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ROOT MULTIPLICITIES FOR A CLASS OF QUASI AFFINE GENERALIZED KAC MOODY ALGEBRAS $QAGGA_2^{(1)}$ OF RANK 4

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ABSTRACT

In this paper the concept of quasi affine generalized Kac Moody algebras (GKM) $QAGGA_2^{(1)}$ are defined; the general form of connected Dynkin diagrams associated with this particular class of $QAGGA_2^{(1)}$ are classified. Then, root multiplicities for some families of quasi affine generalized Kac Moody algebras are computed; First, we consider a general family $QAGGA_2^{(1)}$ of symmetrizable Generalized Generalized Cartan Matrices (GGCM) of quasi affine type, with one imaginary simple root, of rank 4 which are obtained from the affine GCM $A_2^{(1)}$ of rank 3. The real simple and imaginary simple roots for 3 different classes under this family $QAGGA_2^{(1)}$ are explicitly computed; Finally the root multiplicities of all roots of GKM algebras $QAGGA_2^{(1)}$ associated with these Borcherds Cartan matrices of order 4, which are obtained as extensions of the affine family $A_2^{(1)}$ are then determined.

Key words: *Generalized Kac Moody algebras, real, imaginary roots, quasi affine Kac Moody algebras, highest weight module, root multiplicity.*

AMS Classification: *17B67.*

I. INTRODUCTION

The Generalized Kac Moody algebras (GKM algebras, shortly) was introduced by Borcherds in [3]. Almost all the results that are true for Kac Moody algebras (KM algebras in short) can be generalized to the GKM algebras also; The main difference in the structure of both, are the existence of imaginary simple roots in GKM algebras. Determination of the dimensions of the root spaces of GKM algebras explicitly, is an interesting open problem. Using the basic structure and representation theory of Kac Moody algebras, the study can be extended to GKM algebras; Sthanumoorthy, Lilly and Uma Maheswari computed root multiplicities for some extended hyperbolic Generalized Kac Moody algebras in [23]. Sthanumoorthy *et.al* determined the root properties for some GKM algebras, whose Generalized Cartan matrix (GGCM) is obtained by extending the KM GCM of finite, affine and hyperbolic types and determined the root multiplicities([17]-[22]).

In [29] and [30], root structure and root multiplicity of a special GKM algebra EB_2 was obtained.
Multiplicities of simple imaginary roots for the Borcherds algebra $gII_{9,1}$ was given in [1] by Barwald and Gebert. A closed form root multiplicity formula for the roots of GKM algebras was obtained by Kang and *et.al* ([11]-[14]). Applications of the dimension formula to various classes of graded Lie algebras were discussed in [14] and the dimension formula was also derived in [6] – [8]. In [28], Uma Maheswari defined quasi affine Kac Moody algebras, which belong to the indefinite class of Kac Moody algebras and studied the structure of a specific family in the quasi affine type $QAC_2^{(1)}$. The Extended hyperbolic type of indefinite Kac Moody algebras were introduced by Sthanumoorthy and Uma Maheswari and the structure of $EHA^{(1)}{}_1$ and $EHA^{(2)}{}_2$ were determined in ([24]-[27]).

In this paper we extend the definition of quasi affine KM algebras to quasi affine GKM algebras; The general form of connected Dynkin diagrams associated with the symmetrizable GKM algebras are classified for $QAGGA_2^{(1)}$. We consider a general family $QAGGA_2^{(1)}$ of GGCM of quasi affine type, with one imaginary simple root, of rank 4 which are obtained from the affine GCM $A_2^{(1)}$ of rank 3,. We identify the real simple and imaginary simple roots, for 3 different classes under this family $QAGGA_2^{(1)}$. Finally we determine explicitly the root multiplicities of all roots of GKM algebras associated with the Borcherds Cartan matrices of order 4, which are obtained as extensions of the affine family $A_2^{(1)}$.

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2. PRELIMINARIES

Let us first see the basic definitions and results of GKM algebras (For a detailed study on Kac Moody and Generalized Kac Moody algebras, root multiplicities one can refer to [2], [3], [5], [7], [9], [13]).

Definition 2.1:[3] A Borcherds Cartan matrix (BKM) is a real matrix $A = (a_{ij})_{i,j=1}^{n}$ satisfying the conditions:

i) a_{ij} = 0 or $a_{ij} \le 0$ for all i ε I ii) $a_{ij} \le 0$ for i≠j, a_{ij} ε Z if a_{ii} = 2 , iii) a_{ij} = 0 implies a_{ji} = 0.

