International Journal of Mathematical Archive-7(1), 2016, 188-191
Available online through www.ijma.info ISSN 2229 – 5046

A FIXED POINT THEOREM ON S-METRIC SPACES WITH A WEAK S –METRIC

SARADA KESIRAJU[1], V. JHANSI RANI*[2]
Department of Mathematics and Statistics,
RBVRR Women's College, Narayanguda, Hyderabad– 500 027, India.

V. SIVA RAMA PRASAD
Nalla Malla Reddy Engineering College,
Divyanagar, Ghatkesar, Ranga Reddy District, Telangana – 501 301, India.

(Received On: 11-01-16; Revised & Accepted On: 31-01-16)

ABSTRACT

Very recently, Sedghi, Shobe and Aliouche [2] have introduced S-metric space as a generalization of the metric space and proved a Fixed Point Theorem similar to Banach contraction principle in metric spaces. In this paper we define the concept of a weak S-metric on a S-metric space and establish a Fixed Point Theorem on S-metric space with a weak S-metric. We also deduce the theorem proved in [2] from our result.

***Key Words**: S-metric space and weak S-metric.*

***AMS Mathematics Subject Classification:** 54H25, 47H10.*

1. INTRODUCTION

In 2012, Sedghi, Shobe and Aliouche [2] introduced the notion of S-metric space as a generalization of a metric space as follows:

1.1 Definition ([2] Definition 2.1): Let X be a non empty set. A *S-metric* on X is a function $S: X^3 \to [0, \infty)$ satisfying the conditions given below for x, y, z, a $\in X$:

i) $S(x, y, z) \geq 0$
ii) $S(x, y, z) = 0$ if and only if $x = y = z$
iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

A set X with a S-metric defined on it is called a *S- metric space* and is denoted by (X, S)

1.2. Examples:

(i) Let (X, d) be a metric space. Define $S_d: X^3 \to [0, \infty)$ by $S_d(x, y, z) = d(x, z) + d(y, z)$ for x, y, z $\in$ X. Then (X, S_d) is a S- metric space.
(ii) Suppose $X_0 = \{0, \infty\} \cup \{\frac{1}{n}: n \geq 1\}$ and define $S: X_0^3 \to [0, \infty)$ by $S(x,y,z) = |x - z| + |y - z|$ for x,y,z $\in X_0$. Then (X_0,S) is a S-metric space.

For other examples see [2].

The following definitions given in [2] are needed:

1.3. Definition ([2], Definition2.8): Let (X, S) be a S-metric space. A sequence $\{x_n\}$ in X is said to

(i) *converge to* $x \in X$ if to each $\in > 0$ there is a natural number n_0 such that $S(x_n, x_n, x) < \in$ for all $n \geq n_0$ and in this case we write $\lim_{n\to\infty} x_n = x$ in (X,S) or $x_n \to x$ as n→ ∞ in (X,S)
(ii) be a *Cauchy sequence* if to each $\in > 0$ there is a natural number n_0 such that $S(x_m, x_m, x_n) < \in$ for all m,n≥ n_0

Corresponding Author: V. Jhansi Rani*[2]
Department of Mathematics and Statistics,
RBVRR Women's College, Narayanguda, Hyderabad– 500 027, India.

1.4. Remark: It has been proved that every sequence that converges in (X, S) has unique limit ([2], Lemma 2.10) and that it is a Cauchy sequence ([2], Lemma2.11).

1.5. Definition: A S-metric space is said to be *complete* if every Cauchy sequence in it converges. If (X, S) is a S-metric space then

1.6: S(x, x, y) = S(y, y, x) for all x, y∈ X ([2], Lemma2.5) and Lemma 2.12 of [2] is:

1.7: If $\{x_n\}$ and $\{y_n\}$ are sequences in (X, S) converging respectively to x and y then

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x,x,y)$$

In [1], Kada, Suzuki and Takahashi introduced the concept of a weak distance in a metric space. Analogously we define *weak S-metric* in a S-metric space and use it to prove a fixed point theorem for a selfmap on a S-metric space with a *weak S-metric*. Also we show that a theorem proved in [2] is a particular case of our result.

