, then $\gamma_{ctd}(B_4(K_{1,n})) = n - 2$, $n \geq 2$.

Theorem 3.6: If W_p is the wheel on p vertices, then $\gamma_{ctd}(B_4(W_p)) = 3p - 9$, where $p \geq 5$.

Proof: Let $v, v_1, v_2, \ldots, v_{p-1}$ be vertices of W_p and let $e_i = (v, v_i)$, $i = 1, 2, \ldots, p-1$,
$e_{i, i+1} = (v_i, v_{i+1})$, $i = 1, 2, \ldots, p-2$ and $e_{p-1, 1} = (v_{p-1}, v_1)$. $| V(B_4(W_p))| = 3p - 1$.

Then $v, v_1, v_2, \ldots, v_{p-1}, e_{i, i+1}, e_{p-1, 1} \in V(B_4(W_p))$. Let vi, vj be two nonadjacent vertices in W_p. and let e_{i1}, e_{i2}, e_{i3} be the edges incident with v_i and e_{j1}, e_{j2}, e_{j3} be the edges incident with v_j. Then $e_{i1}, e_{i2}, e_{i3}, e_{j1}, e_{j2}, e_{j3} \in V(B_4(W_p))$. Let $D = \{e_{i1}, e_{i2}, e_{i3}, e_{j1}, e_{j2}, e_{j3}, v_i, v_j\} \subseteq V(B_4(W_p))$. Then each vertex in D is adjacent to atleast one vertex in $V(B_4(W_p)) - D$ and $\langle V(B_4(W_p)) - D\rangle \cong S_{3,3}$.

Therefore $V(B_4(W_p)) - D$ is a minimum ctd-set of $B_4(W_p)$ and hence $\gamma_{ctd}(B_4(W_p)) = | V(B_4(W_p)) - D| = 3p - 1 - 8 = 3p - 9$.

Theorem 3.7: If G is a graph obtained from $K_1 + T$ with one pendant edge attached at the vertex of K_1, where T is any tree on $p - 2$ vertices, then $\gamma_{ctd}(B_4(G)) \leq 2p - \Delta(T) - 4$.

Proof: If G is a graph as stated in the Theorem, then $\gamma_{ctd}(G) = 2$. Number of vertices in G is p and the number of edges in G is $|E(T)| + p - 2 + 1 = p - 3 + p - 1 = 2p - 4$. Hence number of vertices in $B_4(G)$ is $p + 2p - 4 = 3p - 4$. Let $V(K_1) = \{v\}$ and u be the pendant vertex adjacent to v in G, $e = (u, v)$ and let $v_1, v_2, \ldots, v_{p-2}$ be the vertices of T with $deg_T(v_i) = \Delta(T)$. Let $e_i = (v, v_i)$, $i = 1, 2, \ldots, p - 2$. Then $v_1, v_2, \ldots, v_{p-2}, e_1, e_2, \ldots, e_{p-2}, u, v, e \in V(B_4(G))$. If $D = \{v, v_i, e, e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{p-2}\} \cup N_T(v_i)$ $(\subseteq V(B_4(G)))$, then the subgraph of $B_4(G)$ induced by D is isomorphic to $S_{p-2, \Delta(T)}$ and $|D| = p + \Delta(T)$. Since the subgraph of $B_4(G)$ induced by vertices of G is complete, v, v_i are adjacent atleast one vertex of G in $V(B_4(G)) - D$. Also $e, e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{p-2}$ are adjacent to $u, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{p-2}$ respectively in $V(B_4(G)) - D$. Therefore, $V(B_4(G)) - D$ is a dominating set of $B_4(G)$ and since $D \cong S_{p-2, \Delta(T)}$, $V(B_4(G)) - D$ is a ctd-set of B4(G). Hence $\gamma_{ctd}(B_4(G)) \leq | V(B_4(G)) - D| = 3p - 4 - (p + \Delta(T)) = 2p - \Delta(T) - 4$. Equality holds, if T is a path on $p - 2$ vertices.

Remark 3.3: If T is a star, then $\Delta(T) = p - 3$ and hence $\gamma_{ctd}(B_4(G)) \le 2p - (p - 3) - 4 = p + 7$.

Theorem 3.8: If G is a graph obtained from a tree by joining each of the vertices of the tree to the vertices of K_2 such that $deg(v) \ge 2$, for all $v \in V(K_2)$, then $\gamma_{ctd}(B_4(G)) \le 2p - 4$.