We assume that the BKM is symmetrizable. Let the real simple and real imaginary roots be denoted by I^{re} = {i ε I / a_{ii} =2} and I^{im} = {i ε I / $a_{ii} \le 0$}; Let the charge $\underline{m}$ = { m_i ε $Z_{>0}$ / i ε I} be a collection of positive integers such that m_i =1 for all i ε I^{re}.

The Generalized Kac-Moody algebra g=g(A, $\underline{m}$) associated with a symmetrizable BKM matrix A = $(a_{ij})_{i,j=1}^{n}$ of charge $\underline{m}$ = (m_i / i ε I} is the Lie algebra generated by the elements h_i, d_i, e_{ik}, f_{ik}, i ε I ,k=1,…, m_i with the following defining relations :

$[h_i, h_j] = [d_i, d_j] = [h_i, d_j] = 0$, $[h_i, e_{jl}] = a_{ij}e_{jl}$, $[h_i, f_{jl}] = -a_{ij}f_{jl}$,

$[d_i, e_{jl}] = \delta_{ij} e_{jl}$, $[d_i, f_{jl}] = -\delta_{ij} f_{jl}$, $[e_{ik}, f_{jl}] = \delta_{ij}\delta_{kl} h_i$

$$(ad\ e_{ik})^{(1-a_{ij})}(e_{jl}) = (ad\ f_{ik})^{(1-a_{ij})}(f_{jl}) = 0 \ \ if \ \ a_{ii} = 2, i \ne j,$$

$$[e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0 \ \ if \ \ a_{ij} = 0, (i, j \in I, k = 1,..., m_i, l = 1,..., m_j).$$

The subalgebra h = $(\oplus Ch_i) \oplus (\oplus Cd_i)$ is called the Cartan subalgebra of g. For each i ε I, we define a linear functional α_i ε h*, by α_i (h_j) = a_{ij}, α_i (d_j) = $a_{ij}\delta_{ij}$,i,j εI . α_i's are called the simple roots of g.

The GKM algebra g = g(A, $\underline{m}$) has the root space decomposition

$$g = \bigoplus_{\alpha \in Q} g_\alpha, \text{ where } g_\alpha = \{x \in g / [h, x] = \alpha(h)x,\ for\ all\ h \in H\}.$$

Let $Q_+ = \sum_{i=1}^{n} Z_+ \alpha_i$. Q has a partial ordering " $\le$ " on h* defined by $\alpha \le \beta$ if $\beta - \alpha \in Q_+$, where $\alpha, \beta \in$ Q.

Definition 2.2: [9] Define $r_i \in$ End (H^*) as $r_i(\alpha) = \alpha - \langle \alpha_i^v, \alpha \rangle \alpha_i$ where $\langle \alpha_i^v \alpha \rangle = \alpha(\alpha_i^v)$ and i∈ I^{re}. For each i, r_i is an invertible linear transformation of H^* and r_i is called a fundamental reflection. Define the Weyl group W to be the subgroup of aut(H^*) generated by {r_i,i∈ I^{re}}. Let $\Delta (= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = \Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$. Let $\rho \in h^*$ be a linear functional, with ρ (h_i) = $a_{ii}/2$ for all i∈ I.

Definition 2.3: [9] In Kac Moody algebras the Dynkin diagrams are defined as follows: To every GGCM A is associated a Dynkin diagram S(A) defined as follows: S(A) has n vertices and vertices i and j are connected by max {$|a_{ij}|$,$|a_{ji}|$} number of lines if $a_{ij}.a_{ji} \le 4$ and there is an arrow pointing towards I if $|a_{ij}|$>1. If $a_{ij}.a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair ($|a_{ij}|$,$|a_{ji}|$) of integers. For GKM algebras, in addition we have the following: If a_{ii} = 2, i^{th} vertex will be denoted by a white circle and if a_{ii} = 0, i^{th} vertex will be denoted by a crossed circle. If a_{ii} = -k, k > 0, i^{th} vertex will be denoted by a white circle with –k written above the circle within the parenthesis.

Let P^+ = {λ ε h*/ λ(h_i) ≥ 0 for all i ε I, λ(h_i) is a positive integer if a_{ii} = 2}. Let V be the irreducible highest weight module over G with the highest weight λ. Let T be the set of all imaginary simple roots counted with multiplicities.

