

## CONNECTED TWO OUT DEGREE EQUITABLE DOMINATION POLYNOMIAL OF A GRAPH

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### ABSTRACT

Let  $G = (V, E)$  be a connected graph. A dominating set  $D$  of  $G$  is called connected two out degree equitable dominating set if for any  $u, v \in D$ , such that  $|od_D(u) - od_D(v)| \leq 2$  and the induced sub graph  $\langle D \rangle$  is connected. The minimum cardinality of connected two out degree equitable dominating set is called connected two out degree equitable domination number and it is denoted by  $\gamma_{c2oe}(G)$ . In this paper we introduce the connected two out degree equitable domination polynomial of  $G$ . The connected two out degree equitable domination polynomial of connected graph  $G$  of order  $p$  is the polynomial  $D_{c2oe}(G, x) = \sum_{i=\gamma_{c2oe}}^p d_{c2oe}(G, i)x^i$ , where  $d_{c2oe}(G, i)$  is the number of connected two out degree equitable dominating set of  $G$  of size  $i$  and  $\gamma_{c2oe}(G)$  is the connected two out degree equitable domination number and we compute this polynomial and its roots for some standard graphs.

**Key words:** Connected, two out degree, equitable, polynomial.

**Mathematics Subject Classification:** 05C69.

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### 1. INTRODUCTION

Throughout this paper we will consider only a simple connected graph finite and undirected without loops and multiple edges. In general, we use  $\langle X \rangle$  to denote the sub graph induced by the set of vertices of  $N(v)$  denote the open neighborhood of a vertex. A set  $D$  of vertices in a graph  $G$  is a dominating set if every vertex in  $V - D$  is adjacent to some vertices in  $D$ . The minimum cardinality of such dominating set is called domination number and it is denoted by  $\gamma(G)$ . For more detail about domination number and related parameter we refer [7], [8]. A dominating set  $D$  of  $G$  is called a connected dominating set if the induced sub graph  $\langle D \rangle$  is connected. The minimum cardinality of such dominating set is called domination number and it is denoted by  $\gamma_c(G)$ [3].

Let  $G$  be a graph, and  $v$  be a vertex in  $D$ . The out degree of  $v$  with respect to  $D$  is denoted by  $od_D(v) = |N(v) \cap V - D|$ . A dominating set  $D$  of  $G$  is called two out degree equitable dominating set if for any  $u, v \in D$ , such that  $|od_D(u) - od_D(v)| \leq 2$ . The minimum cardinality of two out degree equitable dominating set is called two out degree equitable domination number and it is denoted by  $\gamma_{2oe}(G)$ [1]. A two out degree equitable dominating set  $D$  of  $G$  is called a connected two out degree equitable dominating set if the induced sub graph  $\langle D \rangle$  is connected. The minimum cardinality of such connected two out degree equitable dominating set is called connected two out degree equitable domination number and it is denoted by  $\gamma_{c2oe}(G)$ [5]. The domination polynomial of a graph is introduced by Saeid Alikhani and Yee-hock Peng[6]. The connected domination polynomial is introduced by B.V. Dhanajaya Murthy, G.Deepak and N.D.Soner[2]. The connected domination polynomial of connected graph  $G$  of order  $p$  is the polynomial  $D_c(G, x) = \sum_{i=\gamma_c}^p d_c(G, i)x^i$ , where  $d_c(G, i)$  is the number of connected dominating set of  $G$  of size  $i$  and  $\gamma_{c2oe}(G)$  is the connected domination number.

In this paper motivated by connected domination polynomial of a graph [2]. We introduce connected two out degree equitable domination polynomial of  $G$ . we compute the connected two out degree equitable domination polynomial and its roots form some standard and special graphs.

