

UNIQUENESS OF SOME GENERAL CLASS
OF DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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ABSTRACT

We discuss some general class of differential polynomials of meromorphic functions and obtain two theorems which improve a result of Abhijit Banerjee [17].

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1. INTRODUCTION AND MAIN RESULTS

Let f and g be two nonconstant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities and if we do not consider the multiplicities, then f and g are said to share the value a IM (ignoring multiplicities). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \rightarrow \infty$, outside a set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

Due to Nevanlinna [9], it is well known that if f and g share four distinct values CM, then f is a Mobius transformation of g .

Yang and Hua showed that similar conclusions hold for certain types of differential polynomials when they share only one value. They proved the following result.

Theorem 1.1[12]: Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$ th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.

Corresponding to entire functions, Xu and Qu proved the following result.

Theorem 1.2[10]: Let f and g be two nonconstant entire functions, $n \geq 12$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a IM, then either $f = dg$ for some $(n+1)$ th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

To state the next result, we require the following definition.

Definition 1.3[4, 5]: Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} - \{\infty\}$, denote by $E_k(a; f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$ say that f, g share the value a with weight k .

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The definition implies that if f, g share a value a with weight k then z_0 is an a -point of f with multiplicity $m(\leq k)$ if and only if it is an a -point of g with multiplicity $m(\leq k)$ and z_0 is an a -point of f with multiplicity $m(> k)$ if and only if it is an a -point of g with multiplicity $n(> k)$ where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Since $E_k(a; f) = E_k(a; g)$ implies $E_p(a; f) = E_p(a; g)$ for any integer $p(0 \leq p < k)$, clearly if f, g share (a, k) , then f, g share (a, p) for any integer $p(0 \leq p < k)$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) , respectively.

Theorem 1.4[5]: Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share $(a, 2)$, then either $f = dg$ for some $(n+1)$ th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Theorem 1.5[17]: Let f and g be two nonconstant meromorphic functions such that $n > 22 - [5\theta(\infty; f) + 5\theta(\infty; g) + \min\{\theta(\infty; f), \theta(\infty; g)\}]$, where n is an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g'$ share $(a, 0)$, then either $f = dg$ for some $(n+1)$ th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Theorem 1.6[17]: Let f and g be two nonconstant meromorphic functions and $n > \max\{8, 12 - \{3\theta(\infty, f) + 3\theta(\infty, g)\}\}$ an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g'$ share $(a, 1)$, then either $f = dg$ for some $(n+1)$ th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Here in this paper, we partially extend Theorem 1.5 and Theorem 1.6 to a more general class of differential polynomials as.

Theorem 1.7: Let f and g be two nonconstant meromorphic functions such that $n > km + 3m + 4k + 14$, where n, m and k be positive integers. If for $a \in \mathbb{C} - \{0\}$, $f^n (f^m)^{(k)}$ and $g^n (g^m)^{(k)}$ share $(a, 0)$, then either $f^n (f^m)^{(k)} g^n (g^m)^{(k)} \equiv a^2$ or $f^n (f^m)^{(k)} \equiv g^n (g^m)^{(k)}$.

Theorem 1.8: Let f and g be two nonconstant meromorphic functions such that $n > km + 3m + 3k + 11$, where n, m and k be positive integers. If for $a \in \mathbb{C} - \{0\}$, $f^n (f^m)^{(k)}$ and $g^n (g^m)^{(k)}$ share $(a, 1)$, then either $f^n (f^m)^{(k)} g^n (g^m)^{(k)} \equiv a^2$ or $f^n (f^m)^{(k)} \equiv g^n (g^m)^{(k)}$.

Definition 1.9[3]: For $a \in \mathbb{C} - \{\infty\}$, denote by $N(r, a; f | = 1)$ the counting function of simple a -points of f . For a positive integer m , denote by $N(r, a; f | \leq m)$ ($N(r, a; f | \geq m)$) the counting function of those a -points of f whose multiplicities are not greater (less) than m where each a -point is counted according to its multiplicity.

$\bar{N}(r, a; f | \leq m)$ ($\bar{N}(r, a; f | \geq m)$) are defined similarly, where in counting the a -points of f we ignore the multiplicities.

Also $N(r, a; f | < m)$, ($N(r, a; f | > m)$) $\bar{N}(r, a; f | < m)$ and $\bar{N}(r, a; f | > m)$ are defined analogously.

Definition 1.10[5]: Denote by $N_2(r; a; f)$ the sum $\bar{N}(r, a; f) + \bar{N}(r, a; f | \geq 2)$.

Definition 1.11[1, 15, 16]: Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p , a 1-point of g with multiplicity q . Denote by $\bar{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, denote by $N_E^1(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$, and denote by $N_E^2(r, 1; f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way, one can define $\bar{N}_L(r, 1; g)$, $N_E^1(r, 1; g)$, $N_E^2(r, 1; g)$.