2. WEAK S-METRIC AND ITS PROPERTIES

2.1. Definition: Suppose (X, S) is a S-metric space. A *weak S-metric* on X is a function $p: X^3 \to [0,\infty]$ satisfying the conditions given below:

- (a^1) $p(x,y,z) \le p(a,a,x) + p(a,a,y) + p(a,a,z)$ for $x,y,z,a \in X$
- (b^1) For each $x \in X, p(x,x,.): X \to [0,\infty)$ is lower continuous
- (c^1) To each $\in > 0$ there is a $\delta > 0$ such that $p(a,a,x) < \delta$, $p(a,a,y) < \delta$ and $p(a,a,z) < \delta$ for some $a \in X$ imply that $S(x,y,z) < \in$.

2.2. Examples:

(i) If (X, S) is a S-metric space and $p: X^3 \to [0,\infty]$ is defined by $p(x,y,z) = S(x,y,z)$ for x, y, z $\in X$ then p is a weak S-metric. In fact,(a^1) holds in view of Definition 1.1(c) and (1.6); (b^1) holds in view of (1.7) and finally for a given $\in > 0$, taking $\delta = \frac{\epsilon}{3}$ it is easy to verify (c^1) in view of Definition 1.1(c).
That is every S-metric on a set X is a weak S- metric

(ii) Let (X_0,S) be the S-metric space given in Example 1.2(ii)
Define $p: X_0^3 \to [0,\infty)$ by p(x, y, z) = y+2z for x,y,z ∈ X_0. Then p is a weak S-metric. To verify this, for x, y, z, a ∈ X note that p(a, a, x) + p(a, a, y) + p(a, a, z) = 2(x+y+z) +3a ≥ y+2z = p(x, y, z) which gives (a^1); if $\{y_n\}$∈ X tends to y in (X_0,S) then (b^1) holds.
since $\lim_{n\to\infty} p(x,x,y_n) = \lim_{n\to\infty}(x+2y_n) = x+2y = p(x,x,y)$ and finally for $\in > 0$ taking $\delta = \frac{\in}{4}$ we find that p(a, a, x)$< \delta$, p(a,a,y) $< \delta$ and p(a,a,z) $< \delta$ imply 2(x+y+z)+3a$< 3\delta < \in$ so that S(x,y,z) = $| x + y - 2z <$2x+y+z+3a$< \epsilon$ proving (c^1)

2.3. Remark: For a weak S-metric p on a S-metric space (X, S) observe that p(x, y, z) = 0 need not imply x = y = z. Therefore p(x, x, y) and p(y, y, x) need not be equal for x, y ∈ X.

For instance, in Example 2.2(ii), note that p(a,0,0)=0 for all a ∈ X_0

2.4. Lemma: *Suppose (X, S) is a S-metric space and p is a weak S-metric on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in [0,∞) such that $lim_{n\to\infty}\,\alpha_n = lim_{n\to\infty}\,\beta_n$ = 0 and x,y,z ∈ X. Then*

- (i) $p(x_n,x_n,y) \le \alpha_n$ *and* $p(x_n,x_n,y) \le \beta_n$ *for every* $n \ge 1$ *imply* y=z. *In particular* $p(x,x,y) = p(x,x,z)$ *implies* y=z
- (ii) $p(x_n,x_n,y_n) \le \alpha_n$ *and* $p(x_n,x_n,z) \le \beta_n$ *for every* $n \ge 1$ *imply that* $y_n \to y$ *as* $n \to \infty$ *in* (X,S)
- (iii) $p(x_m,x_m,x_n) \le \alpha_n$ *for all* $m > n \ge 1$ *implies* $\{x_n\}$ *is a Cauchy sequence in* (X,S)
- (iv) $p(y,y,x_n) \le \alpha_n$ *for every* $n \ge 1$ *implies* $\{x_n\}$ *is a Cauchy sequence in* (X,S)

