International Journal of Mathematical Archive-7(1), 2016, 84-92
Available online through www.ijma.info ISSN 2229 – 5046

ON THE EXISTENCE OF GENERALISED FIX POINTS

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(Received On: 17-12-15; Revised & Accepted On: 27-01-16)

ABSTRACT

Using the idea of generalised iterations of functions we prove fix point theorems for certain class of complex functions.

Key words: *Generalised fix points, Exact factor order, Complex functions.*

AMS Subject Classification: *30D60.*

1. INTRODUCTION AND DEFINITIONS

A single valued complex function $f(z)$ is said to belong to class I if $f(z)$ is entire transcendental and class II if it is regular in the complex plane punctured at a, b $(a \neq b)$ and has an essential singularity at b and a singularity at a and if $f(z)$ omits the values a and b except possible at a.

For arbitrary complex function $f(z)$, the iterations are defined inductively by

$$f_0(z) = z \text{ and } f_{n+1}(z) = f(f_n(z));\ n = 0,1,2,... .$$

A point α is called a fix point of $f(z)$ of order n if α is a solution of $f_n(z) = z$ and a fix point of exact order n if α is a solution of $f_n(z) = z$ but not a solution of $f_k(z) = z$, $k = 1,2,3,...,n-1$. With this definition of iteration, for functions of class I, Baker [1] proved the following theorem.

Theorem: A [1] If $f(z)$ belongs to class I, then $f(z)$ has fix points of exact order n except for at most one value of n.

In [4], Bhattacharyya extended Theorem A to functions in class II as follows.

Theorem: B [4] If $f(z)$ belongs to class II, then $f(z)$ has an infinity of fix points of exact order n, for every positive integer n.

In 1997, Lahiri and Banerjee [7] introduced a new concept of iteration called relative iterations (defined below) and using this, proved the result of Bhattacharyya [4].

Let $f(z)$ and $\varphi(z)$ be functions of the complex variable z. Let

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(\varphi(z)) = f(\varphi_1(z))\\
f_3(z) &= f(\varphi(f(z))) = f(\varphi(f_1(z)))\\
&\dots \quad \dots \quad \dots\\
f_n(z) &= f(\phi(f(\phi...(f(z) \text{ or } \varphi(z) \text{ according as } n \text{ is odd or even })...)))\\
&= f(\varphi_{n-1}(z)) = f(\varphi(f_{n-2}(z)))
\end{aligned}$$

and so

$$\varphi_1(z) = \varphi(z)$$
$$\varphi_2(z) = \varphi(f(z)) = \varphi(f_1(z))$$
$$\varphi_3(z) = \varphi(f_2(z)) = \varphi(f(\varphi_1(z)))$$
$$\cdots \quad \cdots \quad \cdots$$
$$\varphi_n(z) = \varphi(f_{n-1}(z)) = \varphi(f(\varphi_{n-2}(z))).$$

If $f(z)$ and $\varphi(z)$ are functions in class II, then so are $f_n(z)$ and $\varphi_n(z)$.

A point α is called a fix point of $f(z)$ of order n with respect to $\varphi(z)$ if $f_n(\alpha) = \alpha$ and a fix point of exact order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha, k = 1,2,\ldots,n-1$. Such point α is also called relative fix point.

Theorem: C [7] If $f(z)$ and $\varphi(z)$ belong to class II, then $f(z)$ has an infinity of relative fix points of exact order n for every positive integer n, provided $\dfrac{T(r,\varphi_n)}{T(r, f_n)}$ is bounded.

Recently, Banerjee and Mandal [2] introduced the concept of relative fix point of exact factor order and using this concept proved analogous theorem of Lahiri and Banerjee [7].

A point α is called a relative fix point of $f(z)$ with respect to $\varphi(z)$ of exact factor order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha$ and $\varphi_k(\alpha) \neq \alpha$ for all divisors $k(k < n)$ of n.

Theorem: D [2] If $f(z)$ and $\varphi(z)$ belong to class II, then $f(z)$ has an infinity of relative fix points of exact factor order n for every positive integer n provided $\dfrac{T(r, f_{n-1})}{T(r, f_n)}$ is bounded.

In [3], Banerjee and Mondal introduced another type of iteration called generalised iteration which runs as follows.

