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Weakly fuzzy α-I-open sets and a New Decomposition of Fuzzy Continuity via Ideals

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ABSTRACT

In this paper, we introduced and discussed the characterizations and properties of weakly fuzzy α-I-open sets, weakly fuzzy b-I-open sets and their corresponding continuous functions. Also, we obtained a new decomposition of fuzzy continuity.

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Keywords: *Weakly fuzzy α - I -open set; Weakly fuzzy b- I -open set; fuzzy strong t- I -set; fuzzy strong $\mathcal{B}_I$-set; Weakly fuzzy α- I -continuity.*

1. INTRODUCTION AND PRELIMINARIES

Fuzziness is one of the most important and useful concepts in the modern scientific studies. In 1965, Zadeh [13] first introduced the notion of fuzzy sets. In 1945, Vaidyanathaswamy [10] introduced the concepts of ideal topological spaces. In 1990, Jankovic and Hamlett [5] have defined the concept of I -open set via local function in ideal topological space. In 1997, Mahmoud [7] and sarkar [9] independently presented some of the ideal concepts in the fuzzy trend and studied many other properties. Decomposition of fuzzy continuity is one of the many problems in the fuzzy topology. It becomes very interesting when decomposition is done via fuzzy topological ideals.

Throughout this paper, X represents a nonempty fuzzy set and fuzzy subset A of X, denoted by $A \leq X$, is characterized by a membership function in the sense of Zadeh [13]. The basic fuzzy sets are the empty set, the whole set and the class of all fuzzy subsets of X which will be denoted by 0, 1 and I^X, respectively. A subfamily τ of I^X will denote topology of fuzzy sets on I^X as defined by Chang [3]. By (X, τ), we mean a fuzzy topological space in Chang's sense. A fuzzy point in X with support $x \in X$ and value $\alpha(0 < \alpha \leq 1)$ is denoted by x_α. For a fuzzy subset A of X, Cl(A), Int(A) and 1 – A will, respectively, denote the fuzzy closure, fuzzy interior and fuzzy complement of A. A nonempty collection I of fuzzy subsets of X is called a fuzzy ideal [9] if and only if

1. $B \in I$ and $A \leq B$, then $A \in I$ (heredity),
2. if $A \in I$ and $B \in I$ then $A \vee B \in I$ (finite additivity).

The triple (X, τ, I) means a fuzzy topological space with a fuzzy ideal I and fuzzy topology τ. For (X, τ, I), the fuzzy local function of $A \leq X$ with respect to τ and I is denoted by $A^*(\tau, I)$ (briefly A^*) and is defined as $A^*(\tau, I) = \wedge\{x \in X: A \wedge U \notin I \text{ for every } U \in \tau(x)\}$. While A^* is the union of the fuzzy points x such that if $U \in \tau(x)$ and $E \in I$, then there is at least one $y \in X$ for which $U(y) + A(y) - 1 > E(y)$. Fuzzy closure operator of a fuzzy set in (X, τ, I) is defined as $Cl^*(A) = A \vee A^*$. In (X, τ, I), the collection $\tau^*(I)$ means an extension of fuzzy topological space than τ via fuzzy ideal which is constructed by considering the class $\beta = \{U - E : U \in \tau, E \in I\}$ as a base [9]. This topology of fuzzy sets is considered as generalization of the ordinary one.

First, we shall recall some definitions used in the sequel.

Lemma 1.1: [9] Let (X, τ, I) be fuzzy ideal topological space and A, B subsets of X. The following properties hold:

(a) If $A \leq B$, then $A^* \leq B^*$,
(b) $(A \vee B)^* = A^* \vee B^*$,
(c) $A^* = Cl(A^*) \leq Cl(A)$,
(d) if $U \in \tau$, then $U \wedge A^* \leq (U \wedge A)^*$,
(e) if $U \in \tau$, then $U \wedge Cl^*(A) \leq Cl^*(U \wedge A)$.

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Let A be a subset of a fuzzy topological space (X, τ). The complement of a semi-open fuzzy set is said to be semi-closed [11] fuzzy set. The intersection of all semi-closed fuzzy sets containing A is called the semi-closure [11] of a fuzzy set A and denoted by sCl(A). The semi-interior [11] of a fuzzy set A, denoted by sInt(A), is defined by the union of all semi-open fuzzy sets contained in A.