Proof: Let $V(K_2) = \{u, v\}$ and $deg_G(u) = m$ and $deg_G(v) = n$, where $m, n \ge 2$ and $m + n \le p$ and let T be a tree on $p - 2$ vertices. Then $|E(G)| = |E(T)| + m - 1 + n - 1 + 1 = p - 3 + m + n - 1 = p + m + n - 4$. Therefore $|V(B_4(G)| = |V(G)| + |E(G)| = 2p + m + n - 4$. Let $u_1, u_2, \ldots, u_{m-1}$ be the vertices of T adjacent to u and let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of T adjacent to v in G.

If $e_i = (u, u_i)$ and $f_j = (v, v_j)$ ($i = 1,2,\ldots,m-1$, $j=1, 2, \ldots, n-1$), then $D = \{u, v, e_1, e_2,\ldots, e_{m-1}, f_1, f_2, \ldots,f_{n-1}\} \subseteq V(B_4(G))$ and $|D| = m + n$. Also $\langle D\rangle \cong S_{m-1, n-1}$ in $B_4(G)$. Each vertex in D is adjacent to atleast one vertex in $V(B_4(G)) - D$ and $\langle D\rangle$ is a tree in $B_4(G)$. Therefore $V(B_4(G)) - D$ is a ctd-set of $B_4(G)$ and hence $\gamma_{ctd}(B_4(G)) \le |V(B_4(G)) - D| = 2p - 4$.

Remark 3.4: If each vertex of the tree is adjacent to both u and v in G, then $\gamma_{ctd}(B_4(G)) = 2p - 4$.

In the similar lines, the following theorem can be proved

Theorem 3.9: If G is a graph obtained from a tree by joining each of the vertices of the tree to the vertices of $2K_1$ such that $deg(v) \ge 1$, for all $v \in V(2K_1)$, then $\gamma_{ctd}(B_4(G)) \le 2p - 5$.

Theorem 3.10: For $n \ge 3$, $\gamma_{ctd}(B_4(P_n + K_1)) = 2n - 3$.

Proof: Let $v_1, v_2, \ldots, v_n \in V(C_n)$, $v \in V(K_1)$ and let $e_i = (v, v_i)$, $i = 1, 2, \ldots, n$; $e_{i,i+1} = (v_i, v_{i+1})$, $i = 1, 2, \ldots, n-1$. $|V(B_4(P_n + K_1))| = |V(P_n)| + |V(K_1)) + |E(P_n))| + n = n + 1 + n - 1 + n = 3n$. Let $D = \{v, v_{n-1}, e_1, e_2, \ldots, e_{n-2}, e_n, e_{n-2,n-1}, e_{n-1,n}\}$. Then D is a subset of $V(B_4(P_n + K_1)$ and $|D| = n + 3$. Each vertex in D is adjacent to atleast one vertex in $V(B_4(P_n + K_1)) - D$. Also $\langle D\rangle \cong S_{n-1, 2}$. Therefore $V(B_4(P_n + K_1)) - D$ is a minimum ctd-set of $B_4(P_n + K_1)$ and hence $\gamma_{ctd}(B_4(P_n + K_1)) = |V(B_4(P_n + K_1)) - D| = 3n - (n + 3) = 2n - 3$.

Theorem 3.11: For $n \ge 5$, $\gamma_{ctd}(B_4(C_n + K_1)) = 2n - 2$.

Proof: Let $v_1, v_2, \ldots, v_n \in V(C_n)$, $v \in V(K_1)$ and let $e_i = (v, v_i)$, $i = 1, 2, \ldots, n$; $e_{i,i+1} = (v_i, v_{i+1})$, $i = 1, 2, \ldots, n-1$; $e_{n,1} = (v_n, v_1)$. $|V(B_4(C_n + K_1))| = |V(C_n)| + |V(K_1)) + |E(C_n))| + n = 3n + 1$.