For F $\subset$T, we write F $\perp$ T if λ(h_i) = 0 for all α_i ε F. For J $\subset I^{re}$, $\Delta_J = \Delta \cap (\sum Z\alpha_i), \Delta^{\pm}_J = \Delta_J \cap \Delta^{\pm}$, $\Delta^{\pm}(J) = \Delta^{\pm} \setminus \Delta_J^{\pm}$.

$$Q_J = Q \cap (\sum Z\alpha_i), Q^{\pm}_J = Q_J \cap Q^{\pm}, Q^{\pm}(J) = Q^{\pm} \setminus Q^{\pm}_J .$$

We define

$g_0^{(J)} = h \oplus (\underset{\alpha\in\Delta_J}{\oplus}), g_{\pm}^{(J)} = \underset{\alpha\in\Delta^{\pm}(J)}{\oplus} g_\alpha$. We get the triangular decomposition:

$g = g_-^{(J)} \oplus g_0^{(J)} \oplus g_+^{(J)}, where\ g_0^{(J)}$ is the Kac Moody algebra associated with the GCM $A_J = (a_{ij})_{i,j\varepsilon J}$.

$g_-^{(J)}\ and\ g_+^{(J)}$ represent the direct sum of irreducible highest weight and lowest weight modules respectively over $g_0^{(J)}$; $W_J = \langle r_j / j\varepsilon J\rangle$ be the subgroup of W generated by the simple reflections.

Let W(J) ={ w ε W/ w $\Delta^- \cap \Delta^+ \subset \Delta^+(J)$.

Proposition 2.4: [13] [16] $H_k^{(J)} = \underset{\substack{w\in W(J)\\ F\subset T\\ l(w)+|F|=k}}{\oplus} V_J(w(\rho - s(F) - \rho)$

where $V_J(\mu)$ denotes the irreducible highest weight module over $g_0^{(J)}$ with highest weight μ ; s(F) denotes the sum of elements in F. Let the homology space $H^{(J)} = \sum_{\substack{w\in W(J)\\ F\subset T\\ l(w)+|F|\geq 1}} (-1)^{l(w)+|F|+1} V_J(w(\rho - s(F)) - \rho)$; $P(H^{(J)})=\{\alpha\ \varepsilon\ Q^-(J) / \dim H_\alpha \neq 0\}$

with d(i)=dim $H_{\tau_i}^{(J)}$ for i=1,2,… Let $T^{(J)}(\tau) = \{n=(n_i)_{i\geq 1}/n_i\ \varepsilon Z_{\geq 0}, \Sigma\, n_i \tau_i = \tau\}$; Let $|n|=\Sigma\, n_i$ and define the Witt partition function $W^{(J)}(\tau) = \sum_{n\in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod (d(i))^{n_i},\ for\ \tau\in Q^-(J)$.

The following theorem gives the closed form root multiplicity formula for all symmetrizable GKM algebras

Theorem 2.5: [11][12][13] Let α ε $\Delta^-(J)$ be a root of a symmetrizable GKM algebra g. Then dim $g_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) W^{(J)}(\alpha/d) = \sum_{d|\alpha}\frac{1}{d}\mu(d) \sum_{n\in T^{(J)}(\alpha/d)} \frac{(|n|-1)!}{n!}\prod d(i)^{n_i}$ where μ is the classical Mobius function.

The Kostants formula given by Liu in [15] was repeatedly used by Kang in determining the root multiplicities ([12,[13]).

Proposition 2.6: [7][13] Suppose that a Borchers-Cartan matrix $A=(a_{ij})_{i,j=1}^n$ of charge $\underline{m} = (m_i\ \varepsilon\ I^{re} / i\ \varepsilon\ I\}$ satisfies : i)I^{re} is finite, ii) $a_{ij}\neq 0$ for all i,j in I^{im}. Let $J = I^{re}$ and the corresponding decomposition of the GKM algebra

g=g(A, $\underline{m}$) = $g_-^{(J)} \oplus g_0^{(J)} \oplus g_+^{(J)}$. Then the algebra $g_-^{(J)} = \underset{\alpha\in\Delta^-(J)}{\oplus} g_\alpha (respectively, g_+^{(J)} = \underset{\alpha\in\Delta^+(J)}{\oplus} g_\alpha)$ is isomorphic to the free Lie algebra generated by the space $V = \underset{i\in I^{im}}{\oplus} V_J(-\alpha_i)^{\oplus m_i} (respectively, V^* = \underset{i\in I^{im}}{\oplus} V_J^*(-\alpha_i)^{\oplus m_i})$ where $V_J(\mu)$ (resp. $V_{J^*}(\mu)$) denotes the irreducible highest weight (resp. lowest weight) module over the Kac Moody algebra $g_o(J)$ with highest weight μ (resp. lowest weight - μ). With these assumptions, the GKM algebra g= $g_-^{(J)} \oplus g_0^{(J)} \oplus g_+^{(J)}$ is isomorphic to the maximal graded Lie algebra with local part $V \oplus g_0^{(J)} \oplus V^*$.

