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RINGS WITH (a, b, a) AND COMMUTATORS IN THE RIGHT NUCLEUS

[1]M. MANJULA DEVI*, [2]K. SUVARNA

[1,2]Department of Mathematics,
Sri Krishnadevaraya University, Anantapuramu-515003, (A.P.), India.

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ABSTRACT

In this paper we show that if R is a nonassociative simple ring satisfying (a, b, a) and (R, R) are in the right nucleus N_r, then (a, b, a) and (R, R) are in the left and middle nuclei of R. Using these properties we prove that (a, b, a) and (R, R) are in the center C of R. Also it is shown that R is commutative.

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INTRODUCTION

Thedy [2] studied rings with commutators in the nuclei. In [1] Kleinfeld obtained Thedy's hypothesis in nonassociative semiprime rings with (x, y, x) and commutators in the left nucleus. In this paper we consider a nonassociative ring R with (a, b, a) and (R, R) are in the right nucleus N_r. We prove that if R is a simple ring, then (a, b, a) and (R, R) are in the left and middle nuclei of R. Using these properties we show that (a, b, a) and (R,R) are in the center C of R. Also it is shown that R is commutative.

PRELIMINARIES

The associator is defined by $(x, y, z) = (xy)z - x(yz)$ and the commutator $(x,y) = xy - yx$ for all x, y, z in R. We define the left nucleus $N_l = \{n \in R \,/\, (n, R, R) = 0\}$, the right nucleus, $N_r = \{n \in R \,/\, (R, R, n) = 0\}$ and the middle nucleus, $N_m = \{n \in R \,/\, (R, n, R) = 0\}$.

The nucleus N of R is defined as
$N = \{n \in R \,/\, (n, R, R) = (R, n, R) = (R, R, n) = 0\}$, that is, $N = N_l \cap N_m \cap N_r$ and the center $C = \{c \in N \,/\, (c, R) = 0\}$.

Throughout this paper we consider a ring R with
(i) $(a, b, a) \subset N_r$ and (ii) $(R, R) \subset N_r$.

In every ring the following identity holds:
$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z. \quad (1)$$

If $S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y)$, then $(xy, z) + (yz, x) + (zx, y) = S(x, y, z)$.

Consequently, using (ii) we have
$$S(x, y, z) \subset N_r. \quad (2)$$

Moreover, in every ring we have the identity
$(xy, z) = x(y, z) + (x, z)y + S(x, y, z) - (x, z, y) - (y, z, x)$.

A linearization of (i) implies
$(x, z, y) + (y, z, x) \subset N_r$.

Corresponding Author: [1]M. Manjula Devi, [1,2]Department of Mathematics, Sri Krishnadevaraya University, Anantapuramu-515003, (A.P.), India.*

Then combining this with the above equation (2) and (ii), we obtain
$x(y, z) + (x, z)y \subset N_r$. (3)

Suppose that $n \in N_r$. Then with z = n in (1), we obtain
(w, x, yn) = (w, x, y)n.

Combining this with (ii), yields
(w, x, yn) = (w, x, y)n = (w, x, ny) . (4)

A combination of (3) and (4) yields
(r, s, x(y, z) + (x, z)y) = 0, that is
(r, s, x(y, z)) = - (r, s,(x, z)y). Thus
(r, s, x) (y, z) = - (r, s, y) (x, z) (5)

MAIN RESULTS

We assume that R is a simple ring. Then we know that the ideal of R is equal to zero or the ideal is equal to R. To prove the main results we consider R as a simple ring satisfying (i) and (ii). First we prove that following lemmas:

Lemma 1: If $T = \{t \in N_r / (R, R, R)\ t=0\}$, then T is an ideal of R and (R, R, R)T=0.

Proof: By substituting t for n in (4), we obtain
(w, x, ty) = (w, x, yt) = (w, x, y)t = 0.

Thus $Rt \subset N_r$ and $tR \subset N_r$. Suppose that $t \in T$ and $Z \in R$

But (1) multiplied on the right by t gives
w(x, y, z).t + (w, x, y)z.t = 0.

So (w, x, y)z.t = - w(x, y, z).t = - w.(x, y, z)t = 0.

We have (w, x, y)z.t = (w, x, y).zt.
$\therefore$ (w, x, y).zt = 0.

From (5) we get
(w, x, y) (t, z) = - (w, x, t) (y, z) = 0

This implies (w, x, y).tz = (w, x, y).zt

Using (w, x, y).zt = 0, we obtain (w, x, y).tz = 0

Thus T is an ideal of R and (R, R, R)T = 0. This completes the proof of the lemma.

Lemma 2: If R is a simple ring, then T = 0.

Proof: From lemma1 we know that T is an ideal of R.

Since R is simple, either T = 0 or T=R.

If T=R, then (R, R, R)R=0. Since R is nonassociative, this is not possible.

So, $T \neq R$. Thus T=0.

Lemma 3: If R is a simple ring, then
((a, b, a), R) = 0, (6)

((a, b),y, z) = 0, (7)

and ((a, b, a),y, z) = 0. (8)

Proof: Using (5) we see that
(x, y, z) ((a, b, a), c) = - (x, y,(a, b, a)) (z, c) = 0, because of (i).

This implies $((a, b, a), c) \subset T$.

Then using Lemma 2, we have $((a, b, a), c) = 0$,
i.e. $((a, b, a), R) = 0$.

This proves (6).

By the linearization of (i), we obtain
$(r, s,((a, b), y, x)) = (r, s,(x, y, (a, b)) = 0$.

Thus $((a, b), y, x)$ is an element of N_r. Also (1) implies
$((a, b)x, y, z) – ((a, b), xy, z) + ((a, b), x, yz) = (a, b) (x, y, z) + ((a, b), x, y)z$

By forming the associators both sides, we get
$(r, s,((a, b)x, y, z)) – (r, s, ((a, b), xy, z)) + (r, s, ((a, b), x, yz)) = (r, s, (a, b) (x, y, z)) + (r, s, ((a, b), x, y)z)$

Hence $(r, s, ((a, b) x, y, z)) = (r, s, (a, b) (x, y, z)) + (r, s, ((a, b), x, y)z)$

Since $(a, b) \subset N_r$ because of (ii) and using (4), we get
$(r, s, (a, b) (x, y, z)) = (r, s, (x, y, z)) (a, b)$ also
$(r, s, ((a, b) x, y, z)) = - (r, s, (z, y, (a, b)x)) = - (r, s, (z, y, x) (a, b) = - (r, s, (z, y, x)) (a, b)$

Thus