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CONNECTED ENTIRE DOMINATION IN GRAPHS

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ABSTRACT

The vertices and edges of a graph G are called the elements of G. A set X of elements in G is an entire dominating set if every element not in X is an entire dominating set if every element not in X is either adjacent or incident to at least one element in X. An entire dominating set X of G is a connected entire dominating set if the induced subgraph ⟨X⟩ is connected. The connected entire domination number $\varepsilon_c(G)$ of G is the minimum cardinality of a connected entire dominating set in G. In this paper, we initiate a study of this parameter and present some bounds and some exact values for $\varepsilon_c(G)$. Also Nordhaus Gaddum type results are obtained.

Keywords: *entire dominating set, connected entire dominating set, connected entire domination number.*

Mathematics Subject Classification: *05C.*

1. INTRODUCTION

All graphs considered here are finite, undirected without loops and multiple edges. Any undefined term in this paper may be found in Kulli [1].

Let $G = (V, E)$ be a graph with $|V|= p$ vertices and $|E| = q$ edges. A set D of vertices in a graph G is a dominating set if every vertex in $V – D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. Recently many new domination parameters are given in the books by Kulli [2, 3, 4].

The vertices and edges of a graph G are called the elements of G. A set X of elements in G is an entire dominating set if every element not in X is either adjacent or incident to at least one element of X. The entire domination number $\varepsilon(G)$ of G is the minimum cardinality of an entire dominating set in G. This concept was introduced by Kulli [5] and was studied, for example, in [6, 7, 8, 9]. Many other domination parameters in dominating theory were studied, for example, in [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

Let D be a subset of the vertex set of G. The induced subgraph $\langle D\rangle$ is the maximal subgraph of G with the vertex set D. A dominating set D of G is a connected dominating set if the induced subgraph $\langle D\rangle$ is connected. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set in G. Many other connected domination parameters in domination theory were studied, for example, in [27, 28, 29, 30].

Let F be a subset of the edge set of G. The induced subgraph $\langle F\rangle$ is the subgraph of G with the vertex set V_1 and edge set F, where V_1 is the set of all vertices incident with the edges of F. An edge dominating set F of G is a connected edge dominating set if the induced subgraph $\langle F\rangle$ is connected. The connected edge domination number $\gamma'_c(G)$ of G is the minimum cardinality of a connected edge dominating set in G. This concept was introduced by Kulli and Sigarkanti in [31] and was studied, for example, in [32].

Let $\lfloor x \rfloor$ denote the least integer less than as equal to x. Let $\overline{G}$ be the complement of G.

In [33], Kulli and Sigarkanti introduced the concept of connected entire domination. In this paper, we study this parameter.

We note that $\varepsilon_c(G)$ is defined only for connected graphs G.

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2. CONNECTED ENTIRE DOMINATION NUMBER

We need the following definition.

Definition 1 [32]: Let $G = (V, E)$ be a graph. Let $D \subseteq V$ and $F \subseteq E$. The subgraph $\langle X\rangle$ induced by $X = D \cup F$ is the subgraph of G with the vertex set $D \cup D_1$ where D_1 is the set of all vertices incident with the edges in F and the edge set $F \cup F_1$ where $F_1 = E(\langle D\rangle)$.

We now define the concept of connected entire domination in graphs.

Definition 2: An entire dominating set X of G is a connected entire dominating set if the induced subgraph $\langle X\rangle$ is connected. The connected entire domination number $\varepsilon_c(G)$ of G is the minimum cardinality of a connected entire dominating set in G.

3. EXACT VALUES

We obtain exact values of $\varepsilon_c(G)$ for some standard graphs.

Proposition 3: For any complete graph K_p with $p \geq 2$ vertices, $\varepsilon_c(K_p) = p - 1$.

Proof: Let V be the vertex set of K_p and v be a vertex of K_p. Then $V - \{v\}$ is a minimum connected entire dominating set of K_p. Thus $\varepsilon_c(K_p) = | V - \{v\}| = p - 1$.