Theorem 2.7:[4][8][10] Let $V(\Lambda_0)$ be the basic representation of the affine Kac-Moody algebra $A_n^{(1)}$, and let λ be a weight of $V(\Lambda_0)$. Then, $\dim(V(\Lambda_0))\lambda = p^{(n)}\left(1 - \frac{(\lambda,\lambda)}{2}\right)$, where the function $p^{(n)}(m)$ are defined by $\sum_{m=0}^{\infty} p^{(n)}(m) q^m = \frac{1}{\phi(q)^n} = \frac{1}{\prod_{j\geq 1}(1-q^j)^n}$

Remark 2.8: The above formula can be restated as $\dim(V(\Lambda_0))\lambda = p^{(n)}(m)$, where $\lambda=\Lambda_0 - m\delta$, Λ_0 is the highest weight, δ is the null root and $m\in Z_+$

Definition 2.9: [28] Let $A = (a_{ij})_{i,j=1}^n$, be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S(A) to be of Quasi Affine (QA) type if S(A) has a proper connected sub diagram of affine type with n-1 vertices. The GCM A is of QA type if S(A) is of QA type. We then say the Kac-Moody algebra g(A) is of QA type.

3. ROOT MULTIPLICITIES FOR A CLASS OF QAGGA$_2^{(1)}$ WITH ONE IMAGINARY SIMPLE ROOT WHOSE ASSOCIATED GGCM IS $\begin{pmatrix} -k & -a & 0 & 0 \\ -a & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$ **where k, a are non negative integers.**

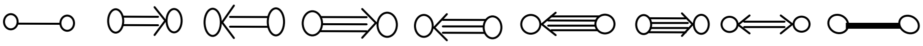
We first define the class of generalized Kac Moody algebras of quasi affine type.

Definition 3.1: We say a GGCM $A = (a_{ij})_{i,j=1}^{n}$ is of Quasi Affine type if A is of indefinite type and the Dynkin diagram associated with A has a connected, proper sub diagram of affine type , whose GCM is of order n-1. We then say the associated Dynkin diagram and the corresponding GKM algebra to be of quasi affine type.

Note that the GGCM of extended hyperbolic type forms a subclass of this quasi affine type and not every quasi affine GGCM is of extended hyperbolic type.

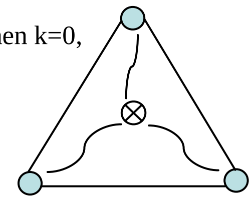
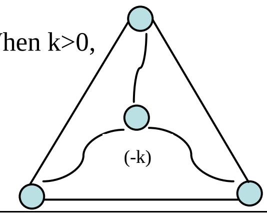
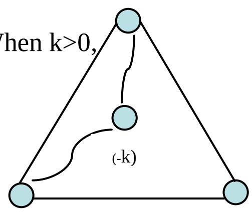
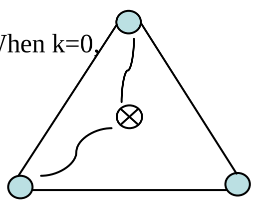
Proposition 3.2: The general form of connected non-isomorphic Dynkin diagrams associated GGCM, of quasi affine type QAGGA$_2^{(1)}$ can be classified as in the following Table 1 and any Dykin diagram of QAGGA$_2^{(1)}$ is one of the types of 438 Dynkin diagrams listed in Table 1.

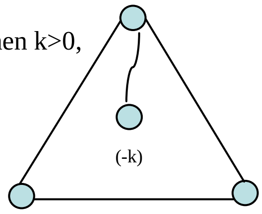
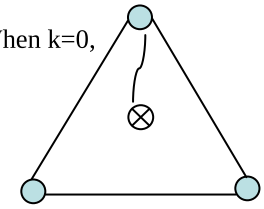
Proof: We start with the affine Dynkin diagram $A_2^{(1)}$ of Kac Moody type; Extend this diagram with an additional 4th vertex so that we have the possible Dynkin diagrams and the associated GGCM, of quasi affine type QAGGA$_2^{(1)}$. Here  can represent one of the possible 9 edges:



The different possibilities of adding the fourth vertex to the existing affine diagrams are discussed in the following table; the extended vertex is represented in the interior of the affine diagram.

