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RANDOM WALKS AND BRAID ARRANGEMENT

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ABSTRACT

In this paper we study the face poset, and a random walk on the regions of the Braid arrangement as a special case when n = 4. The transition matrix for Braid arrangement considered one of the most important findings in this research. In addition to this, we will calculate the face poset, the stationary distribution and the eigenvalues of transition matrix for Braid arrangement.

Key word: *Braid arrangement, face, chamber, random walk, transition matrix, stationary distribution.*

1. INTRODUCTION

An arrangement of hyperplanes is a finite collection of codimension one subspace in a finite dimensional vector space over R(ℂ). Let L = L($\mathcal{A}$) be the intersection poset of $\mathcal{A}$. L is the set of non-empty intersections of hyperplanes in $\mathcal{A}$ ordered by reverse inclusion. By convention L includes V as its unique minimal element. The intersection of the form $x = \bigcap_{H\in\mathcal{A}} H^{\sigma_H}$ is called a face of $\mathcal{A}$, where $\sigma_H \in \{+, -, 0\}$ and $H^0 = H$. The sign vector of x is the sequence $\sigma(x) = (\sigma_H)_{H\in\mathcal{A}}$. A face c is called chamber if $\sigma_H(c) \neq 0$ for all $H \in \mathcal{A}$. The set of faces of $\mathcal{A}$ is called the face poset of $\mathcal{A}$ which is denoted by $\mathcal{F}$. The face poset is partially ordered by:

$$x \leq y \Leftrightarrow \text{for each } H \in \mathcal{A} \text{ either } \sigma_H(x) = 0 \text{ or } \sigma_H(x) = \sigma_H(y).$$

Let J = {+, −, 0}. We may view each face $F \in \mathcal{F}(A)$ as a map $F:\{1, . . . , n\} \to J$ defined by $F(k) = \text{sign}\alpha_H(p)$ for any $p \in F$. Note that F(k) = 0 if and only if $F \subseteq H_K$, and if $F(k) \neq 0$, then the sign indicates whether F is in the positive or negative half-space determined by H_K. Which side is called positive depends on the original choice of α_K. Let $W = J^n$, and let $\pi_H: W \to J$ be the projection onto the k-th coordinate. Define a map $\sigma: \mathcal{F} \to W$ by:

$$\pi_K\sigma(F) = \begin{cases} + \text{ if } F(K) > 0 \\ 0 \text{ if } F(K) = 0 \\ - \text{ if } F(K) < 0 \end{cases}$$

Note that the choice of which half-space to label + or − is arbitrary, but fixed, for more details see [1][2][3][4].

2. THE FACE POSET OF BRAID ARRANGEMENT

The Braid arrangement in R^n consist of $\binom{n}{2}$ hyperplanes $H_{ij} = \{(x_1, \ldots, x_n): x_i = x_j\}$ $(i < j)$. The chambers are associated with a common ordering of the coordinates and so with one of the n! permutations. When n= 4, for example, one of the 24 chambers is the region defined by $x_1 > x_4 > x_2 > x_3$, corresponding to the partition 1423. The hyperplane H_{ij} intersect in the line $x_1 = \ldots = x_n$. The Braid arrangement therefore give rise to an arrangement in (n-1)-dimensional space When n = 4, the resulting arrangement of 6 planes in R^3 may be pictured as in Figure (1). The great circle corresponding to H_{ij} is labeled i-j. (The equator is not one of the great circles of the arrangement).Each chamber is labeled with the associated permutation.

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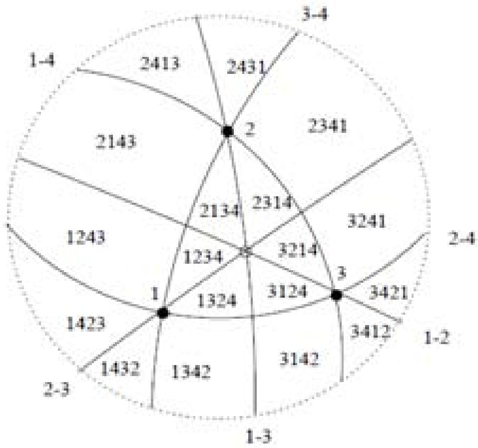

Fig.1: The Braid arrangement when n = 4

The faces of a chamber C are obtained by changing to equalities some of the inequalities defining C. For example, the chamber $x_2 > x_3 > x_1 > x_4$ has a face given by $x_2 > x_3 > x_1 = x_4$, which is also a face of the chamber $x_2 > x_3 > x_4 > x_1$. This common face is represented by edge between 2314 and 2341 in Figure (1).Similarly, the vertex labeled 2 in the figure corresponds to the face $x_2 > x_1 = x_3 = x_4$; it is a face of six chambers, corresponding to the six possible orderings of the indices 1, 3, 4.