The following lemmas are well known.

Lemma 1.2: [11] For a subset A of a fuzzy topological space (X, τ), the following properties hold:
(a) sCl(A) = A ∨ Int(Cl(A)),
(b) sCl(A) = Int(Cl(A)) if A is fuzzy open.

Lemma 1.3: [6] Let (X, τ, I) be an ideal topological space with an arbitrary index Δ, I an ideal of subsets of X and ρ(X) the power set of X. If $\{A_\alpha : \alpha \in \Delta\} \leq \rho(X)$, then the following property holds:

$$\vee_{\alpha\in\Delta} (A_\alpha^*) \leq (\vee_{\alpha\in\Delta} A_\alpha)^*$$

Definition 1.1: [12] A subset A of a space (X, τ, I) is said to be
(a) fuzzy α - I -open if A ≤ Int(Cl*(Int(A))),
(b) fuzzy β - I -open if A ≤ Cl(Int(Cl*(A))).

Definition 1.2: [4] A subset A of a space (X, τ, I) is said to be weakly fuzzy pre- I -open if A ≤ sCl(Int(Cl*(A))), the complement of a weakly fuzzy pre- I -open set will be called weakly fuzzy pre- I -closed.

Definition 1.3: = A subset A of a fuzzy topological space (X, τ) is said to be
(a) fuzzy α -open[2] if A ≤ Int(Cl(Int(A))),
(b) fuzzy pre-open[2] if A ≤ Int(Cl(A)),
(c) fuzzy semi-open[1] if A ≤ Cl(Int(A)).

Definition 1.4: [4] A function f: (X, τ, I) → (Y, σ) is said to be weakly fuzzy pre-I -continuous if inverse image of each fuzzy open set of Y is weakly fuzzy pre- I -open set in X.

2. WEAKLY FUZZY α - I - OPEN SETS

Definition 2.1: A subset A of a space (X, τ, I) is said to be weakly fuzzy α - I -open if A ≤ sCl(Int(Cl*(Int(A)))), the complement of a weakly fuzzy α - I -open set will be called weakly fuzzy α - I -closed.

Proposition 2.1: For a subset of a fuzzy ideal topological space (X, τ, I) the following hold:
(a) Every fuzzy α - I -open set is weakly fuzzy α - I -open.
(b) Every weakly fuzzy α - I -open set is weakly fuzzy pre- I -open.
(c) Every weakly fuzzy α - I -open is fuzzy β - I -open.

Proof: The proof is obvious.

Remark 2.1: The converses of each statement in proposition 2.1 need not be true as the following examples show.

Example 2.1: Let X = {a, b, c} and A, B be fuzzy subsets of X defined as follows:
A(a) = 0.5, A(b) = 0.4, A(c) = 0.6,
B(a) = 0.5, B(b) = 0.5, B(c) = 0.6.

Let τ = {0, A, 1}. If we take I = ρ(X), then B is weakly fuzzy α - I -open but B is not fuzzy α - I -open set.

Example 2.2: Let X = {a, b, c} and A, B, C be fuzzy subsets of X defined as follows:
A(a) = 0.6 , A(b) = 0.4 , A(c) = 0.5 ,
B(a) = 0 , B(b) = 0 , B(c) = 0.5 ,
C(a) = 0.7 , C(b) = 0.6 , C(c) = 0.7 .

Let τ = {0, A, B, 1}. If we take I = {0}, then C is weakly fuzzy pre- I -open but C is not weakly fuzzy α - I -open set.

Example 2.3: Let X = {a, b, c} and A, B be fuzzy subsets of X defined as follows:
A(a) = 0.2, A(b) = 0.3, A(c) = 0.7,
B(a) = 0.1, B(b) = 0.2, B(c) = 0.2.

Let τ = {0, B, 1}. If we take I = {0}, then A is fuzzy β - I -open but A is not weakly fuzzy α - I -open set.

Definition 2.2: A subset A of a fuzzy ideal topological space (X, τ, I) is said to be weakly fuzzy b - I -open if $A \leq Cl^*(Int(Cl(A))) \vee Cl(Int(Cl^*(A)))$, the complement of a weakly fuzzy b - I -open set will be called weakly fuzzy b - I -closed.

