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ERROR ESTIMATE 0F THE FINITE FOURIER TRANSFORM BY THE AVERAGED MODULUS OF SMOOTHNESS

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ABSTRACT

In this paper we will show how we can estimate the error for the Finite Fourier Transform by Averaged modulus of smoothness, for integrable 1-periodic functions defined on aclosed interval and prove results.

1. INTRODUCTION

First, we will define a set of complex sequences of lenght *N+1* by

$$S_{N=}\{C_0,C_1,\ldots\ldots C_n : C_j \in C \quad , \quad j=0,1,\ldots\ldots N\} \tag{1.1}$$

For every $N \in N$, the finite Fourier transform (FFT) is the map from S_N to itself defined by

$$F_{N(\langle C_j \rangle)_k} = \frac{1}{N+1}\sum_{j=0}^{N} c_j \, e^{\frac{-2\pi ijk}{N+1}} \tag{1.2}$$

This sequence is periodic of period *N+1*, it can be extended to all $K \in Z$, it is natural to think $k_{min=-\left[\frac{N+1}{2}\right]}$ to $k_{max=\left[\frac{N}{2}\right]}$

The inverse of (*FFT)* is given as

$$F_N^{-1}(\langle c_j \rangle) = \sum_{k=kmin}^{kmax} c_j \, e^{\frac{2\pi ijk}{N+1}} \tag{1.3}$$

Aprincipal application of the *(FFT*) is to approximately compute samples of the Fourier transform of a function If f is a function defined on [0, 1] then we define

$$\tilde{f}_{N,k} = \frac{1}{N+1}\sum_{j=1}^{N} f\left(\frac{j}{N+1}\right) e^{\frac{-2\pi ijk}{N+1}} \tag{1.4}$$

The sum on the right-hand side of equation (1.4) is Riemann sum for the integral defining the k^{th} Fourier cofficient

$$\hat{f}(k) = \int_o^1 f(x)e^{-2\pi ikx}\, dx \tag{1.5}$$

1. Charles L.Epsten estimat the error of the (*FFT*) by modulus of continuity for continuous and continuous derivatives 1-periodic function on closed interval
2. Auslander and Grunbaum estimate the error of best approximation of (*FFT*) in terms of the sampling models and the frequency but are independent of the function
3. Ivanov obtained some results about approximation of measurable and bounded functions by bernstein polynomials in L_p [0, 1] space
4. Hirstov used a locally global norm for bounded functions and proved that the best one-sided approximations of 2π-periodic bounded function with trigonometric polynomials of degree *n* in the norm L_P

2. SOME USEFUL INFORMATION

Let *f* be an integrable function .on [0, 1] then, we define the Averaged modulus of smoothness

$$\tau_{k\,(f,\delta)_p} = \left\|\omega_{k(f,.;\delta)}\right\|_{L_p} \tag{2.1}$$

$$= \left[\int_a^b \left(\omega_k(f,x;\delta)\right)^p\right]^{\frac{1}{p}}$$

where

$$\omega_k\ (f,x;\delta) = sup\left\{\left|\Delta_h^k f(t)\right| : t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right] \cap [a,b]\right\}$$

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And

$$\Delta_h^k f(t) = \sum_{M=0}^{k} (-1)^{M+k} \binom{k}{m} f(t+Mh)$$

Such that

$$\binom{k}{m} = \frac{k!}{m!(k-m)!}$$

Which called local moduli of smoothness.

