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DEVELOPMENT OF THE SCIENCE OF COMBINATORICS IN ANCIENT INDIA

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ABSTRACT

The topic 'Combinatorics' deals with counting problems, which are usually associated with selection of some objects from a given collection of objects and their arrangements in a certain situation. The principles and methods of solving different combinatorial problems were well-known to the scholars of ancient India since the Vedic period. But no chronological account of contributions of Indian scholars in the field has been made available so far. The present paper is an attempt in this direction.

Keywords: *Combinatorics; Permutation and Combination; Fibonacci numbers; Pascal's triangle; Sanskrit prosody.*

AMS Subject Classification: *Primary 01A32.*

1. INTRODUCTION

Importance of the topic 'Combinatorics' in mathematics cannot be over-emphasised. The subject deals with counting problems, which are usually associated with selection of some objects from a given collection of objects and their arrangements in a certain situation. The treatment and techniques of solution of such problems are based on Addition and Multiplication principles, Induction, Inclusion-Exclusion, Bijection, Pigeon Hole and such other principles. Moreover, Recurrence-relation provides a powerful computational tool and the so-called Fibonacci Sequence is also useful in solving such problems. The combinatorial analysis includes principles of partitions, theorems on choice, theory and construction of designs, etc. Thus the remarkable progress has been made in the field in modern times. Consequently principles and techniques of Combinatorics have found their applications not only in physical science but also in social and biological sciences, operational research, computer science, etc. The subject has become so broad and we have become so much familiar with these methods that we seldom feel necessity to think as to how and when these were invented. There are sufficient evidences to believe that initially these principles were applied by Indian scholars in the field of philosophy, dramatics, medicine, astronomy and such other branches of knowledge. Methods for solving such problems are first found in the Vedas and then in more developed form in later literature, but very little efforts have been made in the past to record its gradual evolution in India. The result is that no chronological account of contributions of Indian scholars in the field has been made available so far. The present paper is an attempt in this direction.

2. CONTRIBUTIONS OF INDIAN SCHOLARS

Study of Combinatorics originated in ancient India as a handmaid of Sanskrit prosody. Its use is also found in respect of medicine and philosophy. In the Vedic literature we find the idea about the number of ways in which different metres (chandas) viz, Anustubha, Tristubha, Jayanti etc. can be varied and as such notions of permutation and combination are traceable here at least four thousand years (BC) ago. Problems of Combinatorics and their solutions are also found in ancient medical treatises, such as Charaka Samhita and Sushruta Samhita (both of the 6th century BC) where it is said that 63 combinatorics may be made out of six different tastes (Rasas) by taking the Rasas one at a time, two at a time and so on. We thus obtain 6, 15, 20, 15, 6 and 1 combinations totaling 63[1]. Pingal's Chanda Sutra (300 BC) is the earliest extant work on the Sanskrit prosody, which considers the method of finding the number of combinations obtainable by taking one letter (Ekaka-Samyoga), two letters (Dwika-Samyoga), etc. out of a given number of letters.

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This work describes a rule for laying down different prastaras, which has been explained clearly by Halayudha (10^{th} century AD) in his commentary on the Chandah Sutra[2]. Besides method of computing the binomial terms, Pingala gives his general Meru-Prastara rule for determining such binomial coefficients. The numbers in the r^{th} row are ${}^{r}c_1$, ${}^{r}c_2$, …, ${}^{r}c_r$ in order. The rule clearly illustrates the formula ${}^{n+1}c_r = {}^{n}c_r + {}^{n}c_{r-1}$. This is supposed to be established by Blaise Pascal (1623-1662), the French mathematician, who arranged binomial co-efficient in a triangular form from top to bottom and in increasing values of n and r from left to right, but the formula was thought of in India many centuries before Pascal.