Table-1

Extended Dynkin diagram of quasi affine type QAGGA$_2^{(1)}$	Corresponding GGCM	Number of possible Dynkin diagrams
When k=0, 	$\begin{pmatrix} 0 & -a_1 & -a_2 & -a_{-3} \\ -b_1 & 2 & -1 & -1 \\ -b_2 & -1 & 2 & -1 \\ -b_3 & -1 & -1 & 2 \end{pmatrix}$	165
When k>0, 	$\begin{pmatrix} -k & -a_1 & -a_2 & -a_{-3} \\ -b_1 & 2 & -1 & -1 \\ -b_2 & -1 & 2 & -1 \\ -b_3 & -1 & -1 & 2 \end{pmatrix}$	165
When k>0, 	$\begin{pmatrix} -k & -a_1 & -a_2 & 0 \\ -b_1 & 2 & -1 & -1 \\ -b_2 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$	45
When k=0, 	$\begin{pmatrix} 0 & -a_1 & -a_2 & 0 \\ -b_1 & 2 & -1 & -1 \\ -b_2 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$	45

When k>0, 	$\begin{pmatrix} -k & -a & 0 & 0 \\ -b & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$	9
When k=0, 	$\begin{pmatrix} 0 & -a & 0 & 0 \\ -b & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$	9

Thus we see that there are 438 types of connected, non isomorphic Dynkin diagrams associated with the GGCM in the family $QAGGA_2^{(1)}$.

Computation of root multiplicities for three families in $QAGGA_2^{(1)}$

We follow the same notations given in the earlier section for the dimension formula.

Case 1: We start with the GKM algebra g= g(A, $\underline{m}$) associated with the Borcherds-Cartan matrix, $\begin{pmatrix} -k & -a & 0 & 0 \\ -a & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$

obtained as an extension of $\mathbf{A_2^{(1)}}$ of charge $\underline{m}$ = (s,1,1,1), where k,s are non negative integers. Note that A is symmetric.

Index set for the simple roots of g is I = {1, 2, 3, 4}; Imaginary simple root = { α_1 }

Real simple roots = { $\alpha_2, \alpha_3, \alpha_4$ }; T = { $\alpha_1, \alpha_1, \ldots, \alpha_1$ } counted s times.

Since (α_1, α_1) = -k < 0, the set F can be either empty or F = { α_1 };

Taking J={2,3,4}, $g_0^{(J)} = g_0 \oplus \mathbb{C} h_1$ where $g_0 = \langle e_2, f_2, e_3, f_3, e_4, f_4 \rangle \approx A_2^{(1)}$ and W(J) = {1}. We get

$H_1^{(J)} = V_J(-\alpha_1) \oplus \ldots \oplus V_J(-\alpha_1)$ (s copies); $H_2^{(J)} = H_3^{(J)} = \ldots = 0$;

Hence $H^{(J)} = V_J(-\alpha_1) \oplus \ldots \oplus V_J(-\alpha_1)$ (s copies) where $V_J(-\alpha_1)$ is the standard representation of $A_2^{(1)}$ with highest weight $-\alpha_1$. Here, $\wedge_0 = -\alpha_1$;

Let $\lambda = \wedge_0 - m\,\delta$; We have $(\lambda, \lambda) = (\wedge_0, \wedge_0) + m^2(\delta, \delta) - 2m(\wedge_0, \delta) = -k - 2m.a$, which implies $m = (-k - (\lambda, \lambda)) / (2a)$.
Identifying $-j\alpha_1 - l\,\alpha_2 - m\,\alpha_3 - n\,\alpha_4 \in Q^-$ with (j,*l*,m,n) $\in Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0}$, the weights of V_J $(-\alpha_1)$ are listed as:
$P(H^{(J)})$={(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)} where dim $H^{(J)}_{(1,0,0,0)}$ = dim $H^{(J)}_{(1,1,0,0)}$ = dim $H^{(J)}_{(1,1,1,0)}$ = dim $H^{(J)}_{(1,1,1,1)}$ = s.

For a weight λ = (1,*l*,m,n),, dim $V_J(-\alpha_1)_\lambda$ **=** $\mathbf{p^{(2)}}$**(** - k –(λ , λ)) / (2a);

For $\lambda = (1,l,m,n) = \alpha_1 + l\alpha_2 + m\alpha_3 + n\alpha_4$, we compute $(\lambda, \lambda) = -k-2al-2ml-2ln-2mn+2(l^2 + m^2 + n^2)$;
(- k – (λ, λ)) / (2a) =1/a $\{al + ml+mn+ln+mn-(l^2 + m^2 + n^2)\}$; (It is to be noted that we choose a such that 1/a $\{al + ml+mn+ln+mn-(l^2 + m^2 + n^2)$ is an integer)

Hence dim $V_J(-\alpha_1)_\lambda = p^{(2)}(1/a\,(al + ml+mn+ln+mn-(l^2 + m^2 + n^2)))$ where the function $p^{(2)}$ are defined by

$$\sum_{m=0}^{\infty} p^2(m)q^m = \frac{1}{\phi(q)^2} = \frac{1}{\prod_{j\geq 1}(1-q^j)^2} \quad \ldots \text{ Equation (3.1)}$$

So dim $H_\lambda^{(J)}$ = s . $\mathbf{p^{(2)}}$**(** - k –(λ , λ)) / (2a) .