Proposition 4: For any cycle C_p with $p\geq 3$ vertices, $\varepsilon_c(C_p) = p - 1$.

Proof: Let V be the vertex set of C_p and $v\in V$. Then $V - \{v\}$ is a minimum connected entire dominating set of C_p. Thus

$$\begin{aligned}\varepsilon_c(C_p) &= |V - \{v\}| \\ &= p - 1.\end{aligned}$$

Proposition 5: For any complete bipartite graph $K_{m,n}$ with $m \leq n$,

$$\begin{aligned}\varepsilon_c(K_{m,n}) &= 1, \quad \text{if} \quad m = 1, \\ &= m+1 \quad \text{if} \quad m \geq 1.\end{aligned}$$

Proof: Let $V=V_1\cup V_2$ be the vertex set of $K_{m,n}$ such that $|V_1| = m$ and $|V_2| = n$. If $m=1$, then $V_1=\{v\}$ and $\{v\}$ is the minimum connected entire dominating set of $K_{m,n}$. Thus

$$\varepsilon_c(K_{m,n}) = 1, \quad \text{if} \quad m = 1.$$

If $m \geq 2$, then for any vertex $v\in V_2$, the set $V_1 \cup \{v\}$ is a minimum connected entire dominating set of $K_{m,n}$. Thus

$$\begin{aligned}\varepsilon_c(K_{m,n}) &= |V_1 \cup \{v\}| \\ &= m + 1, \quad \text{if } m \geq 2.\end{aligned}$$

Proposition 6: For any wheel W_p with $p \geq 4$ vertices,

$$\varepsilon_c(W_p) = \left\lfloor \frac{p}{2} \right\rfloor + 1.$$

Proof: Let $v(W_p) = \{v_1, v_2, \ldots , v_p\}$ and let $\deg v_p = p - 1$ and $\deg v_i = 3$, $1 \leq i \leq p - 1$. We consider the following two cases.

Case-1: Suppose p is odd. Then $X = \{v_2, v_4, \ldots , v_{p-1}\} \cup \{v_p\}$ is a minimum connected entire dominating set of W_p. Thus

$$\begin{aligned}\varepsilon_c(W_p) &= | X | \\ &= \frac{p-1}{2}+1 \\ &= \left\lfloor \frac{p}{2} \right\rfloor + 1.\end{aligned}$$

Case-2: Suppose p is even. Then $Y = \{v_1, v_3, \ldots , v_{p-3}, v_{p-2}\} \cup \{v_p\}$ is a minimum connected entire dominating set of W_p. Thus

$$\begin{aligned}\varepsilon_c(W_p) &= | Y | \\ &= \frac{p}{2}+1\end{aligned}$$

From Case 1 and Case 2, we have

$$\varepsilon_c(W_p) = \left\lfloor \frac{p}{2} \right\rfloor + 1.$$

Proposition 7: For any tree T with $p \geq 3$ vertices,

$$\varepsilon_c(T) = p - n$$

where n is the number of endvertices in T.

Proof: Clearly the set of all nonend vertices is a minimum connected entire dominating set of T. Hence

$$\varepsilon_c(T) = p - n.$$

Corollary 8: For any path P_p with $p \geq 3$ vertices,

$$\varepsilon_c(P_p) = p - 2.$$

Proof: This follows from Proposition 7.

4. BOUNDS

Theorem 9: For any connected graph G with $p \geq 2$ vertices,

$$\varepsilon(G) \leq \varepsilon_c(G) \tag{1}$$

and this bound is sharp.

Proof: Clearly every connected entire dominating set is an entire dominating set. Thus (1) holds.

The graphs $K_{1,p}$, $p \geq 2$ and K_3 achieve this bound.

Theorem 10: For any connected graph G with $p \geq 2$ vertices,

$$\varepsilon_c(G) \leq p - 1 \tag{2}$$

and this bound is sharp.

Proof: Let $v \in V$ be a noncutvertex of G. Then $V - \{v\}$ is a connected entire dominating set of G. Hence

$$\varepsilon_c(G) \leq |V - \{v\}| = p - 1.$$

Equality holds in (2) if $G = K_p$ or C_p.

