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FINITE DIMENSIONAL FUZZY ANTI 2- NORMED LINEAR SPACE

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ABSTRACT

In this paper we have generalized fuzzy anti 2-norm by introducing t-conorm in the earlier definition. The Riesz lemma and a few properties of finite dimensional fuzzy anti 2-normed linear space has been established with respect to t-conorm $\Diamond$.

Keywords: *Fuzzy anti 2- norm, α -2-norm, Riesz lemma.*

INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [11] in 1965 and thereafter several authors applied it different branches of pure and applied mathematics. The concept of fuzzy norm was introduced by Katsaras [9] in 1984. In 1992 Felbin [8] introduced the concept of fuzzy normed linear space. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gähler [6]. Jebril and Samanta [7] gave the definition of fuzzy anti-normed linear space. In 2011, B. Surender Reddy [1] introduced the idea of fuzzy anti 2-normed linear space.

In the present paper we have modified the definition of fuzzy anti 2- normed linear space. The Riesz lemma and important properties of finite dimensional fuzzy anti 2- normed linear space has been established with respect to t-conorm $\Diamond$.

PRELIMINARIES

Definition 2.1[10]: A binary operation $\Diamond : [0,1]\times[0,1] \to [0,1]$ is a t- conorm if $\Diamond$ satisfies the following condition:

(i) $\Diamond$ is commutative and associative,

(ii) $a \Diamond 0 = a, \quad \forall a \in [0,1]$,

(iii) $a \Diamond b \le c \Diamond d$, whenever $a \le c, b \le d$ and $a,b,c,d \in [0,1]$.

Example: (i) $a \Diamond b$ = a+b-ab (ii) $a \Diamond b$ = max {a, b} (iii) $a \Diamond b$ = {a+b, 1}

Definition 2.2[1]: Let X be a linear space over a real field F . A fuzzy subset N^* of $X \times X \times R$ is called a fuzzy anti 2-norm on X if and only if it satisfies,

(Fa2-N1) for all $t \in R$ with $t \le 0$, $N^*(x_1, x_2, t) = 1$

(Fa2-N2) for all $t \in R$ with $t > 0$, $N^*(x_1, x_2, t) = 0$ if and only if x_1 and x_2 are linearly dependent.

(Fa2-N3) $N^*(x_1, x_2, t)$ is invariant under any permutation.

(Fa2-N4) for all $t \in R$ with $t > 0$, $N^*(x_1, cx_2, t) = N^*\left(x_1, x_2, \frac{t}{|c|}\right)$ if $c \ne 0$, $c \in F$

(Fa2-N5) for $s,t \in R$ with $t > 0$ all $N^*(x_1, x_2 + x_2', s+t) \le \max\{N^*(x_1, x_2, s), N^*(x_1, x_2', t)\}$

(Fa2-N6) $N^*(x_1, x_2, t)$ is non-increasing function of $t \in R$ and $\lim\limits_{t\to\infty} N^*(x_1, x_2, t) = 0.$

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Then (X, N^*) is called a fuzzy anti 2-normed linear space. The following condition of fuzzy anti 2-norm N^* will be required later on,

(Fa2-N7) for $t \in R$ with $t > 0$, $N^*(x_1, x_2, t) < 1, \ \forall t > 0 \Rightarrow x_1$ and x_2 are linearly dependent.

Definition 2.3[1]: Let (X, N^*) be a fuzzy anti 2- normed linear space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if $t > 0, \ 0 < r < 1$, $\exists$ an integer $n_0 \in N$ such that $N^*(x_n - x, x_0, t) < r, \ \forall n \geq n_0$.

Definition 2.4[1]: Let (X, N^*) be a fuzzy anti 2- normed linear space. A sequence $\{x_n\}$ in X is said to be cauchy sequence if $t > 0, \ 0 < r < 1$, $\exists$an integer $n_0 \in N$ such that $N^*(x_{n+p} - x_n, x_0, t) < r$, for all $n \geq n_0$. $p = 1,2,3.....$

Definition 2.5[3]: A subset A of a fuzzy anti 2-normed linear space (X, N^*) is said to be bounded iff $\exists$ t > 0, $r \in (0,1)$ $s.t, \ N^*(x, y, t) < r, \ \forall x, y \in A$.

Definition 2.6[3]: Let (X, N^*) be a fuzzy anti 2- normed linear space. A subset B of X is said to be closed if any sequence $\{x_n\}$ in B converges to $x \in B$ that is $\lim_{n \to \infty} N^*(x_n - x, y, t) = 0, \quad \forall \ t > 0 \quad \Rightarrow x, y \in B.$

Definition 2.7[1]: A subset A of a fuzzy anti 2-normed linear space (X, N^*) is said to be compact if any sequence $\{x_n\}$ in A has a subsequence converging to an element of A.