It is useful to encode the system of equalities and inequalities defining a face F by an ordered partition $(B_1, \ldots, B_K)$ of $\{1, \ldots, n\}$. Here $B_1, \ldots, B_K$ are disjoint nonempty sets whose union is $\{1, \ldots, n\}$; they are called the blocks of the partition, and their order counts. For example, the face $x_2 > x_3 > x_1 = x_4$ corresponds to the 3-block ordered partition ({2},{3}, {1, 4}), and the face $x_2 > x_1 = x_3 = x_4$ corresponds to the 2-block ordered partition ({2}, {1, 3, 4}). Notice that there is also a (unique) 1-block ordered partition, corresponding to the face $x_1 = x_2 = x_3 = x_4$. When we pass from R^4 to a 3-dimensional quotient to make the hyperplanes have trivial intersection, this face becomes {0}. It does not show up in Figure (1) because its intersection with the sphere is empty [5][10].

Theorem 2.1 [6]: There is an order-isomorphism $\emptyset_2$: $L(\mathcal{A}_n) \rightarrow \prod n$ between the intersection lattice $L(\mathcal{A}_n)$ for the Braid arrangement $\mathcal{A}_n$ and the set partition lattice $\prod n$.

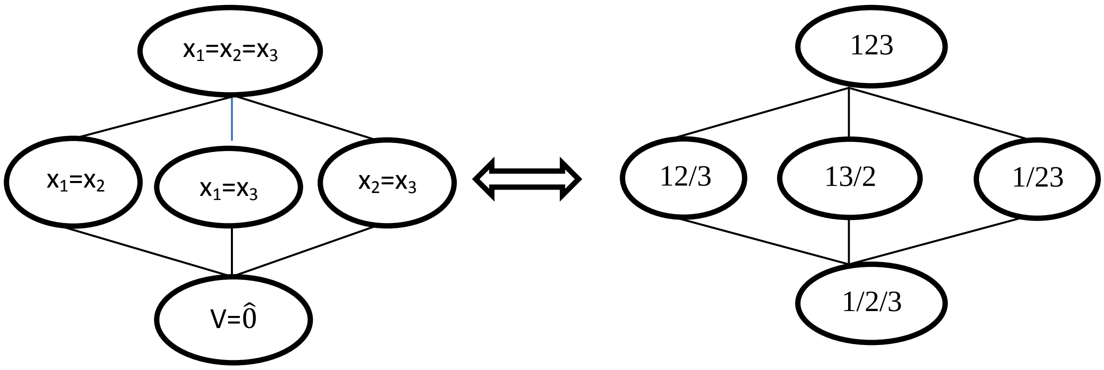

Fig. 2: Hasse diagram between the intersection lattice for the Braid arrangement and the set partition lattice

Example 2.2: In $\mathcal{A}_4$ Braid arrangement, the sign vectors for each face are given in the following Table.

Face	Sign vector $(\sigma_{12},\sigma_{13},\sigma_{14},\sigma_{23},\sigma_{24},\sigma_{34})$	Corresponding Block-Ordered Partition
$x_2 > x_4 > x_3 > x_1$	(+,+,+,+,+,−)	({2},{4},{3},{1})
$x_2 > x_4 > x_1 > x_3$	(+,−,+,+,+,−)	({2},{4},{1},{3})
$x_2 > x_1 > x_4 > x_3$	(+,−,−,+,+,−)	({2},{1},{4},{3})
$x_1 > x_2 > x_4 > x_3$	(−,−,−,+,+,−)	({1},{2},{4},{3})
$x_1 > x_4 > x_2 > x_3$	(−,−,−,+,−,−)	({1},{4},{2},{3})
$x_1 > x_4 > x_3 > x_2$	(−,−,−,−,−,−)	({1},{4},{3},{2})

$x_1 > x_3 > x_4 > x_2$	(−,−,−,−,−,+)	({1},{3},{4},{2})
$x_3 > x_1 > x_4 > x_2$	(−,+,−,−,−,+)	({3},{1},{4},{2})
$x_3 > x_4 > x_1 > x_2$	(−,+,+,−,−,+)	({3},{4},{1},{2})
$x_3 > x_4 > x_2 > x_1$	(+,+,+,−,−,+)	({3},{4},{2},{1})
$x_3 > x_2 > x_4 > x_1$	(+,+,+,−,+,+)	({3},{2},{4},{1})
$x_2 > x_3 > x_4 > x_1$	(+,+,+,+,+,+)	({2},{3},{4},{1})
$x_1 > x_2 > x_3 > x_4$	(−,−,−,+,+,+)	({1},{2},{3},{4})
$x_1 > x_3 > x_2 > x_4$	(−,−,−,−,+,+)	({1},{3},{2},{4})
$x_3 > x_1 > x_2 > x_4$	(−,+,−,−,+,+)	({3},{1},{2},{4})
$x_3 > x_2 > x_1 > x_4$	(+,+,−,−,+,+)	({3},{2},{1},{4})
$x_2 > x_3 > x_4 > x_1$	(+,+,−,+,+,+)	({2},{3},{1},{4})
$x_2 > x_1 > x_3 > x_4$	(+,−,−,+,+,+)	({2},{1},{3},{4})
$x_4 > x_3 > x_2 > x_1$	(+,+,+,−,−,−)	({4},{3},{2},{1})
$x_4 > x_2 > x_3 > x_1$	(+,+,+,+,−,−)	({4},{2},{3},{1})
$x_4 > x_2 > x_1 > x_3$	(+,−,+,+,−,−)	({4},{2},{1},{3})
$x_4 > x_1 > x_2 > x_3$	(−,−,+,+,−,−)	({4},{1},{2},{3})
$x_1 > x_4 > x_3 > x_2$	(−,−,+,−,−,−)	({1},{4},{3},{2})
$x_4 > x_3 > x_1 > x_2$	(−,+,+,−,−,−)	({4},{3},{1},{2})