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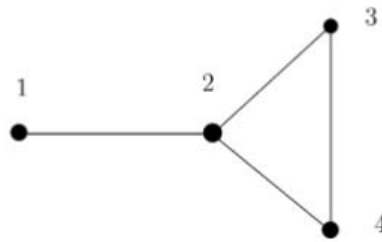
## 2. CONNECTED TWO OUT DEGREE EQUITABLE DOMINATION POLYNOMIAL OF A GRAPH

**Definition: 2.1** Let  $G$  be a connected graph of order  $p$ . Let  $d_{c2oe}(G, i)$  denoted the number of connected dominated set with cardinality  $i$ . Then the connected two out degree equitable domination polynomial  $D_{c2oe}(G, x)$  of  $G$  is defined as  $D_{c2oe}(G, x) = \sum_{i=\gamma_{c2oe}(G)}^p d_{c2oe}(G, i)x^i$  where  $d_{c2oe}(G, i)$  is the number connected two out degree equitable dominating set of  $G$  of size  $i$  and  $\gamma_{c2oe}(G)$  is connected two-out degree equitable domination number of  $G$ . The roots of the connected two out degree equitable domination polynomial are called the connected domination roots of  $G$  and denoted by  $Z(D_{c2oe}(G, x))$ .

**Example: 2.2** let  $G$  be the graph in the Figure 1.  $V(G)=\{1,2,3,4\}$ . Then the connected two out degree equitable domination number is two and the connected two out degree equitable domination number of size two are  $\{1,2\}, \{2,3\}, \{2,4\}$ . The connected two out degree equitable dominating set of size three are  $\{1,2,3\}, \{1,2,4\}, \{2,3,4\}$  and connected two out degree equitable dominating set of size four is  $\{1,2,3,4\}$ .Hence

$$\begin{aligned} D_{c2oe}(G, x) &= x^4 + 3x^3 + 3x^2 \\ &= x^2(x^2 + 3x + 3) \end{aligned}$$

And the connected two out degree equitable dominating roots of  $G$  are 0 with 2 multiplicities and  $\frac{-3 \pm i\sqrt{3}}{2}$



**Theorem: 2.3** For any path  $P_p$ ,  $p \geq 3$  vertices  $D_{c2oe}(P_p, x) = x^p + 2x^{p-1} + x^{p-2}$  and the connected two out degree equitable dominating roots 0 with multiplicity  $p - 2$  and -1 with multiplicity 2.

**Proof:** Let  $G$  be path  $P_p$  with  $p \geq 3$  and let  $P_p = \{v_1, v_2, v_3 \dots \dots v_p\}$ .

The connected two out degree equitable domination number of  $P_p$  is  $p - 2$  and there is only one connected two out degree equitable domination set of order  $p-2$ . That means  $d_{c2oe}(P_p, p - 2) = 1$ .

Also there are only two connected two out degree equitable dominating set of order  $p - 1$  namely  $\{v_2, v_3 \dots \dots v_p\}$  and  $\{v_1, v_2 \dots \dots v_{p-1}\}$ .

Therefore  $d_{c2oe}(P_p, p - 1) = 2$ .

There is only one connected dominating set of order  $n$ . Then  $d_{c2oe}(P_p, p) = 1$

Hence  $D_{c2oe}(G, x) = x^p + 2x^{p-1} + x^{p-2}$  and it is clear that the roots of the polynomial  $x^p + 2x^{p-1} + x^{p-2}$  are 0 with multiplicity  $p - 2$  and  $p - 1$  with multiplicity 2.

**Theorem: 2.4** For an cycle  $C_p$  with  $p$  vertices  $D_{c2oe}(C_p, x) = x^{p-2}(x^2 + px + p)$  And the connected two out degree equitable dominating roots are 0 with multiplicity  $p - 2$  and  $\frac{-p + \sqrt{p^2 - 4p}}{2}, \frac{-p - \sqrt{p^2 - 4p}}{2}$ .

**Proof:** Let  $G$  be cycle  $C_p$  with  $p$  and let  $C_p = \{v_1, v_2, v_3 \dots \dots v_p, v_1\}$ .

The connected two out degree equitable domination number of  $C_p$  is  $p - 2$  and there are  $p$  possibilities connected two out degree equitable domination set of order  $p - 2$ . That means  $d_{c2oe}(C_p, p - 2) = p$ .

Also there are only two connected two out degree equitable dominating set of order  $p - 1$  namely  $\{v_2, v_3 \dots \dots v_p\}$ ,  $\{v_2, v_3, v_4 \dots \dots v_{p-1}\}, \dots \dots \{v_1, v_2, v_3 \dots \dots v_{p-1}\}$

Therefore  $d_{c2oe}(C_p, p - 1) = p$ .