An exponential polynomial of order *M* is a function of the form

$$P(x) = \sum_{-M}^{M} \alpha_k \; e^{2\pi ikx} \qquad \alpha_k \epsilon\, C \tag{2.2}$$

We denote the set of such functions by T_M,if f is an integrable,1-periodic function defined on $[0,1]$,then for each $M \in N$, there is a function $p^* \in T_M$ that is a best approximation to f such that

$$\|f - p^*\|_{p= \min_{p\in T_M} \|f-p\|_p} \tag{2.3}$$

***Proposition* 2.1** [3]: If *f* is an integrable on$[0, 1]$, then for each integer $n \geq 1$, we have

$$\text{F(x)} = P_{n-1}(x) + \varphi_n(f;x) + \sum_{j=0}^{n} \frac{1}{h} \int_0^t \varphi_n\,(f; jh+v) l'_{n,j} \left(\frac{x-v}{h}\right) dv \tag{2.4}$$

where P_{n-1} is a polynomial of degree at most *n-1*

***Proposition* 2.2 [3]:** Let $P_{n-1}^*(f)$ be the interpolation polynomial for f at the point h, 2h,........,nh
i.e

$$P_{n-1}^*(f;x) = \sum_{j=1}^{n} f(ih) l_{n-1,j-1}\left(\frac{x}{h} - 1\right) \tag{2.5}$$

where

$$l_{n,j}\left(\frac{x}{h}\right) = l_{n-1,j-1}\left(\frac{x}{h} - 1\right) + (-1)^{n-j} \binom{n}{j} l_{n,0}\left(\frac{x}{h}\right)$$

Then

$$f(x) - P_{n-1}^*(f;x) = \Delta_h^n f(0) l_{n,0}\left(\frac{x}{h}\right) + \varphi_n(f;x) - \sum_{j=0}^{n} \varphi_n(f;jh) l_{n,j}\left(\frac{x}{h}\right) + h^{-1} \int_o^t \sum_{j=o}^{n} \varphi_n(f;jh+v)\, l'_{n.j}\left(\frac{x-v}{h}\right) dv \tag{2.6}$$

Lemma 2.1 [3]: We have

$$y_n := max\left\{\sum_{j=1}^{n} \binom{n}{j}^{-1} \left|l_{n,j}(x)\right| : 0 \leq x \leq 1\right\} = 1 \tag{2.7}$$

Lemma 2.2:

$$\mu_{n,v} := \sum_{j=o}^{n} \binom{n}{j}^{-1} max\left\{\left|l_{n,j}(x)\right| : v \leq x \leq v+1\right\} \tag{2.8}$$

Then

$$\mu_{n,v} \leq \frac{1+\sigma_v+\sigma_{v+1}}{\binom{n}{v}} \tag{2.9}$$

V = 0, 1, 2,........$\frac{n-1}{2}$
where

$$\sigma_v := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{v} \qquad \sigma_0 := 0$$

In particular, for v = 0, we have

$$\mu_{n,o} = 1 + \sum_{j=1}^{n} \binom{n}{j}^{-1} max\left\{\left|l_{n,j}(x)\right| : o \leq x \leq 1\right\} \leq 2 \tag{2.10}$$

Proposition 2.3 [3]: For any function f which is integrable on [0, 1] we have

$$\left|f(x) - \sum_{j=1}^{n} f(jh) l_{n-1,j-1}\left(\frac{x}{h} - 1\right)\right| \leq \frac{6+7min\,(\sigma_v,\sigma_{n-v})}{\binom{n}{v}} \tau_n(f;h) \tag{2.11}$$

For x ∈ $[vh, (v+1)/h$, h:=1/(n+1), v = 0, 1,.....,n $\sigma_v := 1 + \frac{1}{2} + \cdots + \frac{1}{v}$

$$\sigma_0 := 0$$

3. THE MAIN RESELTS

In this section we state and prove our reselts for an integrabal 1-periodic function