3. FUZZY ANTI 2- NORMED LINEAR SPACE

In this section we have modified the definition of fuzzy anti 2- norm with respect to a t-conorm $\Diamond$ and deduced some important results.

Definition 3.1: Let X be a linear space over a real field F. A fuzzy subset N^* of $X \times X \times R$ is called a fuzzy anti 2-norm on X if and only if it satisfies,

(Fa2-N1) for all $t \in R$ with $t \leq 0, N^*(x_1, x_2, t) = 1$

(Fa2-N2) for all $t \in R$ with $t > 0, N^*(x_1, x_2, t) = 0$ if and only if x_1 and x_2 are linearly dependent.

(Fa2-N3) $N^*(x_1, x_2, t)$ is invariant under any permutation of x_1 and x_2.

(Fa2-N4) for all $t \in R$ with $t > 0$

$$N^*(x_1, cx_2, t) = N^*\left(x_1, x_2, \frac{t}{|c|}\right), \text{ if } c \neq 0, c \in F$$

(Fa2-N5) for $s, t \in R$ with $t > 0$ all $N^*(x_1, x_2 + x_2', s + t) \leq \{N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t)\}$

(Fa2-N6) $N^*(x_1, x_2, t)$ is non-increasing function of $t \in R$ and $\lim_{t \to \infty} N^*(x_1, x_2, t) = 0$

We further assume that for a fuzzy anti 2- normed linear space (X, N^*),

(Fa2-N7) for all $t \in R$ with $t > 0$, $N^*(x_1, x_2, t) < 1, \ \forall t > 0 \Rightarrow x_1$ and x_2 are linearly dependent.

(Fa2-N8) $N^*(x_1, x_2, .)$ is a continuous function on R and strictly decreasing on the subset $\{t : 0 < N^*(x_1, x_2, t) < 1\}$ of R.

(Fa2-N9) $a \Diamond a = a \ , \ \forall a \in [0,1]$.

Remark 3.1: Let N^* be a fuzzy anti 2- norm on X then $N^*(x_1, x_2, t)$ is non-increasing with respect to t for each $x_1, x_2 \in X$.

Proof: Let $t < s$. Then $k = s - t > 0$, we have

$$N^*(x_1, x_2, t) = N^*(x_1, x_2, t) \Diamond 0 \qquad \text{(by property of t-conorm)}$$
$$= N^*(x_1, x_2, t) \Diamond N^*(0,0,k) \geq N^*(x_1, x_2, t + k) = N^*(x_1, x_2, s).$$

Hence Proved

Example 3.1: Let $(X,\|.,.\|)$ be 2-normed linear space and define $a \lozenge b = a+b-ab$. Define $N^*: X \times X \times R \to [0,1]$ by

$$N^*(x_1,x_2,t)=\begin{cases}0, & \text{if } t>\|x_1,x_2\| \\ 1, & \text{if } t\le\|x_1,x_2\|\end{cases}$$

Then N^* is a fuzzy anti 2- norm on X with respect to the t- conorm $\lozenge$ and (X,N^*) is a fuzzy anti 2- normed linear space with respect to the t- conorm $\lozenge$.

Solution:

(i) $\forall x_1,x_2 \in X\times X$ and $\forall t\in R, t\le 0$ we have $N^*(x_1,x_2,t)=1.$

(ii) $\forall t\in R, t\le 0$ if x_1,x_2 are linearly dependent then $\|x_1,x_2\|=0$ so $N^*(x_1,x_2,t)=0.$ Again if $N^*(x_1,x_2,t)=0$ with $t>0 \Rightarrow \|x_1,x_2\|<t, \forall t(>0)\in R \Rightarrow \|x_1,x_2\|=0 \Rightarrow$ x_1,x_2 are linearly dependent.

(iii) It is obvious that $N^*(x_1,x_2,t)$ is invariant under any permutation.

(iv) If $N^*(x_1,cx_2,t)=0 \Leftrightarrow t>\|x_1,cx_2\| \Leftrightarrow t>|c|\|x_1,x_2\| \Leftrightarrow \frac{t}{|c|}>\|x_1,x_2\|$

$$\Leftrightarrow N^*\left(x_1,x_2,\frac{t}{|c|}\right)=0$$

$$N^*(x_1,cx_2,t)=1 \Leftrightarrow t\le\|x_1,cx_2\| \Leftrightarrow t\le|c|\|x_1,x_2\| \Leftrightarrow \frac{t}{|c|}\le\|x_1,x_2\| \Leftrightarrow N^*\left(x_1,x_2,\frac{t}{|c|}\right)=1.$$