Let $f(z)$ and $\varphi(z)$ be two entire functions and $\alpha \in (0,1]$ be any number. Then the generalised iteration of $f(z)$ with respect to $\varphi(z)$ is defined as follows.

$$f_1(z) = (1-\alpha)z + \alpha f(z)$$
$$f_2(z) = (1-\alpha)\varphi_1(z) + \alpha f(\varphi_1(z))$$
$$f_3(z) = (1-\alpha)\varphi_2(z) + \alpha f(\varphi_2(z))$$
$$\cdots \quad \cdots \quad \cdots$$
$$f_n(z) = (1-\alpha)\varphi_{n-1}(z) + \alpha f(\varphi_{n-1}(z))$$

and

$$\varphi_1(z) = (1-\alpha)z + \alpha\varphi(z)$$
$$\varphi_2(z) = (1-\alpha) f_1(z) + \alpha\varphi(f_1(z))$$
$$\varphi_3(z) = (1-\alpha) f_2(z) + \alpha\varphi(f_2(z))$$
$$\cdots \quad \cdots \quad \cdots$$
$$\varphi_n(z) = (1-\alpha) f_{n-1}(z) + \alpha\varphi(f_{n-1}(z)).$$

Note-1: When $\alpha = 1$, generalised iteration reduces to relative iteration.

Clearly if $f(z)$ and $\varphi(z)$ are functions in class II, then so also are $f_n(z)$ and $\varphi_n(z)$.

Now we introduce the following definition.

Definition: 1 A point β is called a generalised fix point of $f(z)$ of order n if $f_n(\beta)=\beta$ and a generalised fix point of exact order n if $f_n(\beta)=\beta$ but $f_k(\beta)\neq\beta, k=1,2,3,...,n-1$. β is called a generalised fix point of $f(z)$ of exact factor order n if $f_n(\beta)=\beta$ but $f_k(\beta)\neq\beta$ and $\varphi_k(\beta)\neq\beta$ for all divisors $k(k<n)$ of n.

Example: 1 Let $f(z)=2z+1$, $\varphi(z)=2z-1$ and $\alpha\in(0,1]$. Then $z=\dfrac{\alpha}{2+\alpha}$ is generalised fix point of exact order 2 of $f(z)$ and $z=-\dfrac{\alpha^2+\alpha+1}{\alpha^2+3\alpha+3}$ is generalised fix point of exact factor order 3 of $f(z)$.

We normalise the functions in class II by taking $a=0$, $b=\infty$ and throughout this paper we consider such type of functions and their generalised iteration unless otherwise stated.

Let $f(z)$ be meromorphic in $r_0\leq|z|<\infty$, $r_0>0$. Following notations given in {[5], pp.88}, the first fundamental theorem takes the form

$$m(r,a,f)+N(r,a,f)=T(r,f)+O(\log r) \tag{1}$$

where the region is always $r_0\leq|z|<\infty$, $r_0>0$.

Further if $f(z)$ is non-constant and $a_1,a_2,...,a_q; q\geq 2,$ be distinct finite complex numbers, $\delta>0$ with $|a_\mu-a_\nu|\geq\delta$ for $1\leq\mu\leq\nu\leq q$, then

$$m(r,f)+\sum_{\nu=1}^{q}m(r,a_\nu,f)\leq 2T(r,f)-N_1(r)+S(r) \tag{2}$$

where $N_1(r)=N\left(r,\dfrac{1}{f'}\right)+2N(r,f)-N(r,f')$

and $S(r)=m\left(r,\dfrac{f'}{f}\right)+\sum_{\nu=1}^{q}m\left(r,\dfrac{f'}{f-a_\nu}\right)+O(\log r)$.

Adding $N(r,f)+\sum_{\nu=1}^{q}N(r,a_\nu,f)$ to both sides of (2) and using (1), we obtain

$$\sum_{\nu=1}^{q}\overline{N}(r,a_\nu,f)\geq(q-1)T(r,f)-\overline{N}(r,f)-S_1(r) \tag{3}$$

where $S_1(r)=O(\log T(r,f))$ and $\overline{N}$ corresponds to distinct roots.

Also since f_n has an essential singularity at ∞, we have {[5], pp.90}, $\dfrac{\log r}{T(r,f_n)}\to 0$ as $r\to\infty$.

The object of this paper is to extend Theorem C and Theorem D to functions in class II, with generalised iteration.