Table-1: Face, sign vectors, and corresponding block-ordered partition for $\mathcal{A}_4$ arrangement

The poset of the Braid arrangement when n = 4 will be large so much, So we will partition it as follows:

1. Figure (5), shows the poset that obtained from $H_{23}\cap H_{24}\cap H_{34}$ in the northern hemisphere.
2. Figure (6), shows the poset that obtained from $H_{13}\cap H_{14}\cap H_{34}$ in the northern hemisphere.
3. Figure (7), shows the poset that obtained from $H_{12}\cap H_{14}\cap H_{24}$ in the northern hemisphere.
4. Figure (8), shows the poset that obtained from $H_{12}\cap H_{13}\cap H_{23}$ in the northern hemisphere.
5. Figure (9), shows the poset that obtained from $H_{23}\cap H_{14}$ in the northern hemisphere.
6. Figure (10), shows the poset that obtained from $H_{12}\cap H_{34}$ in the northern hemisphere.
7. Figure (11), shows the poset that obtained from $H_{13}\cap H_{24}$ in the northern hemisphere.

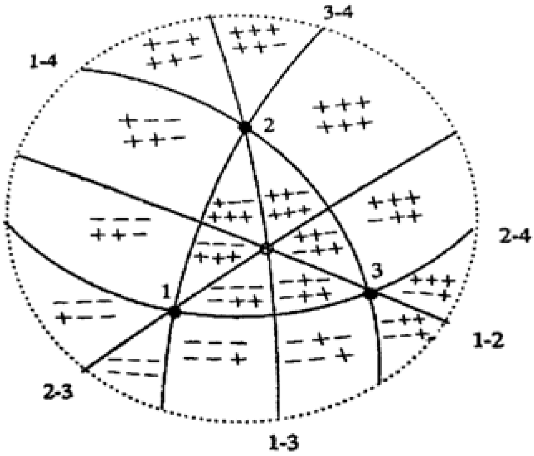

Fig.3: The northern hemisphere of $\mathcal{A}_4$ Braid arrangement

8. Figure (12), shows the poset that obtained from $H_{13}\cap H_{24}$ in the southern hemisphere.
9. Figure (13), shows the poset that obtained from $H_{12}\cap H_{34}$ in the southern hemisphere.
10. Figure (14), show the poset that obtained from $H_{23}\cap H_{14}$ in the southern hemisphere.
11. Figure (15), shows the poset that obtained from $H_{12}\cap H_{13}\cap H_{23}$ in the southern hemisphere.
12. Figure (16), shows the poset that obtained from $H_{12}\cap H_{14}\cap H_{24}$ in the southern hemisphere.
13. Figure (17), shows the poset that obtained from $H_{13}\cap H_{14}\cap H_{34}$ in the northern hemisphere.
14. Figure (18), shows the poset that obtained from $H_{23}\cap H_{24}\cap H_{34}$ in the northern hemisphere.