Proof: For any $\in > 0$ choose a $\delta > 0$ satisfying (c^1) of Definition 2.1 and then find a natural number n_0 such that $\alpha_n < \delta$ and $\beta_n < \delta$ for $n \ge n_0$

- (i) For $n \ge n_0$ we have $p(x_n,x_n,y_n) < \delta$ *and* $p(x_n,x_n,z) < \delta$ so that by (c^1), $S(y,y,z) < \in$. Since $\epsilon > 0$ is arbitrary it follows that S(y, y, z) = 0 giving y = z.
- (ii) For $n \ge n_0$ we get in this case $p(x_n,x_n,y) < \delta$ *and* $p(x_n,x_n,z) < \delta$ and again by (c^1) $S(y_n,y_n,z) < \in$ showing $y_n \to z$ as $n \to \infty$ in (X, S) (see Definition 1.3(i))
- (iii) In this case, $p(x_n,x_n,x_m) < \delta$ for $n \ge n_0$.In particular,$p(x_{n_0},x_{n_0},x_m) < \delta$ and $p(x_{n_0},x_{n_0},x_k) < \delta$ for $m > k > n_0$ which imply by (c^1) that $S(x_m,x_m,x_k) < \in$ whenever $m > k \ge n_0$.That is $\{x_n\}$ is a Cauchy sequence in (X,S) (see Definition 1.3(ii))
- (iv) For $n \ge n_0$, we have $p(y,y,x_n) < \delta$ so that for $m > n \ge n_0, p(y,y,x_m) < \delta$ and $p(y,y,x_n) < \delta$ and hence by (c^1), $S(x_m,x_m,x_n) < \in$ for $m > n \ge n_0$ giving $\{x_n\}$ is a Cauchy sequence in (X, S).

3. MAIN THEOREM

3.1. Theorem: *Suppose (X, S) is a complete S-metric space with a weak S-metric p on it. Suppose* $f: X \to X$ *is a continuous function such that*

(3.2) $p(fx, fx, fy) \leq L.p(x, y, y)$ *for all* $x, y, z \in X$, *for some* $L \in [0,1)$.*Then f has a fixed point* $z \in X$. *Also if* $u \in X$ *is another fixed point of* f *then* $p(u, u, z) = 0$.

Proof: Let $x_0 \in X$ *and* $x_n = fx_{n-1}$ for $n \geq 1$ *so that* $\{x_n\} \in$ X. We now prove that $\{x_n\}$ is a Cauchy sequence in (X,S).

For any integer $k \geq 0$, let $\alpha_k = p(x_k, x_k, x_k)$, $\beta_k = p(x_k, x_k, x_{k+1})$ and $\gamma_n^{n+k} = p(x_{n+k}, x_{n+k}, x_n)$.

Then, by (3.2), $\alpha_k = p(fx_{k-1}, fx_{k-1}, fx_{k-1}) \leq L.p(x_{k-1}, x_{k-1}, x_{k-1}) = L.\alpha_{k-1}$

which on repeated use gives

(3.3) $\alpha_k \leq L.\alpha_{k-1} \leq L^2.\alpha_{k-2} \leq \cdots \leq L^k\alpha_0$ and $\beta_k = p(fx_{k-1}, fx_{k-1}, fx_k) \leq L.p(x_{k-1}, x_{k-1}, x_k) = L.\beta_{k-1}$

which on repeated use gives

(3.4) $\beta_k \leq L.\beta_{k-1} \leq L^2\beta_{k-2} \leq \cdots \leq L^k.\beta_0$