Theorem 2.1: Let (X, τ, I) be a fuzzy ideal topological space and A be a subset of X. Then the following hold:

(a) If A is weakly fuzzy α - I -open set, then A is a weakly fuzzy b - I -open set.
(b) If A is weakly fuzzy α - I -open set, then A is a fuzzy semi open set.
(c) If A is weakly fuzzy α - I -open set, then A is a fuzzy pre open set.

Proof:

(a) Let A be a weakly fuzzy α - I -open set. Then, since $sCl(A) \leq Cl(A)$, we have $A \leq sCl(Int(Cl^*(Int(A)))) \leq Cl(Int(Cl^*(A))) \leq Cl(Int(Cl^*(A))) \vee Cl^*(Int(Cl(A)))$. This shows that A is a weakly fuzzy b - I -open set.
(b) Let A be a weakly fuzzy α - I -open set. Then, since $sCl(A) \leq Cl(A)$, we have $A \leq sCl(Int(Cl^*(Int(A)))) \leq Cl(Int(Cl(Int(A)))) = Cl(Int(A))$. This shows that A is fuzzy semi-open.
(c) Let A be a weakly fuzzy α - I -open set. Then by lemma 1.2, we have $A \leq sCl(Int(Cl^*(Int(A)))) \leq sCl(Int(Cl^*(A))) \leq sCl(Int(Cl(A))) = Int(Cl(Int(Cl(A)))) = Int(Cl(A))$. This shows that A is a fuzzy pre open.

Remark 2.2: The converse of each part in the above theorem need not be true as the following examples show.

Example 2.4: Let X = {a, b, c} and A, B be fuzzy subsets of X defined as follows:

A(a) = 0.2, A(b) = 0.3, A(c) = 0.7,
B(a) = 0.1, B(b) = 0.2, B(c) = 0.2.

Let τ = {0, B, 1}. If we take I = {0}, then A is weakly fuzzy b - I -open and fuzzy semi open but A is not weakly fuzzy α - I -open set.

Example 2.5: Let X = {a, b, c} and A, B, C be fuzzy subsets of X defined as follows:

A(a) = 0.6, A(b) = 0.4, A(c) = 0.5,
B(a) = 0, B(b) = 0, B(c) = 0.5,
C(a) = 0.7, C(b) = 0.6 , C(c) = 0.7.

Let τ = {0, A, B, 1}. If we take I = {0}, then C is fuzzy pre open but C is not weakly fuzzy α - I -open set.

Remark 2.3: By Proposition 2.1, Theorem 2.1, we have the following diagram:

fuzzy α - I -open → fuzzy α -open → fuzzy semi-open
↓ ↑ ↑
fuzzy β - I -open ← weakly fuzzy α - I -open → fuzzy pre-open
↑ ↓ ↓
weakly fuzzy pre - I -open → weakly fuzzy b - I -open

Proposition 2.2: Let (X, τ, I) be a fuzzy ideal topological space. Let A, U and $A_\alpha (\alpha \in \Delta)$ be subsets of X. Then

(a) if A_α is weakly fuzzy α - I -open for each $\alpha \in \Delta$, then $\vee_{\alpha \in \Delta} A_\alpha$ is weakly fuzzy α - I -open,
(b) if A is weakly fuzzy α - I -open and U is fuzzy open i.e. $U \in \tau$, then $(U \wedge A)$ is weakly fuzzy α - I -open.

Proof:

(a) Since A_α is weakly fuzzy α - I -open for each $\alpha \in \Delta$, by using lemmas 1.1, 1.2 and 1.3, we have
(b) $A_\alpha \leq sCl(Int(Cl^*(Int(A_\alpha)))) \leq sCl(Int(Cl^*(Int(\vee_{\alpha \in \Delta} A_\alpha))))$ for each $\alpha \in \Delta$ and hence $\vee_{\alpha \in \Delta} A_\alpha \leq sCl(Int(Cl^*(Int(\vee_{\alpha \in \Delta} A_\alpha))))$. This shows that $\vee_{\alpha \in \Delta} A_\alpha$ is weakly fuzzy α - I -open.
(c) Let A is weakly fuzzy α - I -open and $U \in \tau$. Then by lemmas 1.1 and 1.2, we have $A \wedge U \leq sCl(Int(Cl^*(Int(A)))) \wedge U = Int(Cl(Int(Cl^*(Int(A))))\wedge U) \leq Int(Cl(Int(Cl^*(Int(A)))\wedge U)) = Int(Cl(Int(Cl^*(Int(A))\wedge U))) \leq sCl(Int(Cl^*(Int(A \wedge U))))$.
This shows that $A \wedge U$ is weakly fuzzy α - I -open.