2. LEMMAS

The following lemmas will be needed in the sequel.

Lemma: 1 If f and φ are functions in class II, then for any $r_0>0$ and M, a positive constant $\dfrac{T(r,\varphi(f))}{T(r,f)}>M$, for all large r, except a set of r intervals of total finite length.

This follows from the Lemma of Lahiri and Banerjee [7] simply by taking $n=1$ and $p=1$.

Lemma: 2 If n is any positive integer and $f(z)$ and $\varphi(z)$ are functions in class II, then for any $r_0 > 0$ and M_1, a positive constant

$$\frac{T(r, f_{n+p})}{T(r, f_n)} > M_1 \text{ or } \frac{T(r, \varphi_{n+p})}{T(r, f_n)} > M_1$$

according as p is even or odd, for all large r except a set of r intervals of total finite length.

Proof: For $j = 1,2,...,n$ and for all large r, by using Lemma 1 we get

$$\begin{aligned} T(r, f_{j+1}) &\le T(r,(1-\alpha)\varphi_j) + T(r, \alpha f(\varphi_j)) + O(1) \\ &\le T(r,\varphi_j) + T(r, f(\varphi_j)) + O(1) \\ &= T(r, f(\phi_j))\left[1 + \frac{T(r,\phi_j)}{T(r, f(\phi_j))} + \frac{O(1)}{T(r, f(\phi_j))}\right] \\ &= (1+O(1))T(r, f(\varphi_j)). \end{aligned} \tag{4}$$

Again, $f(\varphi_j(z)) = \frac{1}{\alpha} f_{j+1}(z) - \frac{1-\alpha}{\alpha}\varphi_j(z)$ and so for large r,

$$T(r, f(\varphi_j)) \le T(r, f_{j+1}) + T(r,\varphi_j) + O(1).$$

Therefore

$$\begin{aligned} T(r, f_{j+1}) &\ge T(r, f(\varphi_j)) - T(r,\varphi_j) + O(1) \\ &= T(r, f(\phi_j))\left[1 - \frac{T(r,\phi_j)}{T(r, f(\phi_j))} + \frac{O(1)}{T(r, f(\phi_j))}\right] \\ &= (1+O(1))T(r, f(\varphi_j)). \end{aligned} \tag{5}$$

From (4) and (5), for all large r

$$T(r, f_{j+1}) = (1+O(1))T(r, f(\varphi_j)). \tag{6}$$

Similarly for all large r, we have

$$T(r, \varphi_{j+1}) = (1+O(1))T(r, \varphi(f_j)). \tag{7}$$

In particular,

$$T(r, f_1) = (1+o(1))T(r, f) \text{ and } T(r,\varphi_1) = (1+o(1))T(r,\varphi)).$$

Now we consider the following two cases.

Case-(i): When p is even.

For all large r except a set of r intervals of total finite length, we have from (6) and (7), by using Lemma 1

$$\begin{aligned} \frac{T(r, f_{n+p})}{T(r, f_n)} &= (1+O(1))\frac{T(r, f(\varphi_{n+p-1}))}{T(r, f_n)} \\ &= (1+O(1))\frac{T(r, f(\varphi_{n+p-1}))}{T(r,\varphi_{n+p-1})}\frac{T(r,\varphi_{n+p-1})}{T(r, f_n)} \\ &= (1+O(1))\frac{T(r, f(\varphi_{n+p-1}))}{T(r,\varphi_{n+p-1})}\frac{(1+O(1))T(r,\varphi(f_{n+p-2}))}{T(r, f_n)} \\ &= (1+O(1))\frac{T(r, f(\varphi_{n+p-1}))}{T(r,\varphi_{n+p-1})}\frac{T(r,\varphi(f_{n+p-2}))}{T(r, f_n)} \\ &= (1+O(1))\frac{T(r, f(\varphi_{n+p-1}))}{T(r,\varphi_{n+p-1})}\frac{T(r,\varphi(f_{n+p-2}))}{T(r, f_{n+p-2})}\frac{T(r, f_{n+p-2})}{T(r, f_n)} \end{aligned}$$

$$\begin{aligned}
&\cdots \quad \cdots \quad \cdots\\
&\cdots \quad \cdots \quad \cdots\\
&= (1+O(1))\frac{T(r, f(\phi_{n+p-1}))}{T(r,\phi_{n+p-1})}\frac{T(r,\phi(f_{n+p-2}))}{T(r, f_{n+p-2})}\frac{T(r, f(\phi_{n+p-3})}{T(r,\phi_{n+p-3})}\cdots\frac{T(r,\phi(f_n))}{T(r, f_n)}\\
&> (1+O(1))M^{p}\\
&= M_1, \text{ say}
\end{aligned}$$

where $M_1 = (1+O(1))M^{p}$, a positive constant.

i.e., $\dfrac{T(r, f_{n+p})}{T(r, f_n)} > M_1$ for all large r except a set of r intervals of total finite length.