There is only one connected dominating set of order  $p$ . Then  $d_{c2oe}(C_p, p) = 1$

Hence  $D_{c2oe}(C_p, x) = x^p + px^{p-1} + px^{p-2} = x^{p-2}(x^2 + px + p)$ .

it is clear that the roots of the polynomial  $x^{p-2}(x^2 + px + p)$  are 0 with multiplicity  $p-2$  and  $\frac{-p+\sqrt{p^2-4p}}{2}, \frac{-p-\sqrt{p^2-4p}}{2}$ .

**Theorem: 2.5** For any star graph  $K_{1,p}$  with  $p + 1$  vertices, where  $p \geq 2$ , then

$$D_{c2oe}(K_{1,p}, x) = x \left[ \binom{p}{2} x^{p-2} + px^{p-1} + x^p \right]$$

**Proof:** Let  $G$  be a star graph of size  $p + 1$  and  $p \geq 2$ . By labeling the vertices of  $G$  as  $v_0, v_1, v_2, \dots, v_p$ , where  $v_0$  is the vertex of degree  $p$ , and the connected two out degree equitable domination number of  $K_{1,p}$  is  $p - 1$ .

For the connected two out degree equitable dominating set of size  $n-1$  we need to select the vertex  $v_0$  and  $p - 2$  vertices from the set of vertices  $\{v_1, v_2, \dots, v_p\}$ .

That means there are  $\binom{p}{p-2}$  connected two out degree equitable dominating set of size  $n-1$ .

Similarly for the connected two out degree equitable dominating set of size  $p$  we need to select the vertex  $v_0$  and  $p - 1$  vertices from the set of vertices  $\{v_1, v_2, \dots, v_p\}$ .

That means there are  $\binom{p}{p-1}$  connected two out degree equitable dominating set of size  $p$ .

For the connected two out degree equitable dominating set of size  $n+1$  we need to select the vertex  $v_0$  and  $n$  vertices from the set of vertices  $\{v_1, v_2, \dots, v_p\}$ .

That means there are  $\binom{p}{p}$  connected two out degree equitable dominating set of size  $p + 1$ .

$$\begin{aligned} \text{Hence } D_{c2oe}(K_{1,p}, x) &= \binom{p}{p-2} x^{p-1} + \binom{p}{p-1} x^p + \binom{p}{p} x^{p+1} \\ &= x \left[ \binom{p}{p-2} x^{p-2} + \binom{p}{p-1} x^{p-1} + \binom{p}{p} x^p \right] \\ &= x \left[ \binom{p}{2} x^{p-2} + \binom{p}{1} x^{p-1} + \binom{p}{0} x^p \right] \\ &= x \left[ \binom{p}{2} x^{p-2} + px^{p-1} + x^p \right] \end{aligned}$$

**Theorem: 2.6** For any complete graph  $K_p$  of  $p$  vertices  $D_{c2oe}(K_p, x) = (1 + x)^p - 1 - px$

**Proof:** Let  $G$  be a complete graph  $K_p$ . Then for any  $2 \leq i \leq p$ , it is easy to see that  $d_{c2oe}(K_p, i) = \binom{p}{i}$

$$\begin{aligned} D_{c2oe}(K_p, x) &= \sum_{i=2}^p \binom{p}{i} x^i \\ &= \binom{p}{2} x^2 + \binom{p}{3} x^3 + \binom{p}{4} x^4 + \dots \dots \dots \binom{p}{p} x^p \\ &= \left[ \sum_{i=0}^p \binom{p}{i} x^i \right] - 1 - px \\ &= (x + 1)^p - 1 - px \end{aligned}$$

**Theorem: 2.7** Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs of orders  $p_1$  and  $p_2$  respectively. Then  $D_{c2oe}((G_1 + G_2), x) = D_{c2oe}(G_1, x) + D_{c2oe}(G_2, x) + [(1 + x)^{p_1} - 1][(1 + x)^{p_2} - 1]$