We have $P(H^{(J)}) = \{ \tau_i / i \geq 1\}$, where $\tau_1 = (1,0,0,0)$, $\tau_2 = (1,1,0,0)$, $\tau_3 = (1,1,0,1)$, $\tau_4 = (1,1,1,0)$, etc.

Every root of g is of the form (j, *l*, m, n) for j ≥ 1, *l*, m, n ≥ 0. Hence the Witt partition function $W^{(J)}(\tau)$ becomes

$$W^{(J)}(\tau) = \sum_{n \in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod s \; p^{(2)}\left(\frac{-k-(\tau_i,\tau_i)}{2a}\right)^{n_i}.$$

Thus we have proved the following theorem:

Theorem 3.3: Let g=g(A, $\underline{m}$) be the Generalized Generalized Kac Moody algebra of quasi affine type associated with the Borcherds Cartan matrix A= $\begin{pmatrix} -k & -a & 0 & 0 \\ -a & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$ of charge $\underline{m}$ = (s,1,1,1) where k, s are non negative integers.

Then for any root $\alpha = -k_1\alpha_1 - k_2\alpha_2 - k_3\alpha_3 - k_4\alpha_4$ with k_i's as non negative integers, the root multiplicity of α is given by $\sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n \in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod s \; p^{(2)}\left(\frac{-k-(\tau_i,\tau_i)}{2a}\right)^{n_i}$, where the function $p^{(2)}$ are given by equation 3.1 and μ is the classical Mobius function.

Remark 3.4: Note here that the algebra g is isomorphic to the maximal graded Lie algebra with local part
$H^{(J)} \oplus (\mathbf{A_2^{(1)}} + h) \oplus H^{(J)^*}$.

4. ROOT MULTIPLICITIES FOR A CLASS OF QAGGA$_2^{(1)}$ WITH ONE IMAGINARY SIMPLE ROOT WHOSE ASSOCIATED GGCM IS $\begin{pmatrix} -k & -a & -b & 0 \\ -a & 2 & -1 & -1 \\ -b & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$ obtained as an extension of $\mathbf{A_2^{(1)}}$ of charge $\underline{m}$ = (t,1,1,1),

where k, t are non negative integers. Note that A is symmetric.

Index set for the simple roots of g is I = {1, 2, 3, 4}; Imaginary simple root = { α_1 }

Real simple roots = { $\alpha_2, \alpha_3, \alpha_4$ }; T = { $\alpha_1, \alpha_1, \ldots, \alpha_1$ } counted s times.

Since (α_1, α_1) = - k < 0, the set F can be either empty or F = { α_1 };

Taking J={2,3,4}, $g_0^{(J)} = g_0 \oplus C h_1$ where $g_0 = \langle e_2, f_2, e_3, f_3, e_4, f_4 \rangle \approx A_2^{(1)}$ and W(J) = {1}.

Then $H_1^{(J)} = V_J(-\alpha_1) \oplus \ldots \oplus V_J(-\alpha_1)$ (t copies); $H_2^{(J)} = H_3^{(J)} = \ldots = 0$;

Hence $H^{(J)} = V_J(-\alpha_1) \oplus \ldots \oplus V_J(-\alpha_1)$ (t copies) where $V_J(-\alpha_1)$ denotes the standard representation of $A_2^{(1)}$ with highest weight $-\alpha_1$. Let $\wedge_0 = -\alpha_1$;

Taking $\lambda = \wedge_0 - m\delta$, we have $(\lambda, \lambda) = (\wedge_0, \wedge_0) + m^2(\delta, \delta) - 2m(\wedge_0, \delta) = -k - 2m.(a+b)$, which implies $m = (-k - (\lambda, \lambda)) / (2(a+b))$.

Identifying $-j\alpha_1 - l\,\alpha_2 - m\,\alpha_3 - n\,\alpha_4 \in Q^-$ with $(j,l,m,n) \in Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0}$, the weights of $V_J(-\alpha_1)$ are listed as: $P(H^{(J)})$={(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)} where $\dim H^{(J)}_{(1,0,0,0)} = \dim H^{(J)}_{(1,1,0,0)} = \dim H^{(J)}_{(1,1,1,0)} = \dim H^{(J)}_{(1,1,1,1)}$ = t.

