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Strongly Clean Matrices in $M_2(Z)$: An Intrinsic Characterization

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ABSTRACT

An element of a ring R with identity is called strongly clean if it is the sum of an idempotent and a unit that commute. When R is a projective free ring, a characterization of strongly clean elements in $M_n(R)$ has been given [7]. When R is a principal ideal domain (P.I.D.), towards such a characterization we take an approach which uses well known structure of idempotent matrices in $M_n(R)$. We use this to characterize non triangular strongly clean elements in $M_2(Z)$ in terms of their entries.

Keywords: *Principal ideal domain, strongly clean matrix, intrinsic characterization.*

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1. INTRODUCTION

Let R be a ring with identity. An element a of R is strongly clean if $a = e + u$ where $e^2 = e \in R$ and u is a unit of R with $eu = ue$. In the past few years, several mathematicians have considered the question: when is $M_n(R)$ strongly clean? The authors of [2] and [3] characterized the commutative local rings R for which $M_2(R)$ is strongly clean. For each n, the authors of [1] characterized the commutative local rings R for which $M_n(R)$ is strongly clean. In [7], the authors continued the study of this question for n = 2 and R, a non commutative local ring.

The problem of characterizing a strongly clean element A in $M_n(R)$ which depends only on the entries of A (which we shall call intrinsic characterization) is open. To give such a characterization of strongly clean matrices in $M_n(R)$, it is important to know a similar characterization for idempotent matrices in $M_n(R)$. For $n >$ 2 no such characterization is known. When R is a P.I.D., using well known results concerning free modules over a P.I.D., one can see [Lemma 2.1] that idempotents are similar to a matrix of the form $\begin{bmatrix} I & O \\ O & O \end{bmatrix}$. Using this we get a characterization of strongly clean matrices over P.I.D. (see Theorem 2.2). Thus this approach avoids any intrinsic characterization of idempotent matrices. We shall use this to give an intrinsic characterization of strongly clean matrices in $M_2(Z)$. We note that Theorem 2.2 is a special case of Lemma 2 in [7] which uses Nicholson's criteria [5] for strongly clean endomorphisms of free modules.

In this paper R denotes a P.I.D. and Z as usual the ring of integers. We organize the material of this paper in to two sections. Section 2 deals with some elementary results and section 3 with the main results.

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2. PRELIMINARY RESULTS

We begin the section with recalling the following well known elementary Lemma which is crucial for obtaining a characterization of strongly clean matrices in $M_n(R)$.

Lemma: 2.1 Let R be a Principal Ideal Domain and $E = [e_{ij}]$ be an idempotent matrix of order n over R. Then E is similar to a matrix of the form $\begin{bmatrix} I & O \\ O & O \end{bmatrix}$.

If a matrix *A* in the matrix ring $M_n(R)$ is strongly clean then A = E + U, where E is an idempotent matrix ($\mathrm{E}^2 = \mathrm{E}$) in $M_n(R)$ and U is a unit in $M_n(R)$ with EU = UE. We shall call such a representation of A a **strongly clean representation** of A. Further if E = O or I, we call *A* **trivially strongly clean** otherwise we call A **non-trivially strongly clean**.

The following theorem can easily be deduced from Lemma 2.1 (see also Lemma 2 in [7].)

Theorem: 2.2 Let R be a Principal Ideal Domain and let $A \in M_n(R)$. Then *A* is non-trivially strongly clean if and only if it is similar to a block matrix of the form $\begin{bmatrix} I + U_1 & O \\ O & U_2 \end{bmatrix}$ where U_1 and U_2 are units.

In particular if *A* is of order 2, then *A* is non-trivially strongly clean if and only if it is similar to $\begin{bmatrix} 1 + u_1 & 0 \\ 0 & u_2 \end{bmatrix}$ where $u_1 u_2$ is a unit.

Corollary: 2.3 If R = Z, the ring of integers and *A* is a matrix of order 2, then *A* is non-trivially strongly clean if and only if it is similar to $\begin{bmatrix} 1 + u_1 & 0 \\ 0 & u_2 \end{bmatrix}$ with $u_1 u_2 = \pm 1$.