Theorem 3.1: If $f(x)$ be an integrable, 1-periodic function on $[0,1]$ then the best approximation p^* to f from T_M satisfies

$$\|f - p^*\|_p \leq 6\, \tau_{f\left(\frac{1}{2\pi M}\right)} \tag{3.1}$$

Proof: Since the interpolation polynomial

$$P_{n-1}^{*}(f;x)=\sum_{j=1}^{n} f(jh)l_{n-1,j-1}\left(\frac{x}{h}-1\right)$$

In (2.12) are symmetric with respect to the middle of the interval [0, 1], it is sufficient to prove it only for x ∈ $\left[0,\frac{1}{2}\right]$ or for v = 0,......., [(n-1)/2]. For v = 0, from lemma (2.1) and (2.2) using

$$\| f(x)-p_{n-1}^{*}(f,x)\|=\left\|\Delta_{h}^{n} f(0)l_{n,0}\left(\frac{x}{h}\right)+\varphi_{n}(f;x)-\sum_{j=o}^{n}\varphi_{n}(f;jh)\left(\frac{x}{h}\right)+h^{-1}\int_{0}^{t}\sum_{j=0}^{n}\varphi_{n}(f;jh+v)l_{n,j}^{\prime}\left(\frac{x-v}{h}\right)dv\right\|$$

Using

$$\varphi_{n}(f;x)=\varphi_{n}(f;vh+t$$
$$=\frac{(-1)^{n-v}}{h\binom{n}{v}}\int_{0}^{h}\Delta_{y}^{n} f(x-vy)dy$$

$$\|\varphi_{n}(f;x)\|_{P}\leq\frac{1}{\binom{n}{v}}\tau_{n}(f;h)$$

To obtain

$$\|f(x)-P_{n-1}^{*}(f;x)\|_{P}\leq\left\|\tau_{n}(f;h)\left[\max_{0\leq t\leq 1}\left|l_{n,0}(t)\right|+1+\max_{0\leq t\leq 1}\sum_{j=0}^{n}\frac{1}{\binom{n}{j}}\left|l_{n,j}(t)\right|+\sum_{j=0}^{n}\frac{1}{\binom{n}{j}}\int_{0}^{t}\left|l_{n.j}^{\prime}(v)\right|dv\right]\right\|_{P}$$

By using (2.10) we have

$$\|f(x)-P_{n-1}^{*}(f;x)\|_{P}\leq 6\tau_{n}(f;h)$$

Corollary 3.1: If f is an integrable, 1-periodic function on [0,1] with l integrable 1-periodic dervatives ,then the approximation P^* to f from T_M satisfies

$$\|f-p^{*}\|_{P}\leq 6^{l+1}\frac{\tau_{f}^{l}\left(\frac{1}{2\pi M}\right)}{(2\pi M)^{l}} \tag{3.2}$$

Now, we can state and prove our results for integrable 1-periodic function.

Theorem 3.2: for $M\in N$ and f, an integrable 1-periodic function defined on [0, 1], we have estimates

$$\left|\tilde{f}_{2M,k}-\hat{f}(k)\right|\leq 2\|p^{*}-f\|_{P}\qquad for\ |k|\leq M \tag{3.3}$$

where p* ∈ T_M

Proof: Since $p^*\in T_M$, a simple calculation shows that for $|k|\leq M$ we have

$$\left|\tilde{f}_{2M,k}-\tilde{p}_{2M,k}^{*}+\tilde{p}_{2M,k}^{*}-\hat{f}_{(k)}\right|$$

Then

$$\left|\tilde{f}_{2M,k}-\tilde{p}_{2M,k}^{*}\right|=\left[\left|\int_{0}^{1}\left(\frac{1}{2M+1}\sum_{j=0}^{2M}\left[f\left(\frac{j}{2M+1}\right)-p^{*}\left(\frac{j}{2M+1}\right)\right]e^{\frac{-2\pi ijk}{2M+1}}\right)^{p}dx\right|\right]^{\frac{1}{P}}$$
$$\leq\left[\int_{0}^{1}\left(\left|\frac{1}{2M+1}\sum_{j=0}^{2M}\left[f\left(\frac{j}{2M+1}\right)-p^{*}\left(\frac{j}{2M+1}\right)\right]e^{\frac{-2\pi ijk}{2M+1}}\right|\right)^{P}dx\right]^{\frac{1}{P}}$$

$$\left|\tilde{f}_{2M,k}-\tilde{p}_{2M,k}^{*}\right|\leq\|f-p^{*}\|_{P}$$

In the second line we use the triangle inquality.