**Proof:** From the definition of  $G_1 + G_2$ , if the  $D_1$  be any connected two out degree equitable dominating set of  $G_1$ , then  $D_1$  be any connected two out degree equitable dominating set of  $G_1 + G_2$ . Similarly if the  $D_2$  be any connected two out degree equitable dominating set of  $G_2$ , then  $D_2$  be any connected two out degree equitable dominating set of  $G_1 + G_2$ . Then

$$\begin{aligned} D_{c2oe}((G_1 + G_2), x) &= D_{c2oe}(G_1, x) + D_{c2oe}(G_2, x) + \binom{p_1}{1} \binom{p_2}{1} x^2 + \binom{p_1}{1} \binom{p_2}{2} x^3 + \binom{p_1}{2} \binom{p_2}{1} x^3 + \binom{p_1}{1} \binom{p_2}{3} x^4 + \binom{p_1}{2} \binom{p_2}{2} x^4 \\ &\quad + \binom{p_1}{3} \binom{p_2}{1} x^4 + \binom{p_1}{1} \binom{p_2}{4} x^5 + \binom{p_1}{2} \binom{p_2}{3} x^5 + \binom{p_1}{3} \binom{p_2}{2} x^5 + \binom{p_1}{4} \binom{p_2}{1} x^5 + \dots \dots \dots \\ &\quad + \left[ \binom{p_1}{1} \binom{p_2}{p_1 + p_2 - 1} + \dots \dots \dots \binom{p_1}{p_1 + p_2 - 1} \binom{p_2}{1} \right] x^{p_1 + p_2} \\ &= D_{c2oe}(G_1, x) + D_{c2oe}(G_2, x) + \left[ \binom{p_1}{1} x + \binom{p_1}{2} x^2 + \dots \dots \dots \binom{p_1}{p_1} x^{p_1} \cdot \binom{p_2}{1} x + \binom{p_2}{2} x^2 + \dots \dots \dots \binom{p_2}{p_2} x^{p_2} \right] \end{aligned}$$

$$\begin{aligned}
 &= D_{c2oe}(G_1, x) + D_{c2oe}(G_2, x) + \sum_{i=1}^{p_1} \binom{p_1}{i} x^i \sum_{i=1}^{p_2} \binom{p_2}{i} x^i \\
 &= D_{c2oe}(G_1, x) + D_{c2oe}(G_2, x) + [(1+x)^{p_1} - 1][(1+x)^{p_2} - 1]
 \end{aligned}$$

**Theorem: 2.8** For any complete bipartite graph  $K_{r,t}$ , where  $1 < r < t$  and  $|r - t| \leq 2$  of  $n$  vertices

$$D_{c2oe}(K_{r,t}, x) = [(1+x)^r - 1][(1+x)^t - 1]$$

**Proof:** If  $G$  is complete bipartite graph then  $\gamma_{c2oe}(K_{r,t}) = 2$  with partite set  $V_1$  and  $V_2$ , then any connected two out degree equitable dominating set of  $G$  contains at least one vertex from  $V_1$  and at least one vertex from  $V_2$ .

So we have

$$\begin{aligned}
 D_{c2oe}(K_{r,t}, x) &= \binom{r}{1} \binom{t}{1} x^2 + \binom{r}{1} \binom{t}{2} x^3 + \binom{r}{1} \binom{t}{2} x^3 + \dots \dots \dots \left[ \binom{r}{1} \binom{t}{r+t-1} + \dots \dots \dots \binom{r}{t+r-1} \binom{t}{1} \right] x^{r+t} \\
 &= \left[ \binom{r}{1} x + \binom{r}{2} x^2 + \dots \dots \dots \binom{r}{r} x^r \cdot \left( \binom{t}{1} x + \binom{t}{2} x^2 + \dots \dots \dots \binom{t}{t} x^t \right) \right] \\
 &= \sum_{i=1}^r \binom{r}{i} x^i \sum_{i=1}^t \binom{t}{i} x^i \\
 &= [(1+x)^r - 1][(1+x)^t - 1]
 \end{aligned}$$

**Theorem: 2.9** For any path  $P_p$  of  $p \geq 2$ , vertices  $D_{c2oe}(P_2 \times P_n, x) = (x^2 + x)^p$

**Proof:** The vertices of  $P_2 \times P_p$  is  $V(P_2 \times P_p) = \{(1,1), (1,2), (1,3) \dots (1,p) (2,1), (2,2), (2,3) \dots (2,p)\}$  containing  $2p$  vertices and  $\gamma_{c2oe}(P_2 \times P_p) = p$  for  $p \geq 2$