3. MAIN RESULTS

In what follows **strongly clean always means non-trivially strongly clean**. In this section we give a characterization of strongly clean elements in $M_2(Z)$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z)$. If either b = 0 or c = 0, a characterization of strong cleanness of *A* can be found in [6]. Hence we assume $b \neq 0$ and $c \neq 0$.

By Corollary 2.3 it follows that a matrix $A \in M_2(Z)$ is strongly clean if and only if it is similar to one of $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

Thus for $A \in M_2(Z)$ to be strongly clean, exactly one of the following is necessary

(1) trace (A) = 3 and det(A) = 2
(2) trace (A) = 1 and det(A) = 0
(3) trace (A) = -1 and det(A) = 0
(4) trace (A) = 1 and det(A) = -2

In each of the above cases, we shall obtain intrinsic characterizations for a matrix *A* to be strongly clean.

Case: 1 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z)$ be such that trace (A) = 3 and det (A) = 2.

In view of trace (A) = 3 and det(A) = 2 we can assume that $(a,d)\neq(1,2)$ or $(2,1)$. Otherwise bc = 0 which is not true since *b* and *c* both are non zero according to our assumption.

Lemma: 3.1 Let A be similar to the matrix $\begin{bmatrix}2&0\\0&1\end{bmatrix}$ so that there exists an invertible matrix $C=\begin{bmatrix}c_1&c_2\\c_3&c_4\end{bmatrix}$ such that $CAC^{-1}=\begin{bmatrix}2&0\\0&1\end{bmatrix}$. Then $c_1=\gcd(a-1,c)$, $c_2=\gcd(d-1,b)$, $c_3=\gcd(a-2,c)$ and $c_4=\gcd(d-2,b)$.

Proof: In view of $CAC^{-1}=\begin{bmatrix}2&0\\0&1\end{bmatrix}$, we get the following equations:

$$c_1(a-2)+c_2c=0 \tag{i}$$

$$c_1b+c_2(d-2)=0 \tag{ii}$$

$$c_3(a-1)+c_4c=0 \tag{iii}$$

$$c_3b+c_4(d-1)=0 \tag{iv}$$

and by the invertibility of *C* we also have

$$c_1c_4-c_2c_3=\pm1 \tag{v}$$

In view of the above set of equations and the conditions trace(A)= 3 and det(A) = 2 we have,

Claim: $c_1c_3=\mp c$, $c_2c_4=\pm b$, $c_1c_4=\pm(a-1)$, $c_2c_3=\pm(1-d)$.

Multiplying both sides of (v) by c we get $(c_1c_4)c-(c_2c_3)=\pm c$. Using (i) and (iii) we have $c_1(-c_3(a-1))-(-c_1(a-2))c_3=\pm c$ or $c_1c_3=\mp c$. This proves the first part of the claim.

The other parts of the claim can be proved similarly.

As c_1 and c_2 are co prime (by (v)), by (i) and (ii) respectively we have $c_1 \mid c$ and $c_1 \mid (d-2)$ and hence $c_1 \mid (a-1)$ (as trace (A) = 3).

We show that $c_1=\gcd(a-1,c)$ or equivalently that $\gcd\left(\frac{a-1}{c_1},\frac{c}{c_1}\right)=1$ or $\gcd\left(\frac{a-1}{c_1},c_3\right)=1$ (as $c_1c_3=\mp c$, by the claim).

Consider a prime *p* such that $p \mid \frac{a-1}{c_1}$ and $p \mid c_3$ i.e. $p \mid (a-1)$ and $p \mid c_3$. From (v) we have that c_3 and c_4 are co prime. Therefore by (iv) it follows that $c_3 \mid (d-1)$ and hence by the trace condition $c_3 \mid (a-2)$. Now $p \mid c_3$ and $c_3 \mid (a-2)$ implies that $p \mid (a-2)$. But $p \mid (a-1)$, hence $p = 1$. Thus $c_1=\gcd(a-1,c)$.

The proofs for $c_2=\gcd(d-1,b)$, $c_3=\gcd(a-2,c)$ and $c_4=\gcd(d-2,b)$ are similar.