Definition 1.12(cf. [1]): Let k be a positive integer. Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p , and a 1-point of g with multiplicity q . Denote by $\bar{N}_{f>k}(r, 1; g)$ the reduced counting function of those 1-points of f and g such that $p > q = k$. $\bar{N}_{g>k}(r, 1; f)$ is defined analogously.

Definition 1.13([4, 5]): Let f, g share a value IM. Denote by $\bar{N}_*(r, a; f, g)$ the reduced counting function of those a -points of f whose multiplicities differ from the multiplicities of the corresponding a -points of g .

Clearly

$$\bar{N}_*(r, a; f, g) \equiv \bar{N}_*(r, a; g, f) \text{ and } \bar{N}_*(r, a; f, g) = \bar{N}_L(r, a; f) + \bar{N}_L(r, a; g).$$

Definition 1.14([6]): Let $a, b \in \mathbb{C} \cup \{\infty\}$. Denote by $N(r, a; f|g = b)$ the counting function of those a -points of f , counted according to multiplicity, which are the b -points of g .

Definition 1.15([6]): Let $a, b \in \mathbb{C} \cup \{\infty\}$. Denote by $N(r, a; f|g \neq b)$ the counting function of those a -points of f , counted according to multiplicity, which are not the b -points of g .

2. LEMMAS

In this section, we present some lemmas which will be needed in the sequel. Let f, g, F, G be four nonconstant meromorphic functions. Henceforth, We will denote by h and H the following two functions:

$$h = \left(\frac{f''}{f'} - \frac{2f'}{f-1} \right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1} \right), H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

Lemma 2.1[15]: Let f, g be two nonconstant meromorphic functions such that they share $(1, 0)$ and $h \neq 0$. Then

$$T(r, f) \leq N_2\left(r, \frac{1}{f}\right) + N_2(r, f) + N_2\left(r, \frac{1}{g}\right) + N_2(r, g) + 2\bar{N}\left(r, \frac{1}{f}\right) + 2\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}(r, g) + S(r, f) + S(r, g)$$

Lemma 2.2[17]: Let f, g be two nonconstant meromorphic functions such that they share $(1, 1)$ and $h \neq 0$. Then

$$T(r, f) \leq N_2\left(r, \frac{1}{f}\right) + N_2(r, f) + N_2\left(r, \frac{1}{g}\right) + N_2(r, g) + \frac{1}{2}\bar{N}\left(r, \frac{1}{f}\right) + \frac{1}{2}\bar{N}(r, f) + S(r, f) + S(r, g)$$

Lemma 2.3[18]: Let f and g be two meromorphic functions. If f and g share 1 CM, then one of the following must occur:

- (i) $T(r, f) + T(r, g) \leq 2\left\{N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + N_2(r, f) + N_2(r, g)\right\} + S(r, f) + S(r, g)$
- (ii) either $f \equiv g$ or $fg \equiv 1$.

Lemma 2.4[15]: Let f be a nonconstant meromorphic function. Then

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq k\bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f).$$

3. PROOF OF THEOREM 1.7

Let $F = \frac{f^n (f^m)^{(k)}}{a}$ and $G = \frac{g^n (g^m)^{(k)}}{a}$. Since $f^n (f^m)^{(k)}$ and $g^n (g^m)^{(k)}$ share $(a, 0)$. If possible, we suppose that $H \neq 0$. Then by Lemma 2.1, we obtain

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G) + 2\bar{N}\left(r, \frac{1}{F}\right) + 2\bar{N}(r, F) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, G) + S(r, F) + S(r, G)$$

We see that

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) + N_2(r, F) &\leq N_2\left(r, \frac{1}{f^n (f^m)^{(k)}}\right) + N_2(r, f^n (f^m)^{(k)}) + S(r, f) \\ &\leq N_2\left(r, \frac{1}{f^n}\right) + N_2\left(r, \frac{1}{(f^m)^{(k)}}\right) + 2\bar{N}(r, f^n (f^m)^{(k)}) + S(r, f) \\ &\leq 2\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{(f^m)^{(k)}}\right) + 2\bar{N}(r, f) + S(r, f) \\ &\leq 2\bar{N}\left(r, \frac{1}{f}\right) + mN\left(r, \frac{1}{f}\right) + (k+2)\bar{N}(r, f) + S(r, f) = (k+m+4)T(r, f) + S(r, f). \end{aligned}$$

Therefore,

$$N_2\left(r, \frac{1}{F}\right) + N_2(r, F) \leq (k+m+4)T(r, f) + S(r, f) \tag{3.1}$$

