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# DUALITY IN THE PARAMETRIC SPACES FOR PARAMETRIC NONLINEAR PROGRAMMING PROBLEMS

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**ABSTRACT**

*This paper presents a new concept concerning duality in parametric spaces which utilizes the direct and clear relation between multiobjective nonlinear programming problems (MONLP) and parametric nonlinear programming problems via different scalarization approaches for treating MONLP problems[1], [6], [7].*

*By this concept two parametric nonlinear programming problems are defined; one with parameters in the objective function and the other with parameters in the constraints, the two problems are said to be parametrically dual. The basic notions of set of feasible parameters, the solvability sets and the stability sets of the first and second kinds are defined for both problems and several propositions are presented relating the basic notions for both problems with each other's. Finally, illustrated example is given to clarify the developed results in this paper.*

**Key words:** *Nonlinear programming; Parametric study; Basic notions; Duality; multiobjective problems.*

## 1. INTRODUCTION

In earlier work M. Osman[2], [3] introduced the notions of the set of feasible parameters, the solvability set and the stability sets of the first and second kinds and analyzed these concepts for parametric convex programming problems. The relation to and importance of these results in multiobjective nonlinear programming problems (MONLP) can be seen by the fundamental role which parametric techniques play in multiobjective nonlinear programming. From that time and until now several researchers are working on the same notions and others for different structures of parametric nonlinear programming problems [2], [3], [4], [5].

In this paper two structures are formulated one with parameters in the objective function and the other with parameters in the constraints, the two problems are said to be parametrically dual in the convex case and partially parametrically dual in the non-convex case.

The basic notions are defined for both problems and they are related to each other by several results. The paper is divided into six main sections, the problem formulation is presented in section two, section three is devoted to the definitions of basic notions for both problems, section fourpresents the relations between the basic notions for both problems, section five considers the possibility of reducing the parametric space for one problem. Illustrative example is given to clarify the obtained results. Finally conclusion and points for further researches are presented in section six.

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## 2. PROBLEM FORMULATION

Let us consider the following MONLP problem:

(MONLP):

$\min (f_1(x), f_2(x), \ldots\ldots\ldots\ldots\ldots f_{k+1}(x))$

Subject to

$M=\{x \in R^n / g_r(x) \leq 0,\ r=1,2,\ldots\ldots\ldots\ldots m\}$ } (1)

Where $f_i(x)$, $i= 1,2,\ldots\ldots\ldots k+1$, $g_r(x)$, $r=1,2,\ldots\ldots\ldots\ldots m$ are continuous nonlinear functions on $R^n$

Consider the following two parametric nonlinear programming problems, one with parameters in the objective functions and the second with parameters in the constraints

$P_1(\lambda)$:

$\min \sum_{i=1}^{k+1} \lambda_i\ f_i(x)$

Subject to

$M=\{x \in R^n / g_r(x) \leq 0,\ r=1,2,\ldots\ldots\ldots\ldots m\}$ (2)

Where $\lambda_i \in R, \lambda_i \geq 0,\ i= 1,2,\ldots\ldots k+1, \sum_{i=1}^{k+1} \lambda_i = 1$ }

$P_2(\mathcal{E})$:

$\min f_{k+1}(x)$

Subject to } (3)

$N=\{x \in R^n / f_i(x) \leq \mathcal{E}_i,\ i=1,2,\ldots\ldots k,\ x \in M\}$

$\mathcal{E}_i \in R,\ i=1,2,\ldots\ldots k$

Problems (2), (3) are said to be parametrically dual in $R^{k+1}$, $R^k$.

## 3. BASIC NOTIONS

For problem (2), the following basic notions are defined.

**Definition 1:** (the solvability set $B_1$)

$$B_1 = \{\lambda \in R^{k+1} / \text{problem (2) is solvable}\} \quad (4)$$

**Definition 2:** (the stability set of the first kind)

Assume that for $\bar{\lambda} \in B_1$ an optimal solution is found to be $\bar{x} \in M$, then the stability set of the first kind of problem (2) corresponding to $\bar{x}$ denoted by $S_1(\bar{x})$ is defined by

$$S_1(\bar{x}) = \{\lambda \in B_1 / \bar{x} \text{ solves problem (2)}\} \quad (5)$$

**Definition 3:** (the stability set of the second kind $Q(\sigma(I))$)

Assume that $\sigma(I)$ denotes a side of M defined by:

$$\sigma(I) = \{x \in M / g_r(x) = 0, r \in I \ \ g_r(x) < 0, r \in \{1,2,\ldots.m\} - I \ \} \quad (6)$$

Then, the stability set of the second kind of problem (2) corresponding to the side $\sigma(I)$ denoted by$Q_1(\sigma(I))$ is defined by:

$$Q_1(\sigma(I)) = \{\lambda \in B_1 / \text{a corresponding optimal solution } x' \in \sigma(I) \ \} \quad (7)$$

For problem (3), the following basic notions are defined.