Theorem: 3.2 Let $\alpha=\gcd(a-2,c)$ and $\beta=\gcd(a-1,c)$. Then *A* is similar to $\begin{bmatrix}2&0\\0&1\end{bmatrix}$ if and only if $\alpha\beta=\mp c$.

Proof: Necessity follows from Lemma 3.1 noting that $\alpha=c_3$ and $\beta=c_1$.

For sufficiency, let $C=\begin{bmatrix} -\dfrac{c}{\alpha} & \dfrac{(a-2)}{\alpha} \\ -\dfrac{c}{\beta} & \dfrac{(a-1)}{\beta} \end{bmatrix}$, where $\alpha\beta=\mp c$. Clearly, $c_1=-\dfrac{c}{\alpha}$, $c_2=\dfrac{(a-2)}{\alpha}$, $c_3=-\dfrac{c}{\beta}$ and $c_4=\dfrac{(a-1)}{\beta}$ satisfy equations (i) and (iii) of the Lemma 3.1. Using the conditions trace(A)= 3 and det(A) = 2, one can see that eq. (ii) of Lemma 3.1 follows from eq. (i) and eq. (iv) of Lemma 3.1 follows from (iii).

Hence as $c_1=-\dfrac{c}{\alpha}$, $c_2=\dfrac{(a-2)}{\alpha}$, $c_3=-\dfrac{c}{\beta}$ and $c_4=\dfrac{(a-1)}{\beta}$ (i) and (iii), they also satisfy (ii) and (iv). Furthermore, in view of $\alpha\beta=\mp c$ one can easily see that $\det(C)=\pm 1$. Thus C is invertible i.e. eq. (v) of Lemma 3.1 is also satisfied. This completes the proof.

Remark: 3.3 In the above theorem we can replace α,β by:

(1) $\alpha=\gcd(d-2,b)$, $\beta=\gcd(d-1,b)$ with $\alpha\beta=\pm b$.

(2) $\alpha=\gcd(d-2,b)$, $\beta=\gcd(a-1,c)$ with $\alpha\beta=\pm(a-1)$.

(3) $\alpha=\gcd(a-2,c)$, $\beta=\gcd(d-1,b)$ with $\alpha\beta=\pm(1-d)$.

Case: 2 Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in M_2(Z)$ be such that trace(A)= 1 and det(A) = 0.

In view of trace (A) = 1 and det(A) = 0 we can assume that $(a,d)\neq(0,1)$ or $(1,0)$. Otherwise $bc = 0$ which is not true since b and c both are non zero according to our assumption.

Lemma: 3.4 Let A be similar to the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so that there exists an invertible matrix $C=\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$ such that $CAC^{-1}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $c_1=\gcd(a-1,c)$, $c_2=\gcd(d-1,b)$, $c_3=\gcd(a,c)$ and $c_4=\gcd(d,b)$.

Proof: In view of $CAC^{-1}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we get the following equations:

$$c_1a+c_2c=0 \tag{i}$$
$$c_1b+c_2d=0 \tag{ii}$$
$$c_3(a-1)+c_4c=0 \tag{iii}$$
$$c_3b+c_4(d-1)=0 \tag{iv}$$

and by the invertibility of C we also have
$$c_1c_4-c_2c_3=\pm 1 \tag{v}$$

In view of the above set of equations and the conditions trace(A) = 1 and det(A) = 0, as in Lemma 3.1 we can establish the following :

Claim: $c_1c_3=\pm c$, $c_2c_4=\mp b$, $c_1c_4=\pm d$, $c_2c_3=\mp a$.

As c_1 and c_2 are co prime (by (v)), by (i) and (ii) respectively we have $c_1 \mid c$ and $c_1 \mid d$ and hence $c_1 \mid (a - 1)$ (as trace (A) = 1).

We prove that $c_1=\gcd(a-1,c)$ or equivalently that $\gcd\left(\dfrac{a-1}{c_1},\dfrac{c}{c_1}\right)=1$ or $\gcd\left(\dfrac{a-1}{c_1},c_3\right)=1$ (as $c_1c_3=\pm c$, by the claim).