Let $D = \{v, v_n, e_1, e_2, \ldots, e_{n-1}, e_{n-1,n}, e_{n1}\}$. Then $D \subseteq V(B_4(C_n + K_1)$, and $|D| = n + 3$. v, v_n are adjacent to v_i, $i = 2, 3, \ldots, n-1$ and e_i is adjacent to v_i, $i = 1, 2, \ldots, n-1$, $e_{n-1,n}$, e_{n1} are adjacent to v_n in $V(B_4(C_n + K_1)) - D$. Therefore $V(B_4(C_n + K_1)) - D$ is a dominating set of $V(B_4(C_n + K_1))$ and since $\langle D\rangle \cong S_{n-1, 2}$, $V(B_4(C_n + K_1)) - D$ is a ctd-set of $B_4(C_n + K_1)$. Also there is no other ctd-set containing less than $|V(B_4(C_n + K_1))-D|$ vertices. Therefore $\gamma_{ctd}(B_4(C_n + K_1)) \le |V(B_4(C_n + K_1)) - D| = 3n + 1 - (n + 3) = 2n - 2$, where $n \ge 5$.

Remark 3.5: $\gamma_{ctd}(B_4(C_4 + K_1)) = 5$.

Theorem 3.12: For $m, n \ge 2$, $\gamma_{ctd}(B_4(K_{m,n} + K_1)) = n(m + 1)$

Proof: Let [A, B] be the bipartition of $K_{m,n}$. Assume $A = \{v_1, v_2, \ldots, v_m\}$, $B = \{u_1, u_2, \ldots, u_n\}$ and $V(K_1) = \{v\}$. Let $e_{ij} = (u_i, v_j)$, $e_i = (v, v_i)$ and $f_j = (v, u_j)$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. $|V(B_4(K_{m,n} + K_1))| = |V(K_{m,n})| + |V(K_1)| + |E(K_{m,n})| + |\{e_i, i = 1, 2, \ldots, m\}| + |\{f_j, j = 1, 2, \ldots, n\}| = m + n + 1 + mn + m + n = mn + 2m + 2n + 1$. If $D = \{v, u_1, e_1, e_2, \ldots, e_m, f_2, \ldots, f_n, e_{11}, e_{21}, \ldots, e_{m1}\}$, then $D \subseteq V(B_4(K_{m,n} + K_1))$, $|D| = 2m + n + 1$ and $\langle D\rangle \cong S_{m+n-1, m}$. As in Theorem, $V(B_4(K_{m,n} + K_1)) - D$ is a minimum ctd – set of $B_4(K_{m,n} + K_1)$ and hence $\gamma_{ctd}(B_4(K_{m,n} + K_1)) = |V(B_4(K_{m,n} + K_1)) - D| = (mn + 2m + 2n + 1) - (2m + n + 1) = n(m + 1)$.

Theorem 3.13: For $m, n \ge 3$, $\gamma_{ctd}(B_4(K_{1,n} + K_1)) = n + 3$.

Proof: Let $v_1, v_2, \ldots, v_n, v_{n+1} \in V(K_{1,n})$, where v_1 is the central vertex and $V(K_1) = \{v\}$. Let $e_i = (v, v_i)$, $i = 1, 2, \ldots, n+1$, $e_{1j} = (v_1, v_j)$, $j = 2, 3, \ldots, n+1$. Then $|V(B_4(K_{1,n} + K_1))| = |V(K_{1,n})| + |V(K_1)| + |\{e_i, i = 1, 2, \ldots, n+1\}| + |\{e_{1,j}, j = 2, 3, \ldots, n+1\}| = n + 1 + 1 + n + 1 + n = 3n + 3$.

If $D = \{v, v_1, e_2, e_3, \ldots, e_n, e_{12}, e_{13}, \ldots, e_{1n}\} \subseteq V(B_4(K_{1,n} + K_1))$, then $|D| = 2n$ and $\langle D\rangle \cong S_{n-1,n-1}$. Also $V(B_4(K_{1,n} + K_1)$ $- D$ is a minimum ctd-set of $B_4(K_{1,n} + K_1)$ and hence $\gamma_{ctd}(B_4(K_{1,n} + K_1)) = 3n + 3 - 2n = n + 3$.

Theorem 3.14: For $p \geq 5$, $\gamma_{ctd}(B_4(W_p + K_1)) = 2p - 1$.