We obtain another upper bound on $\varepsilon_c(G)$.

Theorem 11: For any connected graph G with $p \geq 2$ vertices,

$$\varepsilon_c(G) \leq p - \min(n) + 1$$

where n is the number of endvertices in a spanning tree T of G.

Proof: Let T be a spanning tree of G with minimum number of endvertices. If no two endvertices of T are adjacent in G, then the set of all nonendvertices of T is a connected entire dominating set of G. If for every spanning tree T, there is an edge uv in G joining two endvertices u and v of T, then the set of all nonendvertices together with u is a connected entire dominating set of G. Thus

$$\varepsilon_c(G) \leq p - \min(n) + 1.$$

We now obtain a lower bound and an upper bound of $\varepsilon_c(G)$ in terms of $\gamma_c(G)$ and $\gamma'_c(G)$.

Theorem 12: For any nontrivial connected graph G,

$$\left[\gamma_c(G) + \gamma'_c(G)\right]/2 \leq \varepsilon_c(G) \leq \gamma_c(G) + \varepsilon'_c(G).$$

Proof: We obtain the upper bound. Let D be a minimum connected dominating set of G and F be a minimum connected edge dominating set of G. Then $D \cup F$ is a connected entire dominating set of G. Thus

$$\varepsilon_c(G) \leq |D \cup F|$$

or $$\varepsilon_c(G) \leq \gamma_c(G) + \varepsilon'_c(G).$$

We now establish the lower bound. Let $D \cup F$ be a minimum connected entire dominating set of G where $D \subseteq V$ and $F \subseteq E$. For each edge $e = uv$ in F, choose a vertex either u or v or both. Let F' denote the set of all such vertices such that $\langle D \cup F' \rangle$ is connected. Then $D \cup F'$ is a connected dominating set of G.

Thus

$$\gamma_c(G) \le |D \cup F'|$$
or $$\gamma_c(G) \le |D \cup F|$$
or $$\gamma_c(G) \le \varepsilon_c(G). \quad (3)$$

Similarly, for each vertex v in D, choose exactly one edge $e = uv$. Let D' denote the set of all such edges such that $\langle D' \cup F \rangle$ is connected. Then $D' \cup F$ is a connected edge dominating set of G. Thus

$$\gamma'_c(G) \le |D' \cup F|$$
or $$\gamma'_c(G) \le |D \cup F|$$
or $$\gamma'_c(G) \le \varepsilon_c(G). \quad (4)$$

From (3) and (4) we have

$$\left[\gamma_c(G) + \gamma'_c(G)\right]/2 \le \varepsilon_c(G).$$

Theorem 13: Let D be a maximal independent set of G. If every vertex of $V - D$ is a cutvertex and $\langle V - D \rangle$ is connected, then $\gamma_c(G) = \varepsilon_c(G)$.

Proof: Let D be a maximal independent set in G. Then every vertex of D is adjacent to at least one vertex in $V - D$. Hence $V - D$ is a minimum connected dominating set of G. Since D is independent, it implies that $V - D$ is a connected entire dominating set of G. Thus $\varepsilon_c(G) \le \gamma_c(G)$ and since by (3), $\gamma_c(G) \le \varepsilon_c(G)$, it implies that $\gamma_c(G) = \varepsilon_c(G)$.

Theorem 14: Let G be a connected graph with $p \ge 3$ vertices. If $\varepsilon_c(G) = p - 1$, then G is hamiltonian.

Proof: Suppose G is a tree. Then by Proposition 7,

$$\varepsilon_c(G) = p - n \le p - 2$$

which is a contradiction. Thus G contains at least one cycle. We now prove that G does not contain a cutvertex. On the contrary, assume G has a cutvertex v. Without loss of generality, let G_1 and G_2 be the components of $G - v$. If both G_1 and G_2 are nontrivial, then $X = X_1 \cup X_2 \cup \{v\}$ is a connected entire dominating set of G, where X_1 and X_2 are minimum connected entire dominating sets of G_1 and G_2 respectively. Thus

$$\varepsilon_c(G) \le |X|$$
or $$\varepsilon_c(G) \le |X_1 \cup X_2 \cup \{v\}|$$
or $$\varepsilon_c(G) \le p - 2$$

which is a contradiction.