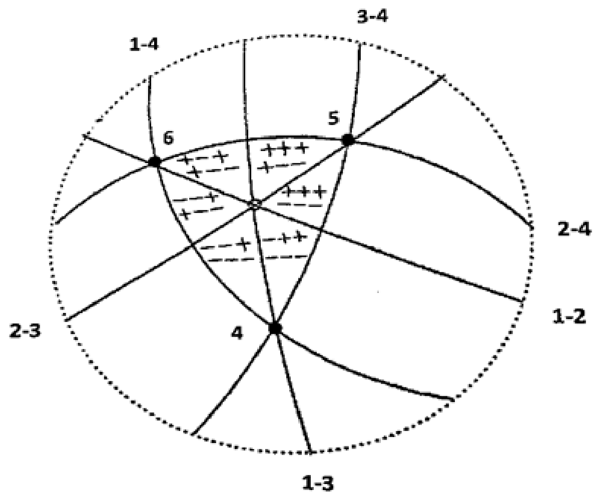

Fig. 4: The southern hemisphere of $\mathcal{A}_4$ Braid arrangements

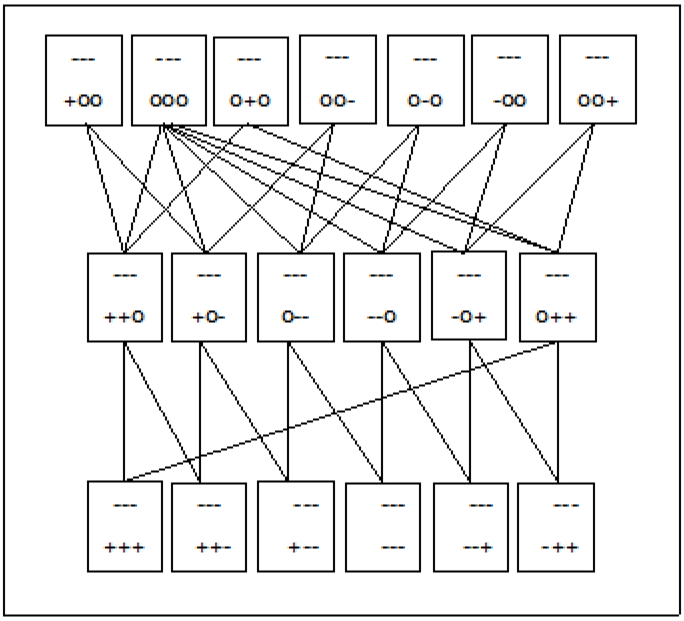

Fig.5

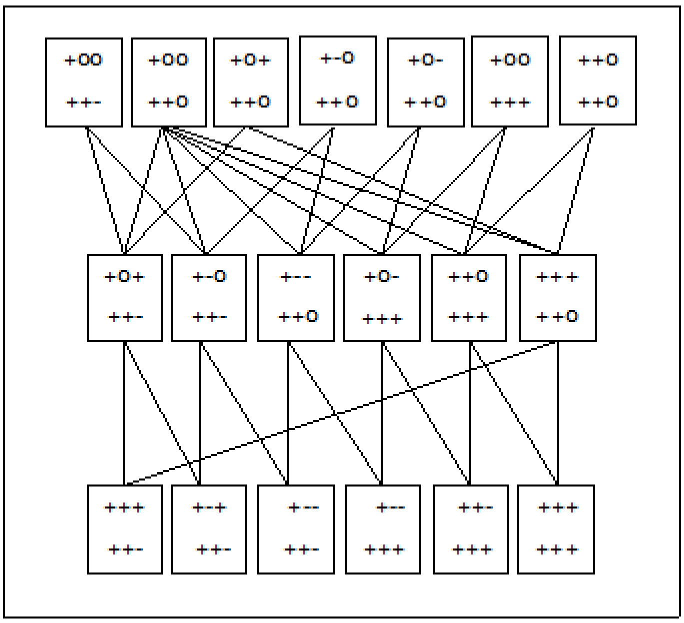

Fig. 6

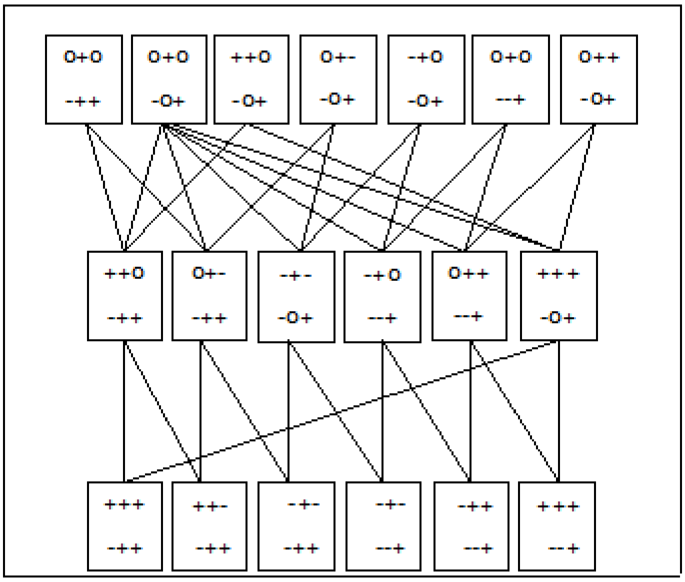

Fig. 7

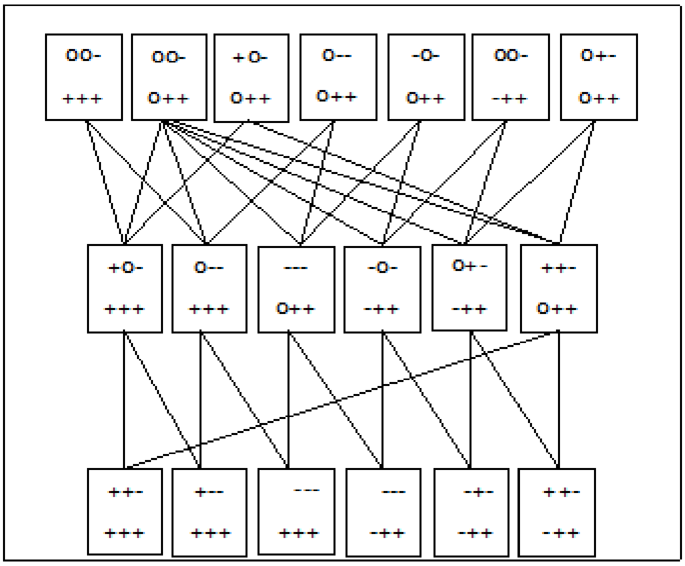

Fig. 8

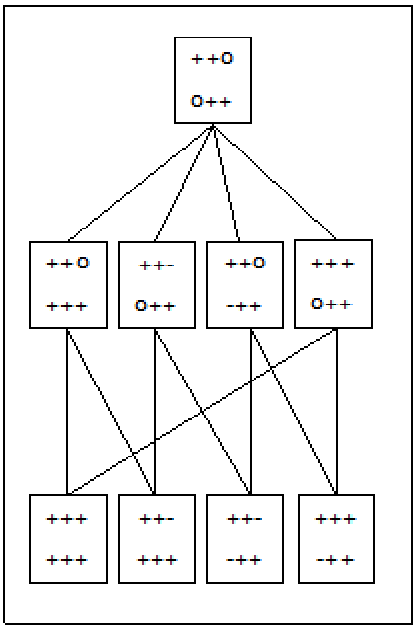

Fig. 9

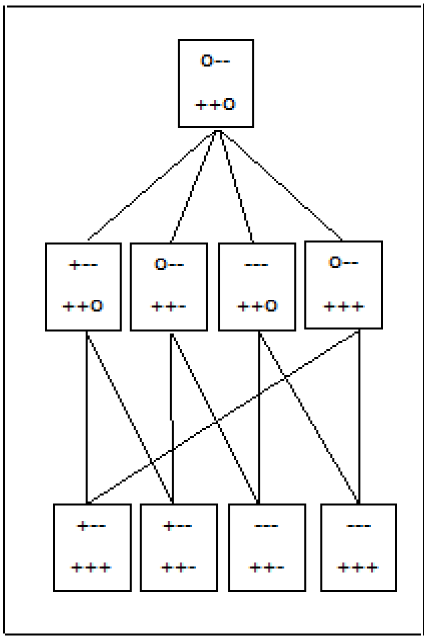

Fig. 10

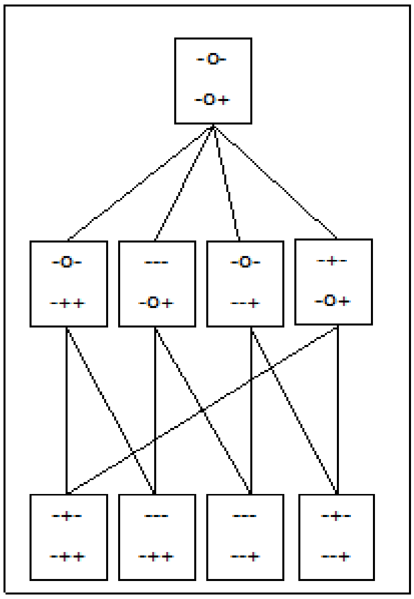

Fig. 11

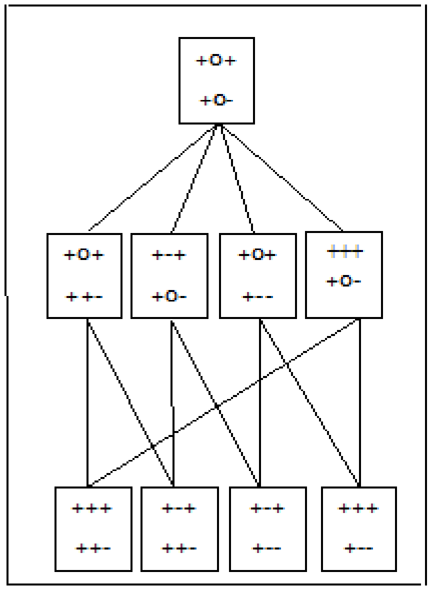

Fig. 12

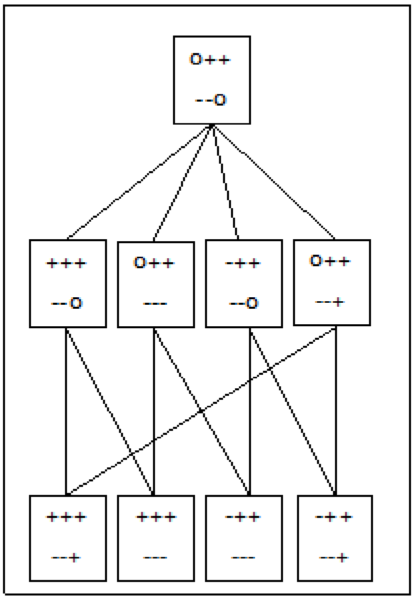

Fig. 13

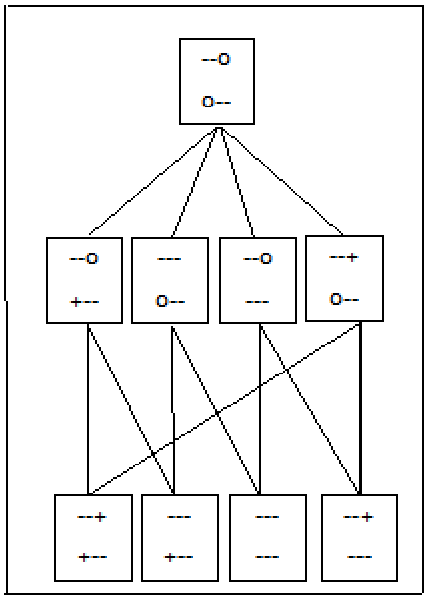

Fig. 14

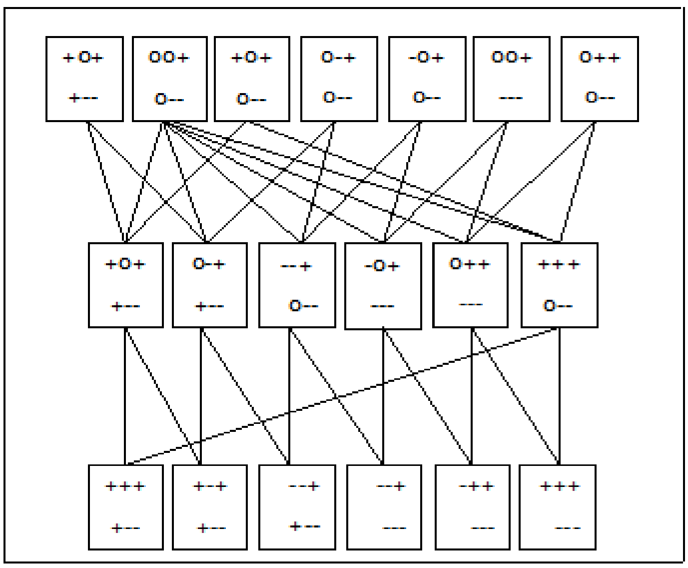

Fig. 15

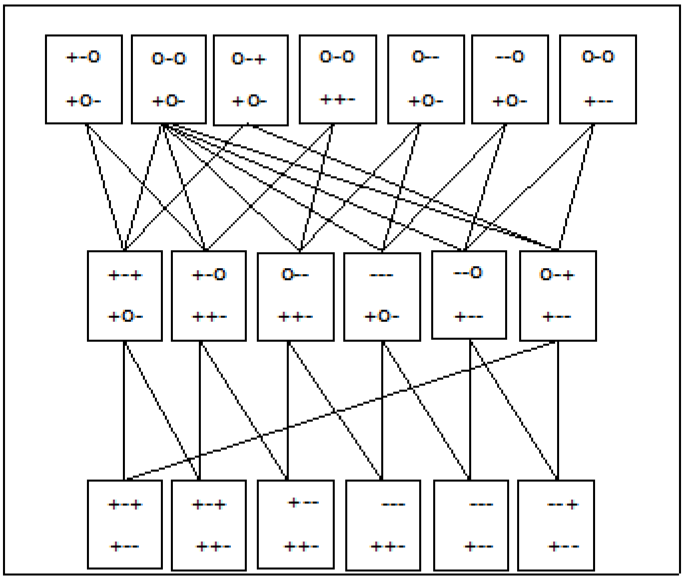

Fig. 16

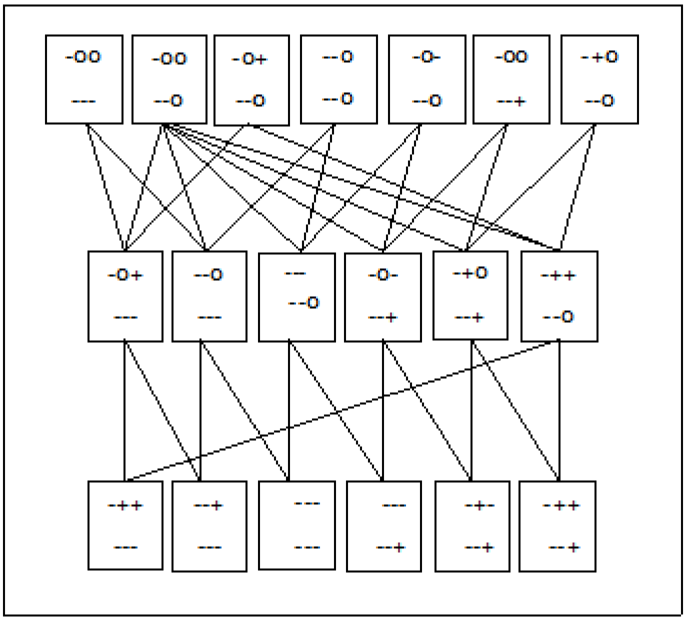