**Definition 4:** (the set of feasible parameters F)

$$F = \{ ⍰ R^k / N \neq \emptyset\} \quad (8)$$

**Definition 5:** (the solvability set $B_2$)

$$B_2 = \{⍰ F / \text{problem (3) is solvable}\} \quad (9)$$

**Definition 6:** (the stability set of the first kind $S_2(\bar{x})$)

$$S_2(\bar{x}) = \{⍰B_2 / \bar{x} \text{ solves problem (3)}\} \quad (10)$$

**Definition 7:** (the stability set of the second kind $Q_2(\sigma(I))$)

$$Q_2(\sigma(I)) = \{⍰B_2 / \text{a corresponding optimal solution } x' \in \sigma(I) \ \} \quad (11)$$

Let $O_1$, $O_2$ denote the set of optimal solutions for problems (2), (3) respectively which are obtained for all $\lambda \epsilon B_1$, $Ɛ \epsilon B_2$.

The sets $B_1$ , $S_1(\bar{x})$ , $Q_1(\sigma(I))$ are said to be parametrically dual to the sets $B_2$, $S_2(\bar{x})$, $Q_2(\sigma(I))$ respectively, under the convexity condition of the set M , and the functions $f_i(x)$, i=1,2,………,k+1.

Otherwise, (In the non-convex case) they are said to be partially parametric dual.

To understand the reasons for these definitions, let's consider the multiobjective nonlinear programing problem (1). It is well known that the efficient solutions of (1) could be generated using either one of the scalarization problems [1] $P_1(\lambda)$, $P_2(Ɛ)$ (defined by problems (2), (3)).

## 4. THE RELATIONS BETWEEN THE EFFICIENT SOLUTIONS OF(MONLP) AND THE OPTIMAL SOLUTIONS OF $P_1(\lambda)$, $P_2(Ɛ)$

These relations are given through the following theorem [2].

**Theorem 1:** If for $\bar{\lambda} \epsilon R^{k+1}$, an optimal solution of $P_1(\bar{\lambda})$ is found to be $\bar{x} \epsilon M$ then $\bar{x}$ is an efficient solution of (MONLP), if either $\bar{\lambda} > 0$, or $\bar{x}$ is the corresponding unique optimal solution.

**Theorem 2:** Under the convexity assumptions (M is convex, $f_i(x)$, i=1, 2,…..,k are convex on M), if $\bar{x}$ is an efficient solution of (MONLP) then there exists$\bar{\lambda} \geq 0$ such that $\bar{x}$ solves problem $P_1(\bar{\lambda})$ .

**Theorem 3:** If for feasible $Ɛ_i$ , i= 1,2,……….k, either$\bar{x} \epsilon M$ solve problem $P_2(Ɛ)$uniquely or $\bar{x} \epsilon M$ solve all the problems $P_r(Ɛ)$, r=1,2,……….k+1 for feasibleƐ, then $\bar{x}$ is an efficient solution of (MONLP) problem where problem $P_r(Ɛ)$ is defined as:

$P_2(Ɛ)$:
Min $f_r(x)$
Subject to k, $Ɛ_i \epsilon I$ } (12)
$f_i(x) \leq Ɛ_i$ , i= 1,2,……….k, $i \neq r$, $x \epsilon M$,
$Ɛ \epsilon R$

**Theorem 4:** If $\bar{x}$ is an efficient solution of (MONLP), then there exists feasible $Ɛ_i$ , i= 1,2,……….k such that$\bar{x}$ solves problem $P_2(Ɛ)$ .

Utilizing the previous theorems, the following propositions relating the basic notions of problems (1), (2) to each other are stated and their proofs could be easily deduced:

**Proposition 1:** From the definitions of the sets $O_1$ and $O_2$, we have $O_1 \subset O_2$ and $O_1 = O_2$ if the convexity assumptions are satisfied.

**Proposition 2:** If $\lambda \epsilon B_1$ then there exists a corresponding $Ɛ \epsilon B_2$ and under convexity assumptions, if $Ɛ \epsilon B_2$, then there exists a corresponding $\lambda \epsilon B_1$ .

**Proposition 3:** If $\bar{\lambda} \epsilon S_1(\bar{x})$, either $\bar{\lambda} > 0$ or $\bar{x}$ solves the problem $P_1(\bar{\lambda})$ uniquely, then there exists a corresponding $\bar{Ɛ} \epsilon S_2(\bar{x})$ and if $\bar{Ɛ} \epsilon S_2(\bar{x})$ , $\bar{x}$ solves uniquely problem $P_2(\acute{\varepsilon})$, then there exists a corresponding$\acute{\lambda} \epsilon S_1(\bar{x})$ if the convexity assumptions are satisfied.

**Proposition 4:** If $\bar{\lambda} \epsilon Q_1(\sigma(I))$, then there exists a corresponding $\bar{Ɛ} Q_2(\sigma(I))$ and if $\bar{Ɛ} \epsilon Q_2(\sigma(I))$, then there exists a corresponding $\acute{\lambda} \epsilon Q_1(\sigma(I))$, if the convexity assumptions are satisfied.