For a weight λ = (1,*l*,m,n),, $\dim V_J(-\alpha_1)_\lambda = \mathbf{p^{(2)}(}-k-(\lambda, \lambda)) / (2(a+b))$;

For $\lambda=(1,l,m,n) = \alpha_1 + l\alpha_2 + m\alpha_3 + n\alpha_4$, we compute $(\lambda, \lambda) = -k-2al-2bm-2ml-2ln-2mn+2(l^2+m^2+n^2)$; Then
$(-k-(\lambda,\lambda))/(2(a+b)) = 1/(a+b)\{al+bm+ml+mn+ln+mn-(l^2+m^2+n^2)\}$;

(It is to be noted that we choose a+b such that $1/(a+b)\{al+bm+ml+mn+ln+mn-(l^2+m^2+n^2)\}$ is an integer)

Hence $\dim V_J(-\alpha_1)_\lambda = p^{(2)}(1/(a+b)(al+bm+ml+mn+ln+mn-(l^2+m^2+n^2)))$ where the function $p^{(2)}$ is defined by

$$\sum_{m=0}^{\infty} p^2(m)q^m = \frac{1}{\phi(q)^2} = \frac{1}{\prod_{j\geq 1}(1-q^j)^2}$$ as given by equation (3.1)

So $\dim H_\lambda^{(J)} = t.\,\mathbf{p^{(2)}}(-k-(\lambda,\lambda))/(2(a+b))$.

We have $P(H^{(J)}) = \{\tau_i / i \geq 1\}$, where $\tau_1 = (1,0,0,0)$, $\tau_2 = (1,1,0,0)$, $\tau_3 = (1,1,0,1)$, $\tau_4 = (1,1,1,0)$, etc.

Every root of g is of the form (j, l, m, n) for $j \geq 1$, l, m, $n \geq 0$. Hence the Witt partition function $W^{(J)}(\tau)$ becomes

$$W^{(J)}(\tau) = \sum_{n\in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!}\prod t\, p^{(2)}\left(\frac{-k-(\tau_i,\tau_i)}{2(a+b)}\right)^{n_i}.$$

Thus we have proved the following theorem:

Theorem 4.1: Let $g=g(A,\underline{m})$ be the Generalized Generalized Kac Moody algebra of quasi affine type associated with the Borcherds Cartan matrix $A=\begin{pmatrix}-k & -a & -b & 0\\ -a & 2 & -1 & -1\\ -b & -1 & 2 & -1\\ 0 & -1 & -1 & 2\end{pmatrix}$ of charge $\underline{m}$ = (t,1,1,1) where k, t are non negative integers.

Then for any root $\alpha = -k_1\alpha_1 - k_2\alpha_2 - k_3\alpha_3 - k_4\alpha_4$ with k_i's as non negative integers, the root multiplicity of α is given by $\sum_{d|\alpha}\frac{1}{d}\mu(d)\sum_{n\in T^{(J)}(\tau)}\frac{(|n|-1)!}{n!}\prod t\, p^{(2)}\left(\frac{-k-(\tau_i,\tau_i)}{2(a+b)}\right)^{n_i}$, where the function $p^{(2)}$ are given by equation 3.1 and μ is the classical Mobius function.

5. ROOT MULTIPLICITIES FOR A CLASS OF QAGGA$_2^{(1)}$ WITH ONE IMAGINARY SIMPLE ROOT WHOSE ASSOCIATED GGCM IS $\begin{pmatrix}-k & -a & -b & -c\\ -a & 2 & -1 & -1\\ -b & -1 & 2 & -1\\ -c & -1 & -1 & 2\end{pmatrix}$

obtained as an extension of $\mathbf{A_2^{(1)}}$ of charge $\underline{m}$ = (r,1,1,1), where k, r are non negative integers. A is a symmetric GGCM.

Index set for the simple roots of g is I = {1,2,3,4}; Imaginary simple root = $\{\alpha_1\}$

Real simple roots = $\{\alpha_2,\alpha_3,\alpha_4\}$; T = $\{\alpha_1,\alpha_1,\ldots,\alpha_1\}$ counted s times.

Since $(\alpha_1,\alpha_1) = -k < 0$, the set F can be either empty or F = $\{\alpha_1\}$;

Taking J={2,3,4}, $g_0^{(J)} = g_0 \oplus \mathbb{C}h_1$ where $g_0 = \langle e_2, f_2, e_3, f_3, e_4, f_4\rangle \approx A_2^{(1)}$ and W(J) = {1}.
Then $H_1^{(J)} = V_J(-\alpha_1) \oplus \ldots \oplus V_J(-\alpha_1)$ (r copies); $H_2^{(J)} = H_3^{(J)} = \ldots = 0$;

Hence $H^{(J)} = V_J(-\alpha_1) \oplus \ldots \oplus V_J(-\alpha_1)$ (r copies) where $V_J(-\alpha_1)$ denotes the standard representation of $A_2^{(1)}$ with highest weight $-\alpha_1$. Let $\wedge_0 = -\alpha_1$;

Taking $\lambda = \wedge_0 - m\delta$, we have $(\lambda,\lambda) = (\wedge_0,\wedge_0) + m^2(\delta,\delta) - 2m(\wedge_0,\delta) = -k - 2m.(a+b+c)$, which implies
$m = (-k-(\lambda,\lambda))/(2(a+b+c))$.