(v) $N^*(x_1,x_2,s)\lozenge N^*(x_1,x_2',t)=N^*(x_1,x_2,s)+N^*(x_1,x_2',t)-N^*(x_1,x_2,s).N^*(x_1,x_2',t).$

If $s>\|x_1,x_2\|$ and $t>\|x_1,x_2'\|$ so $s+t>\|x_1,x_2\|+\|x_1,x_2'\|$ then $N^*(x_1,x_2+x_2',s+t)=0$
and $N^*(x_1,x_2,s)\lozenge N^*(x_1,x_2',t)=0+0-0=0.$

So $N^*(x_1,x_2+x_2',s+t)=N^*(x_1,x_2,s)\lozenge N^*(x_1,x_2',t).$

If $s>\|x_1,x_2\|$ and $t\le\|x_1,x_2'\|$ then $N^*(x_1,x_2,s)\lozenge N^*(x_1,x_2',t)=0+1-0=1.$

If $s\le\|x_1,x_2\|$ and then $N^*(x_1,x_2,s)\lozenge N^*(x_1,x_2',t)=1+0-0=1.$

If $s\le\|x_1,x_2\|$ and $t\le\|x_1,x_2'\|$ then $N^*(x_1,x_2,s)\lozenge N^*(x_1,x_2',t)=1+1-1=1.$
Then in all the above three cases,
$N^*(x_1,x_2,s)\lozenge N^*(x_1,x_2',t)=1\ge N^*(x_1,x_2+x_2',s+t).$
Thus $N^*(x_1,x_2+x_2',s+t)\le N^*(x_1,x_2,s)\lozenge N^*(x_1,x_2',t).$

(vi) From the definition if $t>\|x_1,x_2\|$, then $\lim_{t\to\infty} N^*(x_1,x_2,t)=0$. Thus (X,N^*) is a fuzzy anti 2- normed linear space with respect to the t- conorm $\lozenge$.

Example 3.2: Let $(X,\|.,.\|)$ be 2- normed linear space and define $a \lozenge b = \min\{a+b, 1\}$. Define $N^*: X \times X \times R \to [0,1]$ by

$$N^*(x_1,x_2,t)=\begin{cases}0, & \text{if } t>\|x_1,x_2\| \\ \dfrac{\|x_1,x_2\|}{t+\|x_1,x_2\|}, & \text{if } t\le\|x_1,x_2\|, t>0 \\ 1, & \text{if } t\le 0\end{cases}$$

Then N^* is a fuzzy anti 2- norm on X with respect to the t- conorm $\lozenge$ and (X,N^*) is a fuzzy anti 2- normed linear space with respect to the t- conorm $\lozenge$.

Solution:

(i) From the definition we have $N^*(x_1,x_2,t)=1$ if $\forall t\in R, t\le 0.$

(ii) If $t>0$ and $t \le \|x_1, x_2\|$ the $N^*(x_1, x_2, t) = \dfrac{\|x_1, x_2\|}{t+\|x_1, x_2\|}$ if x_1, x_2 are linearly dependent so $\|x_1, x_2\| = 0$ therefore $N^*(x_1, x_2, t) = 0$.

Conversely, $N^*(x_1, x_2, t) = 0$ then $t > \|x_1, x_2\|, \forall t \Rightarrow \|x_1, x_2\| = 0$, so x_1, x_2 are linearly dependent.

(iii) It is obvious that $N^*(x_1, x_2, t)$ is invariant under any permutation of x_1 and x_2.

(iv) If $N^*(x_1, cx_2, t) = 0 \Leftrightarrow t > \|x_1, cx_2\| \Leftrightarrow t > |c|\ \|x_1, x_2\| \Leftrightarrow \dfrac{t}{|c|} > \|x_1, x_2\|$

$$\Leftrightarrow N^*\left(x_1, x_2, \frac{t}{|c|}\right) = 0$$

If $N^*(x_1, cx_2, t) = \dfrac{\|x_1, cx_2\|}{t+\|x_1, cx_2\|} \Leftrightarrow t \le \|x_1, cx_2\| \Leftrightarrow \dfrac{t}{|c|} \le \|x_1, x_2\|$

$$\Leftrightarrow N^*\left(x_1, x_2, \frac{t}{|c|}\right) = \frac{\|x_1, x_2\|}{\frac{t}{|c|}+\|x_1, x_2\|} = \frac{\|x_1, cx_2\|}{t+\|x_1, cx_2\|}$$