Consider a prime p such that $p \mid \dfrac{a-1}{c_1}$ and $p \mid c_3$ i.e. $p \mid (a-1)$ and $p \mid c_3$. From (v) we have that c_3 and c_4 are co prime. Therefore by (iv) it follows that $c_3 \mid (d-1)$ and hence by the trace condition $c_3 \mid a$. Now $p \mid c_3$ and $c_3 \mid a$ implies that $p \mid a$. But $p \mid (a-1)$, hence $p = 1$. This proves that $c_1 = \gcd(a-1, c)$. The proofs for $c_2 = \gcd(d-1, b)$, $c_3 = \gcd(a, c)$ and $c_4 = \gcd(d, b)$ are similar.

Theorem: 3.5 Let $\alpha = \gcd(a, c)$ and $\beta = \gcd(a-1, c)$. Then A is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ if and only if $\alpha\beta = \pm c$.

Proof: Necessity follows from Lemma 3.4 noting that $\alpha = c_3$ and $\beta = c_1$.

For sufficiency, let $C = \begin{bmatrix} -\dfrac{c}{\alpha} & \dfrac{a}{\alpha} \\ -\dfrac{c}{\beta} & \dfrac{(a-1)}{\beta} \end{bmatrix}$, where $\alpha\beta = \pm c$. Clearly, $c_1 = -\dfrac{c}{\alpha}$, $c_2 = \dfrac{a}{\alpha}$, $c_3 = -\dfrac{c}{\beta}$ and $c_4 = \dfrac{(a-1)}{\beta}$ satisfy equations (i) and (iii) of the Lemma 3.4. Using the conditions trace(A) = 1 and det(A) = 0, one can see that the equations (ii) and (iv) of Lemma 3.4 follow from (i) and (iii) respectively. Therefore, as $c_1 = -\dfrac{c}{\alpha}$, $c_2 = \dfrac{a}{\alpha}$, $c_3 = -\dfrac{c}{\beta}$ and $c_4 = \dfrac{(a-1)}{\beta}$ satisfy (i) and (iii), they also satisfy (ii) and (iv). Furthermore, in view of $\alpha\beta = \pm c$ one can easily check that $\det(C) = \pm 1$. Thus C is invertible i.e. eq. (v) of Lemma 3.4 is also satisfied. This completes the proof.

Remark: 3.6. In the above theorem we can replace α, β by:
(1) $\alpha = \gcd(d, b)$, $\beta = \gcd(d-1, b)$ with $\alpha\beta = \mp b$.
(2) $\alpha = \gcd(d, b)$, $\beta = \gcd(a-1, c)$ with $\alpha\beta = \pm d$.
(3) $\alpha = \gcd(a, c)$, $\beta = \gcd(d-1, b)$ with $\alpha\beta = \mp a$.

Case: 3 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z)$ be such that trace(A) = -1 and det(A) = 0.

In view of trace(A) = -1 and det(A) = 0 we can assume that $(a, d) \neq (0, -1)$ or $(-1, 0)$. Otherwise $bc = 0$ which is not true since b and c both are non zero according to our assumption.

Lemma: 3.7 Let A be similar to the matrix $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ so that there exists an invertible matrix $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$ such that $CAC^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. Then $c_1 = \gcd(a+1, c)$, $c_2 = \gcd(d+1, b)$, $c_3 = \gcd(a, c)$ and $c_4 = \gcd(d, b)$.

Proof: In view of $CAC^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, we get the following equations:

$$c_1 a + c_2 c = 0 \tag{i}$$
$$c_1 b + c_2 d = 0 \tag{ii}$$
$$c_3(a+1) + c_4 c = 0 \tag{iii}$$
$$c_3 b + c_4(d+1) = 0 \tag{iv}$$

and by the invertibility of C we also have

$$c_1 c_4 - c_2 c_3 = \pm 1 \tag{v}$$

In view of the above set of equations and the conditions trace(A) = -1 and det(A) = 0, we have,

Claim: $c_1c_3 = \mp c$, $c_2c_4 = \pm b$, $c_1c_4 = \mp d$, $c_2c_3 = \pm a$.

This claim can be proved as in Lemma 3.1.