The early Jaina canonical as well as literary works treated the subject of permutation and combination in some detail, which was named as Vikalpa or Bhanga. Sthananga Sutra of the 5^{th} century BC speaks of ten topics of mathematics including Vikalpa (Permutation and Combination) as one of them[3]. Bhagavati Sutra (300 BC) also contains such instances where speculation is made about different philosophical categories arising out of the combination of n fundamental objects, one at a time (Eka = Samyoga), two at a time (Dvika-Samyoga), three at a time (Trika-Samyoga) or more at a time. It further forms out of different senses (Karanas) or selections made of males and females and also permutation and combination of various other things which could be put in the modern prevalent form as ${}^{n}c_1 = n$, ${}^{n}c_2 = n(n-1)/2!$, ${}^{n}p_1 = n$, ${}^{n}p_2 = n(n-1)$, ……

A study of Anuyogadvara Sutra (150 BC) makes it clear that the general value of ${}^{r}c_r$ is $r!$ and its proof was also known at that time. Silanka (826 AD), the commentator quotes three rules regarding permutation and combination. According to these rules the total number of permutation of n things taken all at a time = 1.2……n and the total number permutations having the n^{th} thing for beginner = $n!/n = (n – 1)!$. Yatibrisabha (5^{th} century AD) and Virasena (9^{th} century AD) have also considered different cases of permutation and combination. Acharya Nemicandra (10^{th} century AD) gives full details of the method for finding combination by explaining the terms, Sankhya, Prastara, Parivartana, Nasta and Sumuddist for understanding the process[4]. Besides these canonical works, several later works on prosody namely, Acarya Bharata's Natya Sastra (100 BC – 300 AD), Janasrayi Chandoviciti of Janashraya (prior to 600 AD), Vrttajati Samuccaya of Virabarika (600 AD to 900 AD), Jayadeva Chandas (600 AD – 900 AD), Ratna- Manjusa (c. 800 AD), Chandonusasana of Jaykirti (c. 1000 AD), Vrtta-Ratnakara of Kedar Bhatta (c. 1075 AD), Chandonusasana of Hemacandra (1150 AD), Prakrta Pingala (c. 1310 AD), etc. contain mathematical treatment of Sanskrit metres and also include different rules for finding combinatorial and binomial co-coefficients.

As far as mathematical texts are concerned, there are informations regarding several works of ancient India which deal with different methods for solving combinatorial problems. First of all, we find the treatment of the subject in the works of Varahamihira (6^{th} century AD). His method for finding ${}^{n}c_r$ is based on what is called Lostaka-Prastara. Though the rule is mentioned briefly in a condensed form, the so-called Pascal's Triangle is seen formed in this Lostaka-Prastara[5]. Brahmagupta (628 AD), too, has developed a full chapter on the Sanskrit prosody and applied principles for finding various types of metre[6]. The formula for finding ${}^{n}c_r$ or the number of ways in which r things can be selected out of n is specifically indicated by Sridhara (750 AD) in his Patiganita[7]. By the time of Mahaviracarya (850 AD), the subject was treated as an important topic of mathematics. He has established formula ${}^{n}c_r = n!/\{(n-r)!\ r!\}$ and illustrated the rule by giving different examples[8]. Similar rule is given by Aryabhata II (950 AD) in his work Mahasiddhanta[9]. Bhaskaracarya II (1150 AD)[10] has discussed the topic in the last chapter of his mathematical work Lilavati. He has given the name Ankapasa (net or chain of numbers) to the subject combinatorics. General rules for ${}^{n}p_r$ and ${}^{n}c_r$ have been given along with several illustrations (at least 7 in number). One illustration is like: 10! = 36, 28,800 and 4! = 24. The last problem discussed by him deals with the formula for finding the number of n-digit numbers with a given digital sum. According to it the total number of n-digit numbers formed using the digits 1 to 9 will be 9^n.