There is only one connected two out degree equitable domination number  $p$  is  $\{(1,1), (1,2), (1,3) \dots (1,p)\}$  or  $\{(2,1), (2,2), (2,3) \dots (2,p)\}$

To find connected two out degree equitable domination of size  $p+1$  we need to select  $\{(1,1), (1,2), (1,3) \dots (1,p)\}$  or  $\{(2,1), (2,2), (2,3) \dots (2,p)\}$  and select one vertex from the set of vertices  $\{(2,1), (2,2), (2,3) \dots (2,p)\}$  or  $\{(1,1), (1,2), (1,3) \dots (1,p)\}$

That means there are  $\binom{p}{1}$  connected two out degree equitable dominating set of size  $p$

Similarly there are  $\binom{p}{i}$  possibilities to extend the connected two out degree equitable dominating set of size  $p + i$ .

$$\begin{aligned}
 \text{Hence } D_{c2oe}(P_2 \times P_p, x) &= x^p + \binom{p}{1} x^{p+1} + \binom{p}{2} x^{p+2} + \dots \dots \dots + \binom{p}{p} x^{p+p} \\
 &= x^p + \binom{p}{1} x^{p+1} + \binom{p}{2} x^{p+2} + \dots \dots \dots + x^{2p} \\
 &= x^p \left( 1 + \binom{p}{1} x^1 + \binom{p}{2} x^2 + \dots \dots \dots + x^p \right) \\
 &= x^p \sum_{k=0}^p \binom{p}{k} x^k \\
 &= (x^2 + x)^p
 \end{aligned}$$

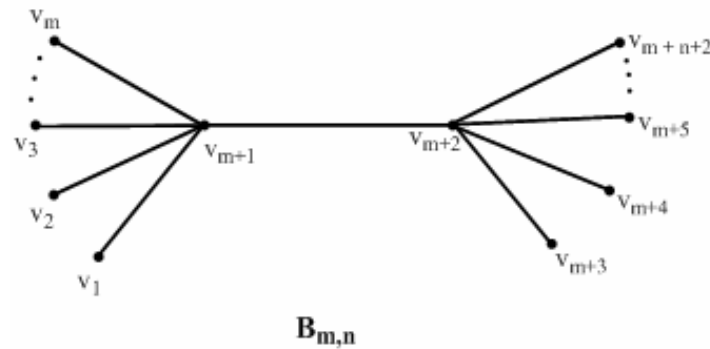
**Theorem: 2.10** For any connected graph  $G$ , with  $p$  vertices  $D_{c2oe}(G \times K_1, x) = (x^2 + x)^p$

**Proof:** Let  $H = G \times K_1$ . Then clearly  $\gamma_{c2oe}(H) = p$  and it is easy to see that there are  $\binom{p}{1}$  connected two out degree equitable dominating set of size  $p$

Similarly there are  $\binom{p}{i}$  possibilities to extend the connected two out degree equitable dominating set of size  $p + i$ .

$$\begin{aligned}
 \text{Hence } D_{c2oe}(G \times K_1, x) &= x^p + \binom{p}{1} x^{p+1} + \binom{p}{2} x^{p+2} + \dots \dots \dots + x^{2p} \\
 &= x^p \left( 1 + \binom{p}{1} x^1 + \binom{p}{2} x^2 + \dots \dots \dots + x^p \right) \\
 &= x^p \left[ \sum_{k=0}^p \binom{p}{k} x^k \right] \\
 &= x^p \sum_{k=0}^p \binom{p}{k} x^k \\
 &= (x^2 + x)^p
 \end{aligned}$$

**Theorem: 2.11** Let  $G$  be a bi-star graph  $B(r, t)$  as in Figure 2 Then  $D_{c2oe}(B(r, t), x) = x^2(1 + x)^{r+t}$  if  $|r - t| \leq 2$



**Proof:** Let  $G$  be a bi-star and with labeling as figure 2. Then  $\gamma_{c2oe}(B(r, t)) = 2$  if  $|r - t| \leq 2$  namely the set  $\{u, v\}$  is only unique minimum connected two out degree equitable dominating set.