Since G contains at least one cycle, it implies that at least one of G_1 and G_2 is nontrivial. Suppose G_1 is nontrivial and G_2 is trivial. Then $X = X_1 \cup \{v\}$ is a connected entire dominating set of G, where X_1 is a minimum connected entire dominating set of G_1.

Thus

$$\varepsilon_c(G) \le |X|$$
or $$\varepsilon_c(G) \le |X_1 \cup \{v\}|$$
or $$\varepsilon_c(G) \le p - 2$$

which is a contradiction. Hence G contains no cutvertices.

Let C_k be a largest cycle in G. Suppose C_k contains fewer than p vertices. Since G contains no cutvertices, it implies that there exists a cycle C_n having at least two vertices in common with C_k. Let $u \in V(C_k)$ and $v \in V(C_n)$ for some $u, v \in V(G)$. Then these two vertices u, v are nonadjacent. If for each u and v, there is an edge $uv \in E(G)$, then $C_k \cup C_n$ contains a cycle of order greater than the order of C_k, which is a contradiction. Thus for some $u \in V(C_k)$ and $v \in V(C_n)$, $uv \notin E(G)$. Therefore $V(G) - \{u, v\}$ is a connected entire dominating set of G, which is a contradiction. Thus C_k contains all the vertices of G. Hence G is hamiltonian.

5. NORDHAUS-GADDUM TYPE RESULTS

Nordhaus-Gaddum type results were obtained for many dominating parameters, for example, in [34, 35, 36, 37].

We now obtain Nordhaus-Gaddum type results.

Theorem 15: For connected graphs G and $\overline{G}$,

$$\varepsilon_c(G)+\varepsilon_c(\overline{G})\le 2(p-1) \tag{5}$$

$$\varepsilon_c(G).\varepsilon_c(\overline{G})\le (p-1)^2. \tag{6}$$

Further the bounds are attained if $G = C_5$.

Proof: By Theorem 10, $\varepsilon_c(G) \le p - 1$. Since $\overline{G}$ is connected and by Theorem 10, $\varepsilon_c(\overline{G})\le (p-1)$. Therefore the inequalities (5) and (6) hold.

Clearly if $G = C_5$, then $\varepsilon_c(G) = 4$ and $\varepsilon_c(\overline{G})=4$. Therefore both bounds in (5) and (6) are attained.

Theorem 16: For connected trees T and $\overline{T}$,

$$\varepsilon_c(T)+\varepsilon_c(\overline{T})\le p(p-3). \tag{7}$$

Furthermore the equality in (7) holds if and only if $T=P_4$.

Proof: By Proposition 7, $\varepsilon_c(T) \le p - 2 = 2(p - 1) - p = 2q - p$. Similarly, $\varepsilon_c(\overline{T})\le 2\overline{q}-p$. Thus $\varepsilon_c(T)+\varepsilon_c(\overline{T})\le 2(q+\overline{q})-2p = p(p-1)-2p = p(p-3)$.

Suppose equality in (7) holds. Thus it implies that $\varepsilon_c(T) = 2p - p$ and $\varepsilon_c(\overline{T})=2\overline{q}-p$. This implies that q, $\overline{q} < p$. Thus $G = P_4$.

Theorem 17: For connected trees T and $\overline{T}$,

$$\varepsilon_c(T).\varepsilon_c(\overline{T})\le (p-2)^2. \tag{8}$$

Furthermore, equality holds if $T=P_4$.

Proof: By Proposition 7, $\varepsilon_c(T)\le p-2$, and also $\varepsilon_c(\overline{T})\le p-2$. Therefore equality (8) holds.

Clearly if $T = P_4$, then $\varepsilon_c(T) = 2$ and $\varepsilon_c(\overline{T})=2$. Therefore equality in (8) holds.

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