Case-(ii): When p is odd.

For all large r except a set of r intervals of total finite length, we have from (6) and (7), by using Lemma 1

$$\begin{aligned}
\frac{T(r,\varphi_{n+p})}{T(r, f_n)} &= (1+O(1))\frac{T(r,\varphi(f_{n+p-1}))}{T(r, f_n)}\\
&= (1+O(1))\frac{T(r,\varphi(f_{n+p-1}))}{T(r, f_{n+p-1})}\frac{T(r, f_{n+p-1})}{T(r, f_n)}\\
&= (1+O(1))\frac{T(r,\varphi(f_{n+p-1}))}{T(r, f_{n+p-1})}\frac{(1+O(1))T(r, f(\varphi_{n+p-2}))}{T(r, f_n)}\\
&= (1+O(1))\frac{T(r,\varphi(f_{n+p-1}))}{T(r, f_{n+p-1})}\frac{T(r, f(\varphi_{n+p-2}))}{T(r, f_n)}\\
&= (1+O(1))\frac{T(r,\varphi(f_{n+p-1}))}{T(r, f_{n+p-1})}\frac{T(r, f(\varphi_{n+p-2}))}{T(r,\varphi_{n+p-2})}\frac{T(r,\varphi_{n+p-2})}{T(r, f_n)}\\
&\cdots \quad \cdots \quad \cdots\\
&\cdots \quad \cdots \quad \cdots\\
&= (1+O(1))\frac{T(r,\phi(f_{n+p-1}))}{T(r, f_{n+p-1})}\frac{T(r, f(\phi_{n+p-2}))}{T(r,\phi_{n+p-2})}\frac{T(r,\phi(f_{n+p-3})}{T(r, f_{n+p-3})}\cdots\frac{T(r,\phi(f_n))}{T(r, f_n)}\\
&> (1+O(1))M^{p}\\
&= M_1, \text{ say}
\end{aligned}$$

where $M_1 = (1+O(1))M^{p}$, a positive constant.

i.e., $\dfrac{T(r,\varphi_{n+p})}{T(r, f_n)} > M_1$ for all large r except a set of r intervals of total finite length.

Lemma: 3 If n is any positive integer and $f(z)$ and $\varphi(z)$ are functions in class II, then for any $r_0 > 0$ and M_1, a positive constant

$$\frac{T(r,\varphi_{n+p})}{T(r,\varphi_n)} > M_1 \quad \text{or} \quad \frac{T(r, f_{n+p})}{T(r,\varphi_n)} > M_1$$

according as p is even or odd, for all large r except a set of r intervals of total finite length.

3. THEOREMS

As soon as the lemmas are obtained, the proof of the following two theorems are analogous of the proof of Theorem C and Theorem D. However, for the sake of completeness and for convenience of readers, we outline the proof.

Theorem: 1 If $f(z)$ and $\varphi(z)$ belong to class II, then $f(z)$ has an infinity of generalised fix points of exact order n for every positive integer n, provided $\frac{T(r,\varphi_n)}{T(r,f_n)}$ is bounded.

Proof: We may assume that $\alpha \neq 1$, because if $\alpha = 1$, the theorem coincides with Theorem C. We consider the function

$$g(z) = \frac{f_n(z)}{z}, \quad r_0 < |z| < \infty$$

then
$$T(r,g) = T(r,f_n) + O(\log r). \tag{8}$$

Assume that $f(z)$ has only a finite number of generalised fix points of exact order n. Now from (3) by taking $q = 2$, $a_1 = 0$, $a_2 = 1$, we obtain for g,

$$T(r,g) \leq \overline{N}(r,\infty,g) + \overline{N}(r,0,g) + \overline{N}(r,1,g) + S_1(r,g) \tag{9}$$

where $S_1(r,g) = O(\log T(r,g))$ outside a set of r intervals of finite length {[6], pp.47}.