(v) $N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t) = \min\{N^*(x_1, x_2, s) + N^*(x_1, x_2', t),\ 1\}$ If $\|x_1, x_2\| \ge s$ and $\|x_1, x_2'\| \ge t$ then

$$N^*(x_1, x_2, s) + N^*(x_1, x_2', t) = \frac{\|x_1, x_2\|}{s+\|x_1, x_2\|} + \frac{\|x_1, x_2'\|}{t+\|x_1, x_2'\|}$$
$$= \frac{\left(t\|x_1, x_2\| + \|x_1, x_2\|\|x_1, x_2'\| + s\|x_1, x_2'\| + \|x_1, x_2\|\|x_1, x_2'\|\right)}{\left(t\|x_1, x_2\| + \|x_1, x_2\|\|x_1, x_2'\| + s\|x_1, x_2'\|\right) + st} \ge 1$$

Since $\|x_1, x_2\|\|x_1, x_2'\| > st.$

In this case $N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t) = 1 \ge N^*(x_1, x_2 + x_2', s+t).$

If $\|x_1, x_2\| \ge s$ and $\|x_1, x_2'\| < t$ then either $\|x_1, x_2 + x_2'\| \ge s+t$ or $\|x_1, x_2 + x_2'\| < s+t.$

Now, $N^*(x_1, x_2, s) + N^*(x_1, x_2', t) = \dfrac{\|x_1, x_2\|}{s+\|x_1, x_2\|} + 0 < 1.$

Hence $N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t) = \dfrac{\|x_1, x_2\|}{s+\|x_1, x_2\|}.$

If $\|x_1, x_2 + x_2'\| \ge s+t$ then consider

$$N^*(x_1, x_2 + x_2', s+t) - N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t) = \frac{\|x_1, x_2 + x_2'\|}{s+t+\|x_1, x_2 + x_2'\|} - \frac{\|x_1, x_2\|}{s+\|x_1, x_2\|}$$
$$\le \frac{\|x_1, x_2\| + \|x_1, x_2'\|}{s+t+\|x_1, x_2\| + \|x_1, x_2'\|} - \frac{\|x_1, x_2\|}{s+\|x_1, x_2\|}$$
$$= \frac{s\|x_1, x_2'\| - t\|x_1, x_2\|}{\left(s+t+\|x_1, x_2\| + \|x_1, x_2'\|\right)\left(s+\|x_1, x_2\|\right)}$$
$$< \frac{st - t\|x_1, x_2\|}{\left(s+t+\|x_1, x_2\| + \|x_1, x_2'\|\right)\left(s+\|x_1, x_2\|\right)},\ \text{Since } \|x_1, x_2'\| < t,$$
$$\le 0,\ \text{Since } s \le \|x_1, x_2\| \text{ so } st < t\|x_1, x_2\| .$$

So, $N^*(x_1, x_2 + x_2', s+t) < N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t)$.

If $\|x_1, x_2 + x_2'\| < s+t$ then

$$N^*(x_1, x_2 + x_2', s+t) = 0 \le \frac{\|x_1, x_2\|}{s+\|x_1, x_2\|} = N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t)$$

If $\|x_1, x_2\| < s$ and $\|x_1, x_2'\| \geq t$ then in the similar way we can show that
$N^*(x_1, x_2 + x_2', s+t) \leq N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t)$.

If $\|x_1, x_2\| < s$ and $\|x_1, x_2'\| \geq t$ then $N^*(x_1, x_2, s) + N^*(x_1, x_2', t) = 0 + 0 < 1$.
Therefore, $N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t) = 0$.

Also $\|x_1, x_2 + x_2'\| \leq \|x_1, x_2\| + \|x_1, x_2'\| < s+t$ and $N^*(x_1, x_2 + x_2', s+t) = 0$.

So $N^*(x_1, x_2 + x_2', s+t) = N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t)$.

So $N^*(x_1, x_2 + x_2', s+t) \leq N^*(x_1, x_2, s) \Diamond N^*(x_1, x_2', t)$

(vi) If $t > \|x_1, x_2\|$ then from the definition $\lim_{t\to\infty} N^*(x_1, x_2, t) = 0$. If x_1, x_2 are not independent and $t \leq \|x_1, x_2\|$ then

$\lim_{t\to\infty} N^*(x_1, x_2, t) = \lim_{t\to\infty} \frac{\|x_1, x_2\|}{t + \|x_1, x_2\|} = 0$.

If x_1, x_2 are linearly dependent and $t \leq \|x_1, x_2\|$ then $\lim_{t\to\infty} N^*(x_1, x_2, t) = 0.$

Hence $\lim_{t\to\infty} N^*(x_1, x_2, t) = 0.\ \forall x_1, x_1 \in X \times X$.

Thus N^* is a fuzzy anti 2-norm on X with respect to the t-conorm $\Diamond$ and (X, N^*) is a fuzzy anti 2-normed linear space with respect to the t-conorm $\Diamond$.