Proof: Let $v_1, v_2, \ldots, v_p$ be vertices of W_p, where v_1 is the central vertex and $v \in V(K_1)$. Let the edges of $B_4(W_p + K_1)$ be denoted by $e_i = (v, v_i)$, $i = 1, 2, \ldots, p$; $e_{1j} = (v_1, v_j)$, $j = 2, 3, \ldots, p$; $(e_i, e_{i+1}) = (vi, v_{i+1})$, $i = 2, 3, \ldots, p-1$ and $e_{p2} = (v_p, v_2)$. $V(B_4(W_p + K_1)) = (V(W_p + K_1)) \cup (E(W_p + K_1))$ and $|V(B_4(W_p + K_1))| = |V(W_p + K_1)| + |E(W_p + K_1)| = (p + 1) + p + (p-1) + (p-2) + 1 = 4p - 1$. If $D = \{ v, v1, e2, e3, \ldots, ep, e12, e13, \ldots, e1p\} \subseteq V(B_4(W_p + K_1))$, then $V(B_4(W_p + K_1)) - D$ is a dominating set of $W_p + K_1$. Also $\langle D \rangle \cong S_{p-1, p-1}$. Therefore $V(B_4(W_p + K_1)) - D$ is ctd-set of $B_4(W_p + K_1)$ and hence $\gamma_{ctd}(B_4(W_p + K_1)) \leq |V(B_4(W_p + K_1)) - D| = 4p - 1 - 2p = 2p - 1$. Also there exists no ctd-set in $B_4(W_p + K_1)$ having $2p - 1$ vertices. Hence $\gamma_{ctd}(B_4(W_p + K_1)) = 2p - 1$.

In the following, tree domination number of $B_4(G)$ is found.

Observation 3.1: $\gamma(B_4(G)) = \gamma_{tr}(B_4(G)) = 1$ if and only if $G \cong K_{1, n} \cup mK_1$, $n, m \geq 1$.

Theorem 3.15: Let G be not a star and $\delta(G) \geq 1$. Then $\gamma_{tr}(B_4(G)) = 2$ if and only if there exists a minimum point cover of G containing two adjacent vertices.

Proof: Let D be a minimum point cover of G containing two adjacent vertices say, u, v. Then each vertex of G in $B_4(G)$ is adjacent to both u and v. Since D is a point cover, each edge in G is incident with atleast one of u and v. Therefore vertices in $B_4(G)$ corresponding to the edges of G are adjacent to atleast one of u and v and hence D is a dominating set of $B_4(G)$. Also $\langle D \rangle \cong K_2$. Therefore D is a tree dominating set of $B_4(G)$. Hence $\gamma_{tr}(B_4(G)) \leq |D| = 2$. Since G is not a star, $\gamma_{tr}(B_4(G)) \geq 2$. Therefore $\gamma_{tr}(B_4(G)) = 2$.

Conversely assume $\gamma_{tr}(B_4(G)) = 2$. Then there exists a tree dominating set D of $B_4(G)$ containing two vertices and D contains atleast one vertex of G, since vertices of line graph L(G) of G in $B_4(G)$ are independent. Let D = { u, v}. Then $\langle D \rangle \cong K_2$ in $B_4(G)$.

Case 1: $u \in V(G)$ and $v \in V(L(G))$
Then $v \in E(G)$ and u is incident v in G. Since D is a dominating set of $B_4(G)$, each edge in G is incident with u. That is, $\alpha_0(G) = 1$ and $G \cong K_{1,n}$, $n \geq 2$. But $\gamma_{tr}(B_4(K_{1, n})) = 1$.

Case 2: $u, v \in V(G)$
Then D is minimum point cover of G containing two adjacent vertices and $\alpha_0(G) = 2$.

Theorem 3.16: For any graph G with $\delta(G) \geq 1$, $\gamma_{tr}(B_4(G)) = 0$ if and only if either $\alpha_0(G) \geq 3$ or all the point covers of G containing two vertices are independent sets of G.

Proof: Assume $\alpha_0(G) \geq 3$. Then there exists a minimum point cover of G containing three vertices. Then D is a dominating set of $B_4(G)$ and but $\langle D \rangle \cong C_3$ in $B_4(G)$ and D will not be tree dominating set of $B_4(G)$. Therefore $\gamma_{tr}(B_4(G)) = 0$

Conversely assume $\gamma_{tr}(B_4(G)) = 0$. If $\alpha_0(G) = 1$, then $\gamma_{tr}(B_4(G)) = 1$. If $\alpha_0(G) = 2$ and if there exists a point cover D of G with $|D| = 2$ and $\langle D \rangle \cong K_2$, then $\gamma_{tr}(B_4(G)) = 2$. Therefore either $\alpha_0(G) \geq 3$ or all the point covers of G containing two vertices are independent sets of G.

CONCLUSION

In this paper, bounds of complementary tree domination number of Boolean function graph $B_4(G)$ are obtained and this number is found for Boolean function graphs of particular graphs. Also a characterization of graphs for which tree domination number is equal to 2 is obtained.

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