$$\left[\left|\int_{0}^{1}\left(\int_{0}^{1}[f(x)-p^{*}(x)]\,e^{-2\pi ikx}\right)^{P}dx\right|\right]^{\frac{1}{P}}\leq\left[\int_{0}^{1}\left|\left(\int_{0}^{1}[f(x)-p^{*}(x)]e^{-2\pi ikx}\right)^{P}\right|dx\right]^{\frac{1}{P}}$$

$$\left|\hat{f}(k)-\tilde{p}_{2M,k}\right|\leq\|f-p^{*}\|_{P}$$

Then

$$\left|\tilde{f}_{2M,k}-\tilde{p}_{2M,k}^{*}+\hat{p}_{2M,k}^{*}-\hat{f}_{(k)}\right|\leq\left|\tilde{f}_{2M,k}-\tilde{p}_{2M,k}^{*}\right|+\left|\hat{f}_{(k)}-\hat{p}_{2M,k}^{*}\right|$$

$$\left|\tilde{f}_{2M,k}-\hat{f}\ (k)\right|\leq\|f-p^{*}\|_{P}+\|f-p^{*}\|_{P}$$
$$\leq 2\|p^{*}-f\|_{P}$$

Corollary 3.2: suppose that f is an integrabal 1-periodic function with $l\geq 0$ an integrabal 1-periodic derivatives; then

$$\left|\tilde{f}_{2M,k}-\hat{f}(k)\right|\leq 12\frac{6^{l}\tau_{f}^{l}\left(\frac{1}{2M\pi}\right)}{(2\pi M)^{l}} \tag{3.4}$$

For $|k|\leq M$

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Proof: From theorem (3.2)

$$\left|\tilde{f}_{2M,k}-\hat{f}(k)\right|\le 2\|p^*-f\|_P$$

Then from corollary (3.2)

$$\|f-p^*\|_P\le 6^{l+1}\ \frac{\tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l}$$

Then

$$\left|\tilde{f}_{2M,k}-\hat{f}(k)\right|\le 2.\,6^{l+1}\ \frac{\tau^l_{f}\left(\frac{1}{2M\pi}\right)}{(2M\pi)^l}$$

$$\le 12\,\frac{6^l\tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l}$$

Corollary 3.3: suppose that *f* is an integrable 1-periodic function then

$$\left|\tilde{f}_{2M,k}\right|\le\left|\hat{f}(k)\right|+6\tau_f\left(\frac{1}{2\pi M}\right) \qquad (3.5)$$

Proof: by theorem (3.2)

$$\left|\tilde{f}_{2M,k}-\hat{f}(k)\right|\le 2\|f-p^*\|_P$$

$$\left|\tilde{f}_{2M,k}-\hat{f}(k)\right|\le\left|\tilde{f}_{2M,k}-\tilde{p}^*_{2M,k}+\tilde{p}^*_{2M,k}-\hat{f}(k)\right|$$
$$\le\left|\tilde{f}_{2M,k}-\tilde{p}^*_{2M,k}\right|+\left|\tilde{p}^*_{2M,k}-\hat{f}(k)\right|$$

$$\left|\tilde{f}_{2M,k}-\tilde{p}^*_{2M,k}\right|\le\ \|f-p^*\|_P$$

Then from theorem (3.1)

$$\|f-p^*\|_p\ \le 6\ \tau_f\ \left(\frac{1}{2\pi M}\right)$$

Then

$$\left|\tilde{f}_{2M,k}\right|\le\ \|f-p^*\|_P+\left|\hat{f}(k)\right|$$
$$\le 6\ \tau_f\ \left(\frac{1}{2\pi M}\right)+\left|\hat{f}(k)\right|$$

Theorem (3.3) there is a universal constant *C* such that, if *f* is an integrabal 1-periodic function, then

$$\left|\tilde{f}_{2M(x)}-f_{2M(x)}\right|\le C\log M\,\|f-p^*\|_P$$

Where $p^*\ \in T_M$ is a best approximation.