The subject of combinatorics with all its relevant aspects has been discussed in detail by Narayana Pandit (14^{th} century AD) in his mathematical text, Ganita Kaumudi (written in 1356 AD)[11]. Like Bhaskaracarya, he, too, has devoted a full chapter to it and named it as Ankapasa (net or chain of numbers). Different topics of combinatorics, namely, permutation and combination, binomial theorem, partitions of a number and their relation, sequences of binomial and polynomial coefficients, uses of numbers containing three sequences and their relations and such other related topics have been dealt with by him. Besides establishing rules for finding ${}^{n}p_r$ and ${}^{n}c_r$ under different conditions and also for finding binomial coefficients, he has given several examples for understanding the process. He has even stated the limitations of these rules and described different types of figures of numbers, namely, Khandameru, Pataka, Sumeru and Matsyameru. Rules established by him are definitely in more advanced form than those given by earlier authorities.

It may also be pointed out that sequences as well as partitions of a number are very much helpful in the solution of combinatorial problems. It is believed that Leonardo Fibonacci (1170 - 1250)[12] is the inventor of the sequence: 0, 1, 2, 3, 5, 8, 13, ….. in which the n^{th} term i.e. u_n is given by $u_{n-1} + u_{n-2}$ but this sequence was well known in India much before his time. Centuries ago Indian authorities on metrical sciences used this sequence in their works on metres. Variations of Matra Vrttas formed this sequences of numbers. As for example, numbers of variations of Matra metres having 1, 2, 3, 4, 5, 6, ….. matras (syllables) are 1, 2, 3, 5, 8, 13, ….. respectively and these are the so-called Fibonacci numbers. The general rule is $\varphi(n) = \varphi(n-1) + \varphi(n-2)$ where $\varphi(n)$ is the number of variations of the metre having n matras (syllables). Acarya Pingala (3^{rd} century BC) is the first authority on metrical sciences in India whose writings indicate a knowledge of the so-called Fibonacci numbers. Similarly other later authorities on metrical sciences, namely, Janasraya, Jayadeva, Kedara Bhatta, Hemacandra and others too stated rules in this connection. Moreover, the process of expansion of Matra Vrttas is a kind of partitioning of a number (the number being the number of syllables in the metre) where the digits take the values 2 and 1 only and the order is relevant, the number of digits in a particular partition being arbitratory[13]. However, ideas of the so-called Fibonacci Numbers were developed further in the mathematical work, Ganita Kaumudi of Narayana Pandita. He makes use of the methods of formations of the Suci Pankti (needle like sequence) in the formation of his figures of numbers named Matsya Meru and his Samasik Pankti (additive sequence) is the tool for the treatment of partition, permutation and combination etc. The so-called Fibonacci numbers are a particular case of the Pankti. Prakrta Pingala (14^{th} century) also contains rules regarding Nasta and Uddist analysis of Matra vrttas which are usual combinatorial problems related to the measurement of morae and letters in verses[14].

3. DEVELOPMENTS IN THE SUBJECT IN RESPECT OF OTHER COUNTRIES

As far as developments in the subject in respect of other countries are concerned, China is probably next to India, where Pascal's triangle was first described by Liu Ju Hsieh (14^{th} century AD) in his book Fu Chi Shih So. In Europe, it appeared not before the 16^{th} century. Newton and E. Moivre (16^{th} century) used this principle for establishing binomial and multinomial theorems while Pascal developed it further. Euler (1764), M.A. Stern (1840), J. Franklin (1881) and others developed different principles, relating to partition as well. G.H. Hardy and S. Ramanujan, the two great scholars of the 20^{th} century, established identities (in 1918) regarding the function $P(n)$, the numbers of partition of n into any number of parts[15].

4. CONCLUSIONS

From a consideration of the above mentioned account it may now be safely concluded that principles and methods of solving different combinatorial problems were well-known to the scholars of ancient India since the Vedic period. Moreover, developments in this regard were made to a great extent in successive ages by the scholars of Sanskrit prosody as well as mathematicians. Besides establishing rules and methods relating to permutation and combination, binomial theorem, binomial co-coefficients, partitions of a number, the so-called Pascal's Triangle and Fibonacci numbers were also thought of here many centuries before them. However, further investigations in this regard are still needed so that the salient achievements of Indian scholars in this field may be properly assessed and re-evaluated.

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