Example 3.3: Let $(X, \|.,.\|)$ be 2-normed linear space and define a $\Diamond$ b = min{a+b, 1}. Define $N^* : X \times X \times R \to [0,1]$ by
$$N^*(x_1, x_2, t) = \begin{cases} \dfrac{\|x_1, x_2\|}{2t - \|x_1, x_2\|}, & \text{if } t > \|x_1, x_2\| \\ 1, & \text{if } t \leq \|x_1, x_2\| \end{cases}$$
Then N^* satisfies all the condition of fuzzy anti 2- norm with respect to t-conorm $\Diamond$ .So N^* is a fuzzy anti 2- norm on X with respect to the t- conorm $\Diamond$ and (X, N^*) is a fuzzy anti 2- normed linear space with respect to the t- conorm $\Diamond$.

Theorem 3.1: Let (X, N^*) be a fuzzy anti 2- normed linear space with respect to a t- conorm $\Diamond$ satisfying (Fa2-N7) and (Fa2-N9). Then for any $\alpha \in (0,1)$ the function $\|x_1, x_2\|_\alpha^* : X \times X \times R \to [0, \infty)$ defined as
$\|x_1, x_2\|_\alpha^* = \wedge \{t > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha\},\ \alpha \in (0,1)$.
is a 2-norm on X. Then $\{\|.,.\|_\alpha^* : \alpha \in (0,1)\}$ is an ascending family of 2-norm on a linear space X.

Proof:
(i) For x_1, x_2 for $t \leq 0$, so $N^*(x_1, x_2, t) \leq 1 - \alpha$ is not possible.
So $\wedge \{t > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha\} \geq 0,\ \alpha \in (0,1) \Rightarrow \|x_1, x_2\|_\alpha^* \geq 0, \alpha \in (0,1)$.

(ii) It is obvious $\wedge \{t > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha\} = 0 \Rightarrow \forall t > 0, N^*(x_1, x_2, t) < 1$
So by (Fa2-N7) x_1 and x_2 are linearly dependent.

Conversely, x_1 and x_2 are linearly dependent
$\Rightarrow \wedge \{t > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha\} = 0, \forall \alpha \in (0,1) \Rightarrow \|x_1, x_2\|_\alpha^* = 0.$

(iii) If $c=0$ it is obvious. If $c\neq 0$ then

$$\begin{aligned}\|x_1,cx_2\|^*_\alpha &= \wedge \ \{s>0: N^*(x_1,cx_2,s)\le 1-\alpha\}\\ &= \wedge \ \left\{s>0: N^*\left(x_1,x_2,\frac{s}{|c|}\right)\le 1-\alpha\right\}\\ &= \wedge \ \{|c|t>0: N^*(x_1,x_2,t)\le 1-\alpha\}\\ &= \wedge |c| \ \{t>0: N^*(x_1,x_2,t)\le 1-\alpha\}\\ &= |c|\,\|x_1,x_2\|^*_\alpha.\end{aligned}$$

(iv) $$\begin{aligned}\|x_1,x_2\|^*_\alpha+\|x_1,x_2'\|^*_\alpha &= \wedge \ \{t>0: N^*(x_1,x_2,t)\le 1-\alpha\}+\wedge \ \{s>0: N^*(x_1,x_2',s)\le 1-\alpha\}, \forall \alpha\in(0,1)\\ &\ge \wedge \ \{t+s>0: N^*(x_1,x_2,t)\le 1-\alpha, N^*(x_1,x_2',s)\le 1-\alpha\}\end{aligned}$$

So $\wedge \ \{t+s>0: N^*(x_1,x_2,t)\Diamond N^*(x_1,x_2',s)\le (1-\alpha)\Diamond(1-\alpha)\}$

$$\ge \wedge \ \{t+s>0: N^*(x_1,x_2+x_2',t+s)\le 1-\alpha\} \qquad \text{by (Fa2-N5) and (Fa2-N9)}$$
$$=\|x_1,x_2+x_2'\|^*_\alpha$$

Hence $\left\{\|.,.\|^*_\alpha : \alpha\in(0,1)\right\}$ is a 2-norm on X.

If $\alpha_1\le\alpha_2$, we have $\{t>0: N^*(x_1,x_2,t)\le 1-\alpha_2\}\subset\{t>0: N^*(x_1,x_2,t)\le 1-\alpha_1\}$

$\Rightarrow \wedge \ \{t>0: N^*(x_1,x_2,t)\le 1-\alpha_2\}\ge \wedge \ \{t>0: N^*(x_1,x_2',t)\le 1-\alpha_1\}$

$\Rightarrow \|x_1,x_2\|^*_{\alpha_2}\ge\|x_1,x_2\|^*_{\alpha_1}$.