Lemma 2.1: [8] Let (X, τ, I) be a fuzzy ideal topological space and A, B subsets of X such that $B \leq A$. Then $B^*(\tau_{|A}, I_{|A}) = B^*(\tau, I) \wedge A$.

Proposition 2.3: Let (X, τ, I) be a fuzzy ideal topological space and $A \leq U \in \tau$. If A is weakly fuzzy α - I -open in (X, τ, I), then A is weakly fuzzy α - I -open in $(U, \tau_{|U}, I_{|U})$.

Proof: Let A be weakly fuzzy α - I -open in (X, τ, I). Since U is fuzzy open, we have $Int_U(A) = Int(A)$ for any subset A of U. By using this fact and lemmas 1.1, 1.2 and 2.1, we have $A \leq sCl(Int(Cl^*(Int(A)))) = Int(Cl(Int(Cl^*(Int(A)))))$ and hence $A = A \wedge U \leq U \wedge Int(Cl(Int(Cl^*(Int(A))))) \leq Int_U(U \wedge Cl(U \wedge Int(Cl^*(Int(A))))) = Int_U(Cl_U(U \wedge Int(Cl^*(Int(A)))))$ $= sCl_U(Int_U(U \wedge Cl^*(Int(A)))) \leq sCl_U(Int_U(Cl^*_U(U \wedge Int(A)))) = sCl_U(Int_U(Cl^*(Int_U(A))))$.

This shows that A is weakly fuzzy α - I -open in (U, $\tau_{|U}$, $I_{|U}$).

Lemma 2.2: Let (X, τ, I) be a fuzzy ideal topological space. If A is weakly fuzzy α - I -open and U is fuzzy open i.e. U ∈ τ, then (U ∧ A) is weakly fuzzy α - I -open in (U, $\tau_{|U}$, $I_{|U}$).

Proof: Since U is fuzzy open i.e. U ∈ τ and A is weakly fuzzy α - I -open, by proposition 2.2, A∧U is weakly fuzzy α - I -open in (X, τ, I). Since U is fuzzy open i.e. U ∈ τ, by proposition 2.3, U∧A is weakly fuzzy α - I -open in (U, $\tau_{|U}$, $I_{|U}$).

Definition 2.3: A subset A of a fuzzy ideal topological space (X, τ, I) is said to be fuzzy strong t- I -set if sCl(Int(Cl*(A))) = Int(A).

Definition 2.4: A subset A of a fuzzy ideal topological space (X, τ, I) is said to be fuzzy strong $\mathcal{B}_I$ -set if A = U ∧ V, where U ∈ τ and V is a fuzzy strong t - I -set.

Proposition 2.4: For a subset A of a fuzzy ideal topological space (X, τ, I), the following conditions are equivalent:
(a) A is a fuzzy open set,
(b) A is weakly fuzzy α - I -open set and fuzzy strong $\mathcal{B}_I$ -set.

Proof: The implications
(a) → (b): is obvious.

(b) → (a): Let A be weakly fuzzy α - I -open, we have A≤ sCl(Int(Cl *(Int(A)))), also A ≤ sCl(Int(Cl*(Int(U ∧ V)))), where A = U ∧ V, U ∈ τ and V is a fuzzy strong t - I -set.
Hence A = U ∧ A ≤ U ∧ sCl(Int(Cl*(Int(U ∧ V)))) ≤ U ∧ sCl(Int(Cl*(Int(U))))∧sCl(Int(Cl*(V))), since V is fuzzy strong t - I -set, therefore A ≤ U ∧ sCl(Int(Cl*(Int(U)))) ∧ Int(V) = U ∧ Int(V) = Int(A). This shows that A is fuzzy open.

Remark 2.4: Weakly fuzzy α - I -open and fuzzy strong $\mathcal{B}_I$ -set are independent to each other, as can be seen from the following examples.