Hence  $d_{c2oe}(B(r, t), \gamma_{c2oe}) = 1$ , and it is obvious that any other connected two out degree equitable dominating set must contain the two vertices  $u$  and  $v$ .

Hence there are  $\binom{r+t}{1}$  possibilities to extend the connected two out degree equitable dominating set of size 3, and there are  $\binom{r+t}{2}$  possibilities to extend the connected two out degree equitable dominating set into connected two out degree equitable dominating set of size 4.

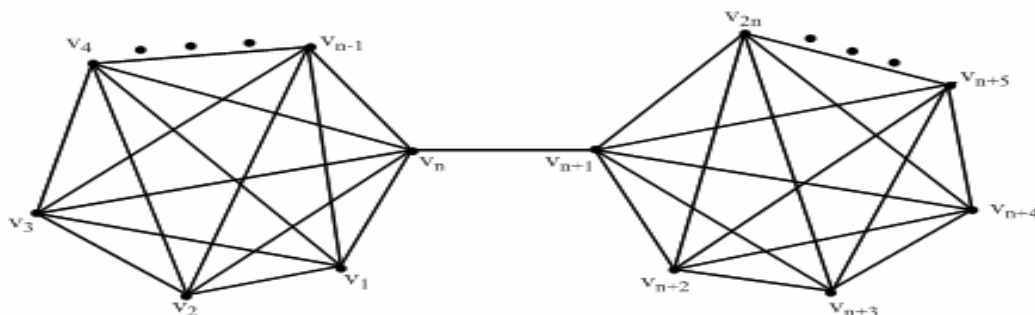
In general it is easy to see that there are  $\binom{r+t}{i-2}$  possibilities to extend the connected two out degree equitable dominating set into connected two out degree equitable dominating set of size  $i$ .

$$\begin{aligned} \text{Therefore } D_{c2oe}(B(r, t), x) &= x^2 + \binom{r+t}{1}x^3 + \binom{r+t}{2}x^4 + \dots + \binom{r+t}{r+t}x^{r+t+2} \\ &= x^2 + \binom{r+t}{1}x^3 + \binom{r+t}{2}x^4 + \dots + x^{r+t+2} \\ &= x^2 \left[ 1 + \binom{r+t}{1}x + \binom{r+t}{2}x^2 + \dots + x^{r+t} \right] \end{aligned}$$

Then  $D_{c2oe}(B(r, t), x) = x^2(1 + x)^{r+t}$  if  $|r - t| \leq 2$

**Theorem: 2.12** For a Barbell graph  $B_n$  with  $2n$  vertices the connected two out degree equitable domination polynomial is  $D_{c2oe}(B_n, x) = x^2(1 + x)^{2n-2}$

**Proof;** Let  $B_n$  be a Barbell graph with  $2n$  vertices. Label the vertices of  $B_n$  as  $v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n}$  is given in figure



Then the set  $\{v_n, v_{n+1}\}$  is the unique minimum connected two out degree equitable dominating set of cardinality of 2.

Therefore  $\gamma_{c2oe}(B_n) = 2$  and  $D_{c2oe}(B_n, 2) = 1$

It is obvious that the other connected two out degree equitable dominating set must contain two vertices  $v_n$  and  $v_{n+1}$

Hence these are  $\binom{2n-2}{1}$  connected two out degree equitable dominating set of cardinality 3 and  $\binom{2n-2}{2}$  connected two out degree equitable dominating set of cardinality 4.

Proceeding like this we obtain  $\binom{2n-2}{2n-2}$  connected two out degree equitable dominating set of cardinality  $2n$  vertices

$$\begin{aligned} \text{Hence } D_{c2oe}(B_n, x) &= x^2 + \binom{2n-2}{1} x^3 + \binom{2n-2}{2} x^4 + \binom{2n-2}{3} x^5 \dots \dots \dots \binom{2n-2}{2n-2} x^{2n} \\ &= x^2 \left[ 1 + \binom{2n-2}{1} x + \binom{2n-2}{2} x^2 + \binom{2n-2}{3} x^3 \dots \dots \dots \binom{2n-2}{2n-2} x^{2n-2} \right] \\ &= x^2 (1+x)^{2n-2} \end{aligned}$$

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