So $\left\{\|.,.\|^*_\alpha : \alpha\in(0,1)\right\}$ is an ascending family of 2- norm on a linear space X. **Hence proved.**

Theorem 3.2: Let (X,N^*) be a fuzzy anti 2-normed linear space satisfying (Fa2-N7) and (Fa2-N8) Also, if $\left\{\|.,.\|^*_\alpha : \alpha\in(0,1)\right\}$ be ascending family of norms of X, defined by $\|x_0,x_0'\|^*_\alpha=\wedge\{t: N^*(x_0,x_0',t)\le 1-\alpha\}, \alpha\in(0,1)$. Then for x_0,x_0' (linearly independent) $\in X$, $\alpha\in(0,1)$ and $s(>0)\in$ R,

$\|x_0,x_0'\|^*_\alpha=s\Leftrightarrow N^*(x_0,x_0',s)=1-\alpha$.

Proof: let $\|x_0,x_0'\|^*_\alpha=s$ then $s>0$. Then $\exists$ a sequence $\{s_n\}_n, s_n>0$ such that $s_n\to s$ as $n\to\infty$ and $N^*(x_0,x_0',s_n)\le 1-\alpha, \forall n\in N$. Therefore $\lim_{n\to\infty} N^*(x_0,x_0',s_n)\le 1-\alpha\Rightarrow N^*\left(x_0,x_0',\lim_{n\to\infty}s_n\right)\le 1-\alpha$

$\Rightarrow N^*\left(x_0,x_0',\|x_0,x_0'\|^*_\alpha\right)\le 1-\alpha, \forall\alpha\in(0,1)$

Let $\alpha\in(0,1)$, x_0,x_0' (linearly dependent) $\in X$ and $s=\|x_0,x_0'\|^*_\alpha=\wedge\{t: N^*(x_0,x_0',t)\le 1-\alpha\}$.

Therefore $N^*(x_0,x_0',s)\le 1-\alpha$ (1)

If possible let $N^*(x_0,x_0',s)<1-\alpha$ then by continuity of $N^*(x_0,x_0',.)$ at s, there exist $s'<s$ such that $N^*(x_0,x_0',s)<1-\alpha$, which is impossible since

$$s=\wedge\{t: N^*(x_0,x_0',s)\le 1-\alpha\}.$$

Thus $N^*(x_0,x_0',s)\ge 1-\alpha$ (2)

From (1) and (2) it follows that $N^*(x_0,x_0',s)=1-\alpha$. Thus

$$\|x_0,x_0'\|^*_\alpha=s\Rightarrow N^*(x_0,x_0',s)=1-\alpha \qquad (3)$$

Next if $N^*(x_0, x_0', s) = 1-\alpha, \alpha \in (0,1)$ then

$$\|x_0, x_0'\|_\alpha^* = \wedge\{t : N^*(x_0, x_0', s) \le 1-\alpha\} = s\ . \tag{4}$$

Hence from (3) and (4) we have for $\alpha \in (0,1)$, x_0, x_0' (linearly independent) $\in X$ and for $s > 0, \|x_0, x_0'\| = s \Leftrightarrow N^*(x_0, x_0', s) = 1-\alpha.$ **Hence proved.**

Theorem 3.3: Let (X, N^*) be a fuzzy anti 2-normed linear space with respect to a t- conorm $\Diamond$ satisfying (Fa2-N7), (Fa2-N8) and (Fa2-N9).

Let $\|x_1, x_2\|_\alpha^* = \wedge\{t : N^*(x_1, x_2, t) \le 1-\alpha\}, \alpha \in (0,1)$. Also, let $N_1^* : X \times X \times R \to [0,1]$ be defined by

$$N^*(x_1, x_2, t) = \begin{cases} \wedge\{1-\alpha : \|x_1, x_2\|_\alpha^* \le t\}, & \text{if are linearly dependent}, t \ne 0 \\ 1, & \text{otherwise} \end{cases}$$

Then $N_1^* = N^*$.

Proof: Let $(x_0, x_0', t_0) \in X \times X \times R$ and $\alpha \in (0,1)$. To prove this we consider the following cases:

Case (i): For any $(x_0, x_0') \in X \times X$ and $t \le 0, N^*(x_0, x_0', t_0) = N_1^*(x_0, x_0', t_0) = 1.$

Case (ii): Let x_0, x_0'(linearly dependent), $t_0 > 0.$
Then $N^*(x_0, x_0', t_0) = 0$ also $\|x_1, x_2\|_\alpha^* = 0$ so $N_1^*(x_0, x_0', t_0) = 0$

Case (iii): Let x_0, x_0'(inearly independent), $t_0 > 0$ such that $N^*(x_0, x_0', t_0) = 1.$ By theorem (3.2).

we have $N^*\left(x_0, x_0', \|x_0, x_0'\|_\alpha^*\right) = 1-\alpha$. Since $N^*(x_0, x_0', t_0) = 1 > 1-\alpha$ it follows that $N^*\left(x_0, x_0', \|x_0, x_0'\|_\alpha^*\right) = 1-\alpha < N^*(x_0, x_0', t_0)$ and since $N^*(x_0, x_0', .)$ is strictly non-increasing.