Fig. 17

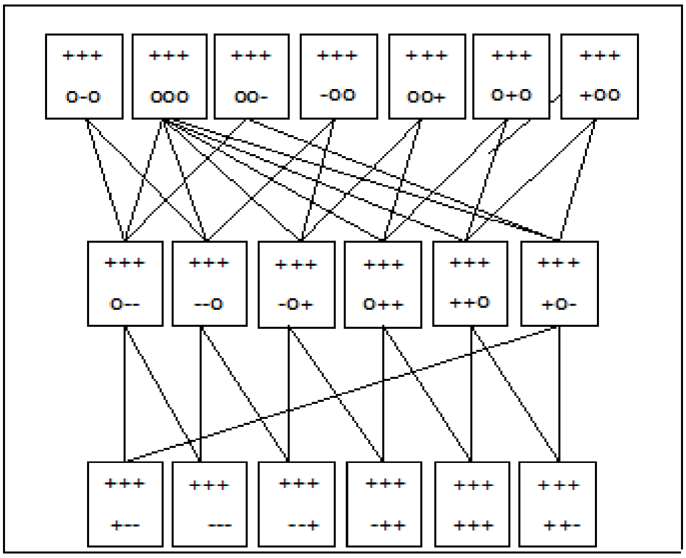

Fig. 18

3. THE TRANSITION MATRIX K

In **[7]** Brown found the transition matrix for a subset of Braid arrangement when n = 4, so we will find the transition matrix for all the regions of Braid arrangement.

3.1. A random walk on the regions [8],

Bidigare, Hanlon, and Rockmore in **[9]** used the projection operators to define the following walk on the regions: If the walk is in region E, choose a vertex v at random and move to the projection $E' = vE$. This walk is described mathematically by its transition matrix K. The rows and columns of K are indexed by the regions, with $K(E, E')$ being the chance of moving from to E' in one step. Thus

$$K(E, E') = 1/a_0 \, \Lambda(E, E'),$$