Since $f_n(z)$ belongs to class II, it has a singularity at $z = 0$ and an essential singularity at $z = \infty$ and $f_n(z) \neq 0, \infty$ in $r_0 < |z| < \infty$.

Also the distinct roots of $g(z) = 0$ in $r_0 < |z| \leq t$ are the roots of $f_n(z) = 0$ in $r_0 < |z| \leq t$. So $\overline{n}(t,0,g) = 0$. Consequently $\overline{N}(r,0,g) = 0$. By similar argument $\overline{N}(r,\infty,g) = 0$.

Further if $g(z) = 1$, then $f_n(z) = z$.

So,
$$\overline{N}(r,1,g) = \overline{N}(r,0,f_n - z) \leq \sum_{j=1}^{n-1} \overline{N}(r,0,f_j - z) + O(\log r),$$

the term $O(\log r)$ arises due to the assumption that $f(z)$ has only a finite number of generalised fix points of exact order n.

Now from (9), we have

$$T(r,g) \leq \sum_{j=1}^{n-1} \overline{N}(r,0,f_n - z) + O(\log r) + O(\log T(r,g))$$

$$\leq \sum_{j=1}^{n-1} T(r,f_j) + O(\log T(r,g)) + O(\log r)$$

$$= T(r,f_n)\left[\begin{array}{l} \frac{T(r,f_{i_1})}{T(r,f_n)} + \frac{T(r,f_{i_2})}{T(r,f_n)} + \cdots + \frac{T(r,f_{i_p})}{T(r,f_n)} \\ + \left\{\frac{T(r,f_{j_1})}{T(r,\phi_n)} + \frac{T(r,f_{j_2})}{T(r,\phi_n)} + \cdots + \frac{T(r,f_{j_q})}{T(r,\phi_n)}\right\}\frac{T(r,\phi_n)}{T(r,f_n)} \\ + \frac{O\left(\log\left\{T(r,f_n)\left(1 + \frac{O(\log r)}{T(r,f_n)}\right)\right\}\right)}{T(r,f_n)} + \frac{O(\log r)}{T(r,f_n)} \end{array}\right],$$

where $i_1, i_2, ..., i_p$ and $j_1, j_2, ..., j_q$ are $(n-1)$ distinct index belong to the set $\{1, 2, 3, ..., n-1\}$ such that $(n-i_p)$'s are even and $(n-j_q)$'s are odd

$$< T(r, f_n)[\frac{n-1}{4n} + \frac{n+1}{4n}] \text{, for all large } r \text{, by Lemma 2 and since } \frac{T(r,\varphi_n)}{T(r, f_n)} \text{ is bounded}$$

$$= \frac{1}{2} T(r, f_n).$$

Therefore, $T(r, g) < \frac{1}{2} T(r, f_n)$ for all large r. This contradicts (8). Hence $f(z)$ has infinitely many generalised fix points of exact order n.

This proves the theorem.

Theorem: 2 If $f(z)$ and $\varphi(z)$ belong to class II, then $f(z)$ has an infinity of generalised fix points of exact factor order n for every positive integer n, provided $\frac{T(r,\varphi_n)}{T(r, f_n)}$ is bounded.

Proof: As in Theorem 1, we assume that $f(z)$ has only a finite number of generalised fix points of exact factor order n. Considering the function $h(z) = \frac{f_n(z)}{z}$, $r_0 < |z| < \infty$ we have

$$T(r, h) = T(r, f_n) + O(\log r). \tag{10}$$

Here also $\overline{N}(r, 0, h) = 0$ and $\overline{N}(r, \infty, h) = 0$.

To calculate $\overline{N}(r, 1, h)$ we consider two cases separately.

Case-(i): When n is even.