So $t_0 < \|x_1, x_2\|_\alpha^*, \forall \alpha \in (0,1)$. So $N_1^*(x_0, x_0', t_0) = \wedge\{1-\alpha : \|x_1, x_2\|_\alpha^* \le t_0\} = 1.$

Thus $N^*(x_0, x_0', t_0) = N_1^*(x_0, x_0', t_0) = 1.$

Case (iv): Let x_0, x_0'(linearly independent), $t_0 > 0$ such that $N^*(x_0, x_0', t_0) = 0.$

As $\|x_0, x_0'\|_\alpha^* = \wedge\{t : N^*(x_0, x_0', t) \le 1-\alpha\}, \alpha \in (0,1).$

As $N^*\left(x_0, x_0', \|x_0, x_0'\|_\alpha^*\right) = 1-\alpha$ as N^* is decreasing

It follows that, $\|x_0, x_0'\|_\alpha^* < t_0, \forall \alpha \in (0,1)$, by (Fa2-N6). Therefore,
$\|x_1, x_2\|_\alpha^* < t_0 \Rightarrow N_1^*(x_0, x_0', t_0) = \wedge\{1-\alpha : \|x_0, x_0'\|_\alpha^* \le t_0\} = 0$,

Thus $N^*(x_0, x_0', t_0) = N_1^*(x_0, x_0', t_0) = 0.$

Case (v): Let x_0, x_0'(linearly independent), $t_0 > 0$, $s.t.$, $0 < N^*(x_0, x_0', t_0) < 1.$

Let, $N^*(x_0, x_0', t_0) = 1-\beta$, as $\|x_0, x_0'\|_\beta^* = \wedge\{t : N^*(x_0, x_0', t) \le 1-\beta\}$

as N^* is non-increasing function of t, we have $\|x_0, x_0'\|_\beta^* \le t_0$

So $N_1^*(x_0, x_0', t_0) \le 1-\beta$. Therefore,

$$N_1^*(x_0, x_0', t_0) \le N^*(x_0, x_0', t_0) \tag{1}$$

As $N^*(x_0, x_0', t_0) = 1-\beta \Leftrightarrow \|x_1, x_2\|_\beta^* = t_0$.

If $\beta < \alpha < 1$ and let $\|x_0, x_0'\|_\beta^* = t_1$ then $N^*(x_0, x_0', t_1) = 1-\alpha < 1-\beta = N^*(x_0, x_0', t_0)$

As $N^*(x_0, x_0', .)$ is monotonically decreasing so $t_0 < t_1$ since $\|x_0, x_0'\|_\alpha^* = t_1 > t_0$.

$$\text{So } N_1^*(x_0, x_0', t_0) > 1-\beta = N^*(x_0, x_0', t_0) \tag{2}$$

So from (1) and (2) we have $N_1^*(x_0, x_0', t_0) = N^*(x_0, x_0', t_0)$. **Hence proved.**

Lemma 3.4: Let (X, N^*) be a fuzzy anti 2-normed linear space with respect to t- conorm $\Diamond$ satisfying (Fa2-N7), (Fa2-N8) and (Fa2-N9), every sequence is convergent if and only if it is convergent with respect to its corresponding α -2-norms, $\alpha \in (0,1)$.

Proof: Let (X, N^*) be a fuzzy anti 2-normed linear space satisfying (Fa2-N7), (Fa2-N8), (Fa2-N9) and $\{x_n\}$ be a sequence in X such that $x_n \to x$ as $n \to \infty$.

$$\lim_{n\to\infty} N^*(x_n - x, x_0, t) = 0, \quad \forall t > 0.$$

Let $0 < \alpha < 1$. So, $\lim_{n\to\infty} N^*(x_n - x, x_0, t) = 0 < 1-\alpha \Rightarrow \exists\ n_0(t)$ such that

$$N^*(x_n - x, x_0, t) < 1-\alpha \quad \forall n \ge n_0(t, \alpha).$$

Now, $\|x_n - x, x_0\|_\alpha^* = \wedge\{t > 0 : N^*(x_n - x, x_0, t) \le 1-\alpha\}$

$$\Rightarrow \|x_n - x, x_0\|_\alpha^* \le t, \quad \forall n \ge n_0(t, \alpha)$$

Since $t > 0$ is arbitrary, $\|x_n - x, x_0\|_\alpha^* \to 0$ as $n \to \infty$, $\forall \alpha \in (0,1)$.