Where $\Lambda(E, E')$ is the number of vertices v of E' such that $vE = E'$.

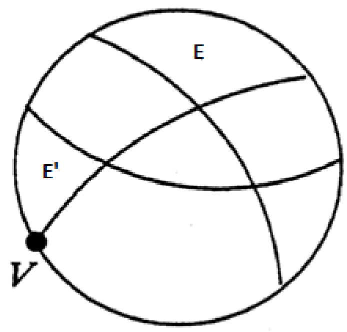

Fig. 19: The projection of E on v

3.2. The transition matrix of Braid arrangement

We should now be able to construct geometrically the transition matrix involving all the regions. For example, starting with the region 1234, you choose at random a proper subset of the places and put those first in the original order. So if you choose places 1 and 3, then you put those first and get 1324. Choosing 1 or 12 or 123 will all lead to 1234 again, so the probability of moving 1234 ⟹ 1234 is 3/14. So choosing subset S leads to permutation pi as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234
pi	1234	2134	3124	4123	1234	1324	1423	2314	2413	3412	1234	1243	1342	2341

So the probability of moving 1234 ⇒ pi is, for example,

1234 ⇒ 1234 3/14
1234 ⇒ 4321 0
1234 ⇒ 3412 1/14
1234 ⇒ 2314 1/14
1234 ⇒ 4312 0

and so forth. Really, what is happening here is the probability of 1234 ⇒ pi is 3/14 if pi has 0 descents, 1/14 if pi has 1 descent, 0 if pi has 2 or 3 descents. Therefore the following matrix is the transition matrix of all regions for Braid arrangement when n = 4.

	2431	2413	2143	1243	1423	1432	1342	3142	3412	3421	3241	2341
2431	3	1	1	1	0	0	0	0	0	0	1	1
2413	1	3	1	1	0	0	0	0	0	0	1	1
2143	1	1	3	1	1	1	0	0	0	0	0	0
1243	1	1	1	3	1	1	0	0	0	0	0	0
1423	0	0	1	1	3	1	1	1	0	0	0	0
1432	0	0	1	1	1	3	1	1	0	0	0	0
1342	0	0	0	0	1	1	3	1	1	1	0	0
3142	0	0	0	0	1	1	1	3	1	1	0	0
3412	0	0	0	0	0	0	1	1	3	1	1	1