Now, $\overline{N}(r,1,h) = \overline{N}(r, 0, f_n - z)$

$$\leq \sum_{j/n,\, j=1}^{n-2} [\overline{N}(r, 0, f_j - z) + \overline{N}(r, 0, \varphi_j - z)] + O(\log r)$$

$$\leq \sum_{j/n,\, j=1}^{n-2} [T(r, f_j) + T(r, \varphi_j) + O(\log r)$$

$$= T(r, f_n)\left[\frac{T(r, f_{j_2})}{T(r, f_n)} + \frac{T(r, f_{j_4})}{T(r, f_n)} + \cdots + \frac{T(r, f_{j_{2q}})}{T(r, f_n)} + \frac{T(r, \phi_{j_1})}{T(r, f_n)} + \frac{T(r, \phi_{j_3})}{T(r, f_n)} + \cdots + \frac{T(r, \phi_{j_{2p-1}})}{T(r, f_n)}\right]$$

$$+ T(r, \phi_n)\left[\begin{array}{l} \frac{T(r, f_{j_1})}{T(r, \phi_n)} + \frac{T(r, f_{j_3})}{T(r, \phi_n)} + \cdots + \frac{T(r, f_{j_{2p-1}})}{T(r, \phi_n)} \\ + \frac{T(r, \phi_{j_2})}{T(r, \phi_n)} + \frac{T(r, \phi_{j_4})}{T(r, \phi_n)} + \cdots + \frac{T(r, \phi_{j_{2q}})}{T(r, \phi_n)} \end{array}\right] + O(\log r)$$

where $j_1, j_3, ..., j_{2p-1}$ are distinct odd divisors of n and $j_2, j_4, ..., j_{2q}$ are distinct even divisors of n and strictly less than n

$$< \frac{n-1}{4n} T(r, f_n) + \frac{n+1}{4n} T(r, \varphi_n) + O(\log r),$$

for all large r, by Lemma 2 and Lemma 3.

Case-(ii): When n is odd.

$$\begin{aligned}
\overline{N}(r,1,h) &= \overline{N}(r,0,f_n - z)\\
&\le \sum_{j/n,\, j=1}^{n-2}[\overline{N}(r,0,f_j - z) + \overline{N}(r,0,\varphi_j - z)] + O(\log r)\\
&\le \sum_{j/n,\, j=1}^{n-2}[T(r,f_j) + T(r,\varphi_j) + O(\log r)\\
&= T(r,f_n)\sum_{j/n,\, j=1}^{n-2}\frac{T(r,f_j)}{T(r,f_n)} + T(r,\varphi_n)\sum_{j/n,\, j=1}^{n-2}\frac{T(r,\varphi_j)}{T(r,\varphi_n)} + O(\log r)\\
&< \frac{n-1}{4n}T(r,f_n) + \frac{n+1}{4n}T(r,\varphi_n) + O(\log r),
\end{aligned}$$

for all large r, by Lemma 2 and Lemma 3.

Thus in any case,

$$\overline{N}(r,1,h) < \frac{n-1}{4n}T(r,f_n) + \frac{n+1}{4n}T(r,\varphi_n) + O(\log r).$$

So,

$$\begin{aligned}
T(r,h) &\le \overline{N}(r,1,h) + S_1(r)\\
&< \frac{n-1}{4n}T(r,f_n) + \frac{n+1}{4n}T(r,\varphi_n) + O(\log r) + O(\log T(r,h))\\
&= T(r,f_n)\left[\frac{n-1}{4n} + \frac{n+1}{4n}\frac{T(r,\phi_n)}{T(r,f_n)} + \frac{O(\log r)}{T(r,f_n)} + \frac{O(\log T(r,h))}{T(r,f_n)}\right]\\
&\le T(r,f_n)\left[\frac{n-1}{4n} + \frac{n+1}{4n} + \frac{O(\log r)}{T(r,f_n)} + \frac{O(\log(T(r,f_n) + O(\log r)))}{T(r,f_n)}\right],
\end{aligned}$$

Since $\dfrac{T(r,\varphi_n)}{T(r,f_n)}$ is bounded

$$= T(r,f_n)\left[\frac{1}{2} + \frac{O(\log r)}{T(r,f_n)} + \frac{O\left(\log\left(T(r,f_n)\left(1 + \frac{O(\log r)}{T(r,f_n)}\right)\right)\right)}{T(r,f_n)}\right]$$

$= \dfrac{1}{2}T(r,f_n)$, for all large r.

Therefore, $T(r,h) < \dfrac{1}{2}T(r,f_n)$ for all large r. This contradicts (10). Hence $f(z)$ has infinitely many generalised fix points of exact factor order n.

This proves the theorem.

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Source of support: Nil, Conflict of interest: None Declared