Conversely, suppose that $\|x_n - x, x_0\|_\alpha^* \to 0$ as $n \to \infty$, $\forall \alpha \in (0,1)$.

Then for $\alpha \in (0,1)$, $\varepsilon > 0$, $\exists\ n_0(\alpha, \varepsilon)$ such that $\|x_n - x, x_0\|_\alpha^* < \varepsilon$, $\forall n \ge n_0(\alpha, \varepsilon), \alpha \in (0,1)$.

Now, $\quad N^*(x_n - x, x_0, \varepsilon) = \wedge\{1-\alpha : \|x_n - x, x_0\|_\alpha^* \le \varepsilon\}$

$$\Rightarrow N^*(x_n - x, x_0, \varepsilon) \le 1-\alpha, \ \forall n \ge n_0(\alpha, \varepsilon), \alpha \in (0,1).$$

$$\Rightarrow \lim_{n\to\infty} N^*(x_n - x, x_0, \varepsilon) = 0, \quad \forall t > 0.$$

Thus x_n converges to x. **Hence Proved**.

Corollary 3.5: Let (X, N^*) be a fuzzy anti 2-normed linear space with respect to a t- conorm $\Diamond$ satisfying (Fa2-N7), (Fa2-N8) and (Fa2-N9). $W \subseteq X$ is closed in (X, N^*) if and only if it is closed with respect to its corresponding α -2-norms, $\alpha \in (0,1)$.

Theorem 3.6 (Riesz lemma): Let W be a closed and proper subspace of a fuzzy anti 2- normed linear space (X, N^*) with respect to a t- conorm $\Diamond$ satisfying (Fa2-N7) (Fa2-N8) and (Fa2-N9). Then for each $\varepsilon > 0$ there exist $y_1, y_2 \in (X - W)^2$ such that $N^*(y_1, y_2, 1) \le 1-\alpha$ and $N^*(y_1 - w, y_2 - w, \varepsilon) \le 1-\alpha$ for all $u(y,1) \le 1-\alpha$ and $w \in W$.

Proof: As $\|x_1, x_2\|_\alpha^* = \wedge\{t : N^*(x_1, x_2, t) \le 1-\alpha\}, \alpha \in (0,1)$ and $\{\|.,.\|_\alpha^* : \alpha \in (0,1)\}$ is an ascending family of fuzzy α -2-norm on a linear space X. Now by Riesz lemma for 2- normed linear space, it follows that for any $\varepsilon > 0$ there exist $y_1, y_2 \in (X-W)^2$ such that $\|y_1, y_2\|_\alpha^* = 1$ and $\|y_1 - w, y_2 - w\|_\alpha^* > 1-\varepsilon$, $\forall w \in W$

Now, from theorem (3.3), for all $u(y,1) \le 1-\alpha$ we have

$$N^*(y_1, y_2, t) = \wedge\{1-\alpha : \|y_1, y_2\|_\alpha^* \le t\}$$
$$\Rightarrow N^*(y_1, y_2, 1) = \wedge\{1-\alpha : \|y_1, y_2\|_\alpha^* \le 1\}$$
$$\Rightarrow N^*(y_1, y_2, 1) \le 1-\alpha$$

Again,

$$N^*(y_1 - w, y_2 - w, t) = \wedge\{1-\alpha : \|y_1 - w, y_2 - w\|_\alpha^* \le t\}$$
$$\Rightarrow N^*(y_1 - w, y_2 - w, \varepsilon) = \wedge\{1-\alpha : \|y_1 - w, y_2 - w\|_\alpha^* \le \varepsilon\}$$
$$\Rightarrow N^*(y_1 - w, y_2 - w, \varepsilon) \le 1-\alpha.$$

Hence proved.

Theorem 3.7: Let (X, N^*) be a fuzzy anti 2-normed linear space with respect to a t- conorm $\Diamond$ satisfying (Fa2-N7), (Fa2-N8) and (Fa2-N9). If the set $\{x_1, x_2 : N^*(x_1, x_2, 1) \le 1-\alpha\}, \alpha \in (0,1)$ is compact then X is a space of finite dimension.

Proof: It can be easily verified that $\{x_1, x_2 : N^*(x_1, x_2, 1) \le 1-\alpha\} = \{x_1, x_2 : \|x_1, x_2\|_\alpha^* \le 1\}, \alpha \in (0,1)$. By applying Riesz lemma 3.6, it can be proved that if for some $\alpha \in (0,1)$ the set $\{x_1, x_2 : \|x_1, x_2\|_\alpha^* \le 1\}$ is compact then X is of finite dimensional. By lemma (3.4), it follows that , for some $\alpha \in (0,1)$, $\{x_1, x_2 : N^*(x_1, x_2, 1) \le 1-\alpha\}$ is compact then X is a space of finite dimensional.

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