Also since $p(x, x, y) \leq 2p(a, a, x) + p(a, a, y)$ *for any* $x, y, a \in X$ (by (a[1]) of Definition 2.1), we have

$$\gamma_n^{n+k} \leq 2p(x_{n+k-1}, x_{n+k-1}, x_{n+k}) + p(x_{n+k-1}, x_{n+k-1}, x_n)$$
$$= 2.\beta_{n+k-1} + \gamma_n^{n+k-1}$$

which on repeated use gives

$$\gamma_n^{n+k} \leq 2\beta_{n+k-1} + 2\beta_{n+k-2} + \cdots + 2\beta_n + \gamma_n^n$$
$$= 2\beta_{n+k-1} + 2\beta_{n+k-2} + \cdots + 2\beta_n + \alpha_n$$

so that by (3.3) and (3.4) we get

(3.5) $\gamma_n^{n+k} \leq 2\beta_0 . L^n(1 + L + \cdots + L^{k-1}) + L^n.\alpha_0 < A_n$,

where $A_n = \frac{2\beta_0}{1-L} . L^n + \alpha_0 . L^n$

Therefore, if m= n+k where $k \geq 0$ *and* $n \geq 1$ then (3.5) shows $p(x_m, x_m, x_n) < A_n$ for all $\geq n \geq 1$ *and since* $A_n \to 0$ as $n \to \infty$(*because* $0 \leq L < 1$), it follows from (iii) of Lemma 2.4 that $\{x_n\}$ is a Cauchy sequence in (X, S).

Now since (X,S) is complete there is a $z \in X$ such that $\lim_{n\to\infty} x_n = z$, and since f is continuous

$$f(z) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = z$$

showing z is a fixed point of f

If $u \in X$ is such that fu = u then by (3.2),

$$p(u, u, z) = p(fu, fu, fz) \leq L.p(u, u, z)$$

Which shows $p(u, u, z) = 0$, since $0 \leq L \leq 1$

Hence the theorem

3.6 Corollary: ([2], Theorem2.1) *If (X, S) is a complete S-metric space and* $f: X \to X$ *is a mapping for which* $S(fx, fx, fy) \leq L.S(x, x, y)$ *holds for all* $x, y \in X$ *where* $0 \leq L < 1$ (such a mapping is called a *contraction on* (X, S) in [2], Definition 2.13) *then f has a unique fixed point* $z \in X$.

Proof: Taking p = S (which is a weak S-metric, see Example 2.2(i)) in the theorem we get a fixed point $z \in X$. Also if $u \in X$ is another fixed point then S(u, u, z) = 0 which gives u = z, proving the uniqueness of the fixed point.

3.7 Corollary: (Banach Contraction principle)**.** *If (X, d) is a complete metric space and* $f: X \to X$ *is a mapping such that* $d(fx, fy) \leq L. d(x, y)$ *for all* $x, y \in X, for\ some\ L \in [0,1)$ *then f has a unique fixed point* $z \in X$.

Proof: Given a complete metric space (X, d), define $S_d: X^3 \to [0, \infty)$ by $S_d(x, y, z) = d(x, z) + d(y, z)$ for $x, y, z \in X$. Then (X, S_d) is a S-metric space and also it is complete. Further since S_d (x, x, y) = 2d(x, y) for any x, y∈ X the conditions of this corollary gives $S_d(fx, fx, fy) \leq L. S_d(x, x, y)$for all $x, y \in X$.Therefore, by Corollary 3.6, f has a unique fixed point $z \in X$

ACKNOWLEDGEMENT

The second author (VJR) thanks the University Grants Commission, Govt. of India, New Delhi for providing financial assistance through minor research project. No F.No:4-4/2014-15(MRP-SEM)/UGC-SERO).

REFERENCES

1. O.Koda,T.Suzuki and W.Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces Math*, *Japon.*44(1996),*no*.2, 381-391
2. Sedghi, N.Shobe and A.Aliouche, *A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik*, 64.3(2012), 258-266.

Source of support: University Grants Commission, Govt. of India, Conflict of interest: None Declared.