3421	0	0	0	0	0	0	1	1	1	3	1	1
3241	1	1	0	0	0	0	0	0	1	1	3	1
2341	1	1	0	0	0	0	0	0	1	1	1	3
2314	1	0	1	0	1	0	0	1	0	0	0	1
3214	0	0	1	0	0	1	0	1	0	1	1	0
3124	1	0	0	1	0	0	0	1	1	0	1	0
1324	0	1	0	1	0	1	1	0	0	0	1	0
1234	0	0	0	1	1	0	1	0	1	0	0	1
2134	0	1	1	0	0	0	1	0	0	1	0	1
4132	0	1	0	0	0	1	1	0	1	0	1	0
4123	0	1	0	1	1	0	0	0	1	0	0	1
4213	0	1	1	0	1	0	1	0	0	1	0	0
4231	1	0	0	0	1	0	0	1	0	1	0	1
4321	1	0	1	0	0	1	0	0	0	1	1	0
4312	1	0	0	1	0	1	0	1	1	0	0	0

Table-2: The transition matrix for all regions of Brid arrangement

2314	3214	3124	1324	1234	2134	4132	4123	4213	4231	4321	4312
1	0	1	0	0	0	0	0	0	1	1	1
0	0	0	1	0	1	1	1	1	0	0	0
1	1	0	0	0	1	0	0	1	0	1	0
0	0	1	0	1	0	0	1	1	1	0	0
1	0	0	0	1	0	0	1	1	1	0	0
0	1	0	1	0	0	1	0	0	0	1	1
0	0	0	1	1	1	1	0	1	0	0	0
1	1	1	0	0	0	0	0	0	1	0	1
0	0	1	0	1	0	1	1	0	0	0	1
0	1	0	0	0	1	0	0	1	1	1	0
0	1	1	1	0	0	1	0	0	0	1	0
1	0	0	0	1	1	0	1	0	1	0	0
3	1	1	0	1	1	0	0	0	1	0	0
1	3	1	1	0	1	0	0	0	0	1	0
1	1	3	1	1	0	0	0	0	0	0	1
0	1	1	3	1	1	1	0	0	0	0	0
1	0	1	1	3	1	0	1	0	0	0	0
1	1	0	1	1	3	0	0	1	0	0	0
0	0	0	1	0	0	3	1	1	0	1	1
0	0	0	0	1	0	1	3	1	1	0	1
0	0	0	0	0	1	1	1	3	1	1	0
1	0	0	0	0	0	0	1	1	3	1	1
0	1	0	0	0	0	1	0	1	1	3	1
0	0	1	0	0	0	1	1	0	1	1	3

3.3. The eigenvalues of transition matrix for Braid arrangement

The intersection for the Braid arrangement with the sphere S^2 is empty. Therefore the Braid arrangement cannot consider as a sphere, and hence the eigenvalues for the transition matrix when n = 4 cannot be calculated using the account eigenvalues laws as in [10].So the following eigenvalues are found by using Matlab program.

13.0409, 5.9191, 0.0961, 0.9440, 6.1623, 6.1623, 6.1623, 1.0000, 1.0000, 1.0000, 1.0000, 6.1623, 6.1623, -0.1623, -0.1623, 3.0000, 3.0000, 3.0000, 3.0000, 3.0000, 3.0000, -0.1623, -0.1623, -0.1623

4. THE STATIONARY DISTRIBUTION ON THE REGIONS

Definition 4.1 [11]: Let $P = \mathcal{P}_{ij}$ be the transition matrix, then the vector π is called stationary distribution for P if $\forall\, j \in W \subset N$ (the set of natural number) it satisfies:
$0 \leq \pi_j \leq 1$.
$\sum_{j \in W} \pi_j = 1$.
$\pi_j = \sum_{i \in W} \pi_i \mathcal{P}_{ij}$.

This equation in matrix notation is $\pi = \pi P$, where $\pi = (\pi_i : i \in W)$ is a row vector.

One way to compute the stationary distribution for P is to solve the linear equations

$$\pi.P = P .$$

For example, let $P = \begin{bmatrix} 6/10 & 1/10 & 3/10 \\ 1/10 & 7/10 & 2/10 \\ 2/10 & 2/10 & 6/10 \end{bmatrix}$, since we know that $\pi.P = P$,

We can write this as a series of equations:

$$6/10\, \pi_1 + 1/10\, \pi_2 + 2/10\, \pi_3 = \pi_1$$
$$1/10\, \pi_1 + 7/10\, \pi_2 + 2/10\, \pi_3 = \pi_2$$
$$\pi_1 + \pi_2 + \pi_3 = 1.$$

Now by solving these equations we get that $\pi \approx (0.2759, 0.3448, 0.3793)$.

In case the random walk on regions the stationary distribution π of the region N represents the chance that the random walk is in E after a large number of steps from any starting region E'.

4.2. The stationary distribution of Braid arrangement

The confusion we are having is due to the fact that in Figure (1), the equator (dotted circle) is not a plane in the braid arrangement, so the regions touching the equator all continue on the bottom (for example the rest of the region labeled 3421 (on the bottom) is antipodal to 1243 on the top. (The entire picture on the bottom is antipodal to the one on the top, so there are 24 complete triangular regions in all).Thus all regions are bounded by 3 sides and so the stationary distribution is uniform (probability=1/24 for each region). This follows from(proposition.1,[8]).

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