As c_1 and c_2 are co prime (by (v)), by (i) and (ii) respectively we have $c_1 \mid c$ and $c_1 \mid d$ and hence $c_1 \mid (a + 1)$ (as trace (A) = -1).

We prove that $c_1 = \gcd(a+1, c)$ or equivalently that $\gcd\left(\frac{a+1}{c_1}, \frac{c}{c_1}\right) = 1$ or $\gcd\left(\frac{a+1}{c_1}, c_3\right) = 1$ (as $c_1c_3 = \mp c$, by the claim).

Consider a prime p such that $p \mid \frac{a+1}{c_1}$ and $p \mid c_3$ i.e. $p \mid (a + 1)$ and $p \mid c_3$. From (v) we have that c_3 and c_4 are co prime. Therefore by (iv) it follows that $c_3 \mid (d + 1)$ and hence by the trace condition $c_3 \mid a$. Now $p \mid c_3$ and $c_3 \mid a$ implies that $p \mid a$. But $p \mid (a + 1)$, hence $p = 1$. This proves that $c_1 = \gcd(a+1, c)$. The proofs for $c_2 = \gcd(d+1, b)$, $c_3 = \gcd(a, c)$ and $c_4 = \gcd(d, b)$ are similar.

Theorem: 3.8. Let $\alpha = \gcd(a, c)$ and $\beta = \gcd(a+1, c)$. Then A is similar to $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ if and only if $\alpha\beta = \mp c$.

Proof: Necessity follows from Lemma 3.7 noting that $\alpha = c_3$ and $\beta = c_1$.

For sufficiency, let $C = \begin{bmatrix} -\frac{c}{\alpha} & \frac{a}{\alpha} \\ -\frac{c}{\beta} & \frac{(a+1)}{\beta} \end{bmatrix}$, where $\alpha\beta = \mp c$. Clearly, $c_1 = -\frac{c}{\alpha}$, $c_2 = \frac{a}{\alpha}$, $c_3 = -\frac{c}{\beta}$ and $c_4 = \frac{(a+1)}{\beta}$ satisfy equations (i) and (iii) of the Lemma 3.7.

Using the conditions trace(A)= -1 and det(A) = 0 it can be seen that the equations (ii) and (iv) of Lemma 3.7 follow from (i) and (iii) respectively. Therefore, as $c_1 = -\frac{c}{\alpha}$, $c_2 = \frac{a}{\alpha}$, $c_3 = -\frac{c}{\beta}$ and $c_4 = \frac{(a+1)}{\beta}$ satisfy (i) and (iii), they also satisfy (ii) and (iv) . Furthermore, in view of $\alpha\beta = \mp c$ one can easily see that $\det(C) = \pm 1$. Thus C is invertible i.e. eq. (v) of Lemma 3.7 is also satisfied. This completes the proof.

Remark: 3.9 In the above theorem we can replace α, β by:

(1) $\alpha = \gcd(d, b)$, $\beta = \gcd(d+1, b)$ with $\alpha\beta = \pm b$.
(2) $\alpha = \gcd(d, b)$, $\beta = \gcd(a+1, c)$ with $\alpha\beta = \mp d$.
(3) $\alpha = \gcd(a, c)$, $\beta = \gcd(d+1, b)$ with $\alpha\beta = \pm a$.

Case: 4 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z)$ be such that trace(A) = 1 and det(A) = -2.

In view of trace (A) = 1 and det(A) = -2 we can assume that $(a, d) \neq (-1, 2)$ or $(2, -1)$. Otherwise $bc = 0$ which is not true since b and c both are non zero according to our assumption.

Lemma: 3.10 Let A be similar to the matrix $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ so that there exists an invertible matrix $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$ such that $CAC^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. Then $3c_1 = \gcd(a+1, c)$, $3c_2 = \gcd(d+1, b)$, $3c_3 = \gcd(a-2, c)$ and $3c_4 = \gcd(d-2, b)$.