Identifying $-j\alpha_1 - l\,\alpha_2 - m\,\alpha_3 - n\,\alpha_4 \in Q^-$ with $(j, l, m, n) \in Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0}$, the weights of $V_J(-\alpha_1)$ are listed as: $P(H^{(J)})=\{(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)\}$ where $\dim H^{(J)}_{(1,0,0,0)} = \dim H^{(J)}_{(1,1,0,0)} = \dim H^{(J)}_{(1,1,1,0)} = \dim H^{(J)}_{(1,1,1,1)} = r$.

For a weight $\lambda = (1,l,m,n)$,, $\dim V_J(-\alpha_1)_\lambda$ **=** $\mathbf{p^{(2)}}$**(** $-k-(\lambda,\lambda))/(2(a+b+c))$;

For $\lambda = (1, l, m, n) = \alpha_1 + l\alpha_2 + m\alpha_3 + n\alpha_4$, we compute $(\lambda, \lambda) = -k-2al-2bm-2cn-2ml-2ln-2mn+2(l^2+m^2+n^2)$; Then **(**$-k-(\lambda,\lambda))/(2(a+b+c)) = 1/(a+b+c)\{al+bm+cn+ml+mn+ln+mn-(l^2+m^2+n^2)\}$;

(It is to be noted that we choose a, b, c such that $1/(a+b+c)\{al+bm+cn+ml+mn+ln+mn-(l^2+m^2+n^2)\}$ is an integer)

Hence $\dim V_J(-\alpha_1)_\lambda = p^{(2)}(1/(a+b+c)(al+bm+cn+ml+mn+ln+mn-(l^2+m^2+n^2))$ where the function $p^{(2)}$ is defined by $\sum_{m=0}^{\infty} p^2(m)q^m = \frac{1}{\phi(q)^2} = \frac{1}{\prod_{j\geq 1}(1-q^j)^2}$ as given by equation (3.1)

So $\dim H^{(J)}_\lambda = r.\,\mathbf{p^{(2)}}$**(** $-k-(\lambda,\lambda))/(2(a+b+c))$.

We have $P(H^{(J)}) = \{\tau_i / i \geq 1\}$, where $\tau_1 = (1,0,0,0)$, $\tau_2 = (1,1,0,0)$, $\tau_3 = (1,1,0,1)$, $\tau_4 = (1,1,1,0)$, etc.

Every root of g is of the form (j,l,m,n) for $j \geq 1$, $l, m, n \geq 0$. Hence the Witt partition function $W^{(J)}(\tau)$ becomes

$$W^{(J)}(\tau) = \sum_{n\in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod t\, p^{(2)}\left(\frac{-k-(\tau_i,\tau_i)}{2(a+b+c)}\right)^{n_i}.$$

Thus we have proved the following theorem:

Theorem 5.1: Let $g=g(A,\underline{m})$ be the Generalized Generalized Kac Moody algebra of quasi affine type associated with the Borcherds Cartan matrix $A=\begin{pmatrix} -k & -a & -b & -c \\ -a & 2 & -1 & -1 \\ -b & -1 & 2 & -1 \\ -c & -1 & -1 & 2 \end{pmatrix}$ of charge $\underline{m} = (r,1,1,1)$ where k, r are non negative integers. Then for any root $\alpha = -k_1\alpha_1 - k_2\alpha_2 - k_3\alpha_3 - k_4\alpha_4$ with k_i's as non negative integers, the root multiplicity of α is given by $\sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n\in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod r\, p^{(2)}\left(\frac{-k-(\tau_i,\tau_i)}{2(a+b+c)}\right)^{n_i}$, where the function $p^{(2)}$ are given by equation 3.1 and μ is the classical Mobius function.

CONCLUSION

In this work, a particular class of quasi affine GKM algebras are defined, the general form of associated Dynkin diagrams given and the root multiplicities for three specific classes in $\mathbf{QAGGA_2^{(1)}}$ are computed; There is further scope for determining the root multiplicities of other GKM algebras of indefinite type , which are quasi affine.

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