Example 2.6: Let X = {a, b, c} and A, B be fuzzy subsets of X defined as follows:
A(a) = 0.2, A(b) = 0.3, A(c) = 0.7,
B(a) = 0.1, B(b) = 0.2, B(c) = 0.2,

Let τ = {0, B, 1}. If we take I = {0}, then A = A ∧ B is fuzzy strong $\mathcal{B}_I$ -set but A is not weakly fuzzy α - I -open set.

Example 2.7: Let X = {a, b, c} and A, B, C be fuzzy subsets of X defined as follows:
A(a) = 0.3, A(b) = 0.2, A(c) = 0.7,
B(a) = 0.8, B(b) = 0.8, B(c) = 0.4,
C(a) = 0.3, C(b) = 0.2, C(c) = 0.4.

Let τ = {0, A, B, A∨B, C, 1}. If we take I = {0}, then C is weakly fuzzy α - I –open set but C = C∧ B is not fuzzy strong $\mathcal{B}_I$ -set. Because C ∈ τ and B is not fuzzy strong t- I -set.

3. DECOMPOSITION OF FUZZY CONTINUITY

Definition 3.1: A function f: (X, τ, I)→ (Y, σ) is called weakly fuzzy α - I – continuous (resp. fuzzy strong $\mathcal{B}_I$ -continuous) if inverse image of each fuzzy open set of Y is weakly fuzzy α - I -open (resp. fuzzy strong $\mathcal{B}_I$ -set) in X.

Definition 3.2: A function f: (X, τ, I)→ (Y, σ) is said to be weakly fuzzy b - I - continuous if inverse image of each fuzzy open set of Y is weakly fuzzy b - I -open set in X .

Theorem 3.1: If a function f: (X, τ, I)→ (Y, σ) is weakly fuzzy α - I -continuous, then f is weakly fuzzy b-I-continuous.

Proof: The proof is immediately follows from Theorem 2.1(a).

Proposition 3.1: For a function f: (X, τ, I) → (Y, σ), the following properties are equivalent:
(a) f is weakly fuzzy α - I -Continuous.
(b) For each x ∈ X and V ∈ τ containing f(x), there exists weakly fuzzy α - I -open set A containing x such that f(A) ≤ V.
(c) The inverse image of each fuzzy closed set in (Y, σ) is weakly fuzzy α - I - closed.

Proposition 3.2: If f: (X, τ, I)→ (Y, σ , J) is a weakly fuzzy α - I –continuous function and g: (Y, σ, J) → (Z, η) is a fuzzy continuous function, then gof: (X, τ, I) → (Z, η) is a weakly fuzzy α - I -continuous function.

Proof: Let V ∈ η. Since g is fuzzy continuous, $g^{-1}(V) \in \sigma$. On the other hand, since f is weakly fuzzy α - I -continuous, we have $f^{-1}(g^{-1}(V))$ is a weakly fuzzy α - I -open set. Since $(gof)^{-1}(V) = f^{-1}(g^{-1}(V))$, we obtain that (gof) is weakly fuzzy α - I -continuous.

Proposition 3.3: Let f: (X, τ, I)→ (Y, σ) is a weakly fuzzy α - I -continuous and U ∈ τ. Then the restriction $f_{|U} : (U, \tau_{|U}, I_{|U}) \rightarrow (Y, \sigma)$ is weakly fuzzy α - I -continuous.

Proof: Let V be any fuzzy open set of (Y, σ). Since f is weakly fuzzy α - I - continuous, $f^{-1}(V)$ is a weakly fuzzy α - I -open set. On the other hand, we have $(f_{|U})^{-1}(V) = f^{-1}(V) \wedge U$ is weakly α - I -open in $(U, \tau_{|U}, I_{|U})$. This shows that $f_{|U} : (U, \tau_{|U}, I_{|U}) \rightarrow (Y, \sigma)$ is weakly fuzzy α - I - continuous.

Proposition 3.4: For a function f: (X, τ, I) → (Y, σ) the following conditions are equivalent:

(a) f is fuzzy continuous,

(b) f is weakly fuzzy α - I -continuous and fuzzy strong $\mathcal{B}_I$ -continuous.

Proof: This is an immediate consequence of Definition 3.1 and Proposition 2.4.

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