## 5. REDUCTION OF THE DIMENSIONALITY SPACE OF PROBLEM $P_1(\lambda)$

Utilizing the condition $\sum_{i=1}^{k} \lambda_i = 1$, problem (2) could be formulated in the following equivalent form $P'_1(\lambda)$:

$P'_1(\lambda) =$
$\min \left(f_{k+1}(x) + \sum_{i=1}^{k} \lambda_i \left(f_i(x) - f_{k+1}(x)\right)\right)$ } (14)
Subject to
M , $\lambda_i \geq 0$ , $\sum_{i=1}^{k} \lambda_i \leq 1$

In this case the dual parametric notions will in spaces of the same dimensionk.

**An illustrative numerical example**

Decompose the parametric spaces according to the stability sets of the first kind for the following dual parametric problems:

$P_1(\lambda)$: $\min[(1-\lambda)(3x-2y)+\lambda(x+y)]$

Subject to

$$x^2+y^2 \le 1$$

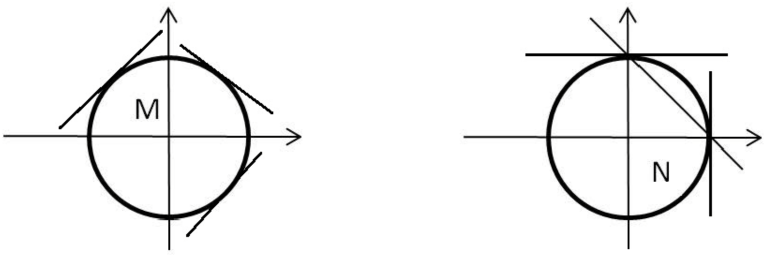
$P_2(\varepsilon)$: $\min(3x-2y)$

Subject to

$$x+y \le \varepsilon,\ x^2+y^2 \le 1$$

The two problems $P_1(\lambda)$, $P_2(\varepsilon)$ are clearly convex.

The basic notions for the two problems $P_1(\lambda)$, $P_2(\varepsilon)$ can be obtained graphically as follows:




Problem $P_1(\lambda)$ Problem $P_2(\varepsilon)$

(i) For the problem $P_2(\varepsilon)$

$F=\{\varepsilon \in R / \varepsilon \ge -\sqrt{2}\}$ and $B_2 = F$

(1) For $-\sqrt{2} \le \varepsilon \le -\frac{1}{\sqrt{13}}$, the optimal solution is $\bar{x} = \frac{\varepsilon + \sqrt{2-\varepsilon^2}}{2}$, $\bar{y} = \frac{-\varepsilon + \sqrt{2-\varepsilon^2}}{2}$

(2) For $\varepsilon \ge -\frac{1}{\sqrt{13}}$, the optimal solution is $\bar{x} = \frac{-3}{\sqrt{13}}$, $\bar{y} = \frac{2}{\sqrt{13}}$ .

This means that $S_2(\bar{x})$ is one point set for each $\varepsilon$ in the range $-\sqrt{2} \le \varepsilon \le -\frac{1}{\sqrt{13}}$.

And $S_2\left(\frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right) = \{\varepsilon \in B_2 / \varepsilon \ge -\frac{1}{\sqrt{13}}\}$.

(ii) For problem $P_1(\lambda)$, $B_1 = \{\lambda \in R / 0 \le \lambda \le 1\}$

(3) For $0 \le \lambda \le 1$, the optimal solution is $\bar{x} = \frac{-(3-2\lambda)}{\sqrt{13-24\lambda+13\lambda^2}}$, $\bar{y} = \frac{2-3\lambda}{\sqrt{13-24\lambda+13\lambda^2}}$

This means that $S_1(\bar{x})$ is one point set for each $\lambda \in B$.

(4) For example, the point $\bar{x} = \bar{y} = -\frac{1}{\sqrt{2}}$ is optimal for the two problems, it is generated from $P_1(\lambda)$ by letting $\lambda=1$, and generated from $P_2(\varepsilon)$ by letting $\varepsilon=-\sqrt{2}$ and therefore $\lambda=1$, $\varepsilon=-\sqrt{2}$ are dual parameters.

(5) Another example the point $\bar{x} = -\frac{3}{\sqrt{13}}$, $\bar{y} = -\frac{2}{\sqrt{13}}$ is optimal for the two problems, it is generated from $P_1(\lambda)$ by letting $\lambda=0$, and generated from $P_2(\varepsilon)$ by taking $\varepsilon$ any point in the set $\{\varepsilon \in B_2/\varepsilon \ge -\frac{1}{\sqrt{13}}\}$, and therefore the point $\lambda=0$ and the set $\{\varepsilon \in B_2/\varepsilon \ge -\frac{1}{\sqrt{13}}\}$ are dual parameters.

**Remark:** For the general case, the notions of the stability set of the first kind and the stability set of the second kind could be obtained using K.T. conditions.

For more details see [2], [3], [4] and [5].

## 6. CONCLUSION AND FURTHER RESEARCHES

In this paper, we introduced a procedure to solve parametric nonlinear programming problems by using the concept of duality in the parametric spaces.

The idea behind presenting the concept of dual parametric problems is to clarify the fruitful relation between the two problems and to discuss the possibility that by solving one of them, the second problem is clearly solved.

It is clear that using other scalarization methods several dual parametric problems could be easily derived.

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