On the similar lines we can write (3.1) for the function G as

$$N_2\left(r, \frac{1}{G}\right) + N_2(r, G) \leq (k+m+4)T(r, g) + S(r, g) \tag{3.2}$$

Since

$$nT(r, f) = T(r, f^n) = T\left(r, \frac{f^n (f^m)^{(k)} a}{a (f^m)^{(k)}}\right)$$

$$\begin{aligned} &\leq T(r, F) + T\left(r, \frac{1}{(f^m)^{(k)}}\right) + T(r, a) + S(r, f) \\ &= T(r, F) + (km + m)T\left(r, \frac{1}{f}\right) + S(r, f) \end{aligned} \tag{3.3}$$

Therefore

$$(n - km - m)T(r, f) \leq T(r, F) + S(r, f) \tag{3.4}$$

Similarly

$$(n - km - m)T(r, g) \leq T(r, G) + S(r, g) \tag{3.5}$$

$$(n - km - m)\{T(r, f) + T(r, g)\} \leq \{T(r, F) + T(r, G)\} + S(r, f) + S(r, g) \tag{3.6}$$

Suppose that

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F)\right\} + 2\left\{N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right\} + 3\left\{\bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G}\right)\right\} \\ &\quad + 3\{\bar{N}(r, F) + \bar{N}(r, G)\} + S(r, f) + S(r, g). \end{aligned}$$

Then from (3.1), (3.2), (3.6) and (3.7) we have

$$\begin{aligned} &(n - km - m)\{T(r, f) + T(r, g)\} \\ &\leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right\} + 3\left\{N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + \bar{N}(r, F) + \bar{N}(r, G)\right\} \\ &\quad + S(r, f) + S(r, g) \\ &\leq 2(k + m + 4)\{T(r, f) + T(r, g)\} \\ &\quad + 3\left\{N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + (k + 1)\left\{\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right)\right\} + k\{\bar{N}(r, f) + \bar{N}(r, g)\} + 2\bar{N}(r, f) \right. \\ &\quad \left. + 2\bar{N}(r, g)\right\} + S(r, f) + S(r, g) \\ &\leq \{4k + 2m + 14\}\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g) \end{aligned}$$

which implies that

$$\{n - km - 3m - 4k - 14\}\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g)$$

a contradiction since $n > km + 3m + 4k + 14$, where $m > k - 1$.

Then by Lemma 2.3, it follows that either

$$F \cdot G \equiv 1 \text{ or } F \equiv G \text{ that is either } f^n (f^m)^{(k)} g^n (g^m)^{(k)} \equiv a^2 \text{ or } f^n (f^m)^{(k)} = g^n (g^m)^{(k)}.$$

4. PROOF OF THEOREM 1.8

Let $F = \frac{f^n (f^m)^{(k)}}{a}$ and $G = \frac{g^n (g^m)^{(k)}}{a}$. Since $f^n (f^m)^{(k)}$ and $g^n (g^m)^{(k)}$ share $(a, 1)$. Suppose that $H \not\equiv 0$. Then by Lemma 2.2, we obtain

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G) + \frac{1}{2}\bar{N}\left(r, \frac{1}{F}\right) + \frac{1}{2}\bar{N}(r, F) + S(r, F) + S(r, G)$$

Suppose that

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right\} + \frac{1}{2}\{\bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{G}\right) \\ &\quad + \bar{N}(r, G)\} + S(r, F) + S(r, G) \end{aligned} \tag{4.1}$$

Then from (3.1), (3.2), (3.6) and (4.1) we have

$$\begin{aligned} &(n - km - m)\{T(r, f) + T(r, g)\} \\ &\leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right\} + \frac{1}{2}\{\bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, G)\} \\ &\quad + S(r, F) + S(r, G) \\ &\leq 2(k + m + 4)\{T(r, f) + T(r, g)\} \\ &\quad + \frac{1}{2}\left\{N\left(r, \frac{1}{f}\right) + (k + 1)\bar{N}\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + N\left(r, \frac{1}{g}\right) + (k + 1)\bar{N}\left(r, \frac{1}{g}\right) + k\bar{N}(r, g) + 2\bar{N}(r, f) \right. \\ &\quad \left. + 2\bar{N}(r, g)\right\} + S(r, f) + S(r, g) \leq (3k + 2m + 11)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g) \end{aligned}$$

which implies that

$$\{n - km - 3m - 3k - 11\}\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g)$$

a contradiction since $n > km + 3m + 3k + 11$, where $m > k - 1$.

Then by Lemma 2.3, it follows that either

$$F \cdot G \equiv 1 \text{ or } F \equiv G \text{ that is either } f^n (f^m)^{(k)} g^n (g^m)^{(k)} \equiv \alpha^2 \text{ or } f^n (f^m)^{(k)} = g^n (g^m)^{(k)}.$$

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