Proof: Let Dirichlet kernel

$$f_{2M}(x)=\int_0^1 D_M\,(x-y)f(y)dy$$

Then

$$\tilde{f}_{2M}(x)=\frac{1}{2M+1}\sum_{j=0}^{2M}D_M\left(x-\frac{j}{2M+1}\right)f\left(\frac{j}{2M+1}\right)$$

We observe that

$$\left|\tilde{f}_{2M(x)}-f_{2M(x)}\right|\le\left|\tilde{f}_{2M(x)}-p^*(x)\right|+\left|p^*(x)-f_{2M(x)}\right|$$
$$\le\left[\left|\int_0^1\left(\frac{1}{2M+1}\sum_{j=0}^{2M}D_M\left(x-\frac{j}{2M+1}\right)\left[f\left(\frac{j}{2M+1}\right)-p^*\left(\frac{j}{2M+1}\right)\right]+\int_0^1 D_M\,(x-y)[p^*(y)-f(y)]dy\right)^P dx\right|\right]^{\frac{1}{P}}$$
$$\le\left[\int_0^1\left|\left(\frac{1}{2M+1}\sum_{j=0}^{2M}D_M\left(x-\frac{j}{2M+1}\right)\left[f\left(\frac{j}{2M+1}\right)-p^*\left(\frac{j}{2M+1}\right)\right]+\int_0^1 D_M\,(x-y)[p^*(y)-f(y)]dy\right)^P\right|\quad dx\right]^{\frac{1}{P}}$$
$$\le\|f-p^*\|_p\,.\left[\frac{1}{2M+1}\sum_{j=0}^{2M}\left|D_M\left(x-\frac{j}{2M+1}\right)\right|+\int_0^1\left|D_M(x-y)\right|dy\right]$$

Both the sum and the integral in the last line are bounded by a constant times log M

Corollary 3.4: if *f* is an integrabal 1-periodic function whose averaged modulus satisfies

$$\tau_f(\delta)=o(|\log\delta|^{-1}) \qquad (3.7)$$

Then the (FFT) partial sum ($\tilde{f}_{2M}$) converges to f on [0, 1] if f has l integrabal periodic derivatives and τ_f^l satisfies the above estimate, then for each 1≤ $j\le l$,the sequence $\langle\partial_x^i\tilde{f}_{2M}\rangle$ converges to ∂_x^j f

Proof: let p*∈ T_M be a best approximation to f, we will use the triangle inquality to conclude that

$$\left|\tilde{f}_{2M(x)}-f_{2M(x)}\right|\le\left|\tilde{f}_{2M(x)}-p^*(x)\right|+\left|p^*(x)-f_{2M(x)}\right|$$

Applying theorem (3.1), we see that

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \le (C \log M + 6)\tau_f\left(\frac{1}{2M\pi}\right) \tag{3.8}$$

The estimate in (3.7) implies the right-hand side of equation (3.8) tends to zero as M tends to infinity. The last inquality tell us that $\left\|\tilde{f}_{2M(x)} - f_{2M(x)}\right\|$ tends to zero as M tends to infinity

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REFERENSES

1. Charles L-Epsten "How Well Does the Finite Fourier Transform Approximate the Fourier Transform" (2004) Wiley periodicals,Inc.
2. Auslender,L.; Grunbaum, F.A."The Fourier transform and discre e Fourier transform”. Inverse problems 5 (1989), 149-164.
3. Popov,V.A (1981) "Averaged local moduli and their applications in approximation and functions spaces", PWN-polish scientific publishers, Warsaw
4. Ivanov, K.G.(1984), "Approximation by Bernstein polynomials in L_p-metric constructive theory", Bulg. Acad. of Sci, 421-429.
5. Hristov, V.H. (1989), "Best one-sided approximation and mean approximation by interpolation polynomials of periodic functions" Math.Balk, New Series, 3, 418-428.

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