Proof: In view of $CAC^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, we get the following equations:

$$c_1(a-2)+c_2c=0 \tag{i}$$
$$c_1b+c_2(d-2)=0 \tag{ii}$$
$$c_3(a+1)+c_4c=0 \tag{iii}$$
$$c_3b+c_4(d+1)=0 \tag{iv}$$

and by the invertibility of C we also have

$$c_1c_4-c_2c_3=\pm 1 \tag{v}$$

In view of the above set of equations and the conditions trace(A)= 1 and det(A) = -2 we have,

Claim: $3c_1c_3 = \mp c$, $3c_2c_4 = \pm b$, $3c_1c_4 = \pm(a+1)$, $3c_2c_3 = \mp(1+d)$.

Multiplying both sides of (v) by c we get $c_1(c_4c)-(c_2c)c_3 = \pm c$ by (i) and (iii) $c_1(-c_3(a+1))-(-c_1(a-2))c_3 = \pm c$ or $3c_1c_3 = \mp c$. This proves the first part of the claim.

The other parts of the claim can be proved similarly.

As c_1 and c_2 are co prime (by (v)), by (i) and (ii) respectively we have $c_1 \mid c$ and $c_1 \mid (d-2)$ and hence $c_1 \mid (a+1)$ (as trace (A) = 1).

We prove that $3c_1 = \gcd(a+1, c)$ or equivalently that $\gcd\left(\frac{a+1}{3c_1}, \frac{c}{3c_1}\right) = 1$

Consider a prime p such that $p \mid \frac{a+1}{3c_1}$ and $p \mid \frac{c}{3c_1}$ i.e. $p \mid \frac{a+1}{3}$ and $p \mid \frac{(a-2)}{3}$ (using eq. (i) of the above Lemma). Therefore $p \mid \frac{[(a+1)-(a-2)]}{3}$ or $p \mid 1$. Hence $p = 1$. Thus $3c_1 = \gcd(a+1, c)$. The proofs for $3c_2 = \gcd(d+1, b)$ $3c_3 = \gcd(a-2, c)$ and $3c_4 = \gcd(d-2, b)$ are similar.

Theorem: 3.11 Let $\alpha = \gcd(a-2, c)$ and $\beta = \gcd(a+1, c)$. Then A is similar to $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ if and only if $\alpha\beta = \mp 3c$.

Proof:. Necessity follows from Lemma 3.10 noting that $\alpha = 3c_3$ and $\beta = 3c_1$.

For sufficiency, let $C = \begin{bmatrix} -\frac{c}{\alpha} & \frac{(a-2)}{\alpha} \\ -\frac{c}{\beta} & \frac{(a+1)}{\beta} \end{bmatrix}$, where $\alpha\beta = \mp 3c$. Clearly, $c_1 = -\frac{c}{\alpha}$, $c_2 = \frac{(a-2)}{\alpha}$, $c_3 = -\frac{c}{\beta}$ and $c_4 = \frac{(a+1)}{\beta}$ satisfy equations (i) and (iii) of the Lemma 3.10.

Using the conditions trace(A) = 1 and det(A) = -2 it can be shown that the equations (ii) and (iv) of Lemma 3.10 follow from (i) and (iii) respectively. Hence, as $c_1 = -\frac{c}{\alpha}$, $c_2 = \frac{(a-2)}{\alpha}$, $c_3 = -\frac{c}{\beta}$ and $c_4 = \frac{(a+1)}{\beta}$ satisfy (i) and (iii), they also satisfy (ii) and (iv). Furthermore, in view of $\alpha\beta = \mp 3c$ one can easily see that $\det(C) = \pm 1$. Thus C is invertible i.e. eq. (v) of Lemma 3.10 is also satisfied. This completes the proof.

Remark: 3.12 In the above theorem we can replace α, β by:

1. $\alpha = \gcd(d-2, b)$, $\beta = \gcd(d+1, b)$ with $\alpha\beta = \pm 3b$.
2. $\alpha = \gcd(d-2, b)$, $\beta = \gcd(a+1, c)$ with $\alpha\beta = \pm 3(a+1)$.
3. $\alpha = \gcd(a-2, c)$, $\beta = \gcd(d+1, b)$ with $\alpha\beta = \mp 3(1+d)$.

In view of the discussion in the beginning of section 3, combining Theorems 3.2, 3.5, 3.8 and 3.11 one gets the desired intrinsic characterizations for $A \in M_2(Z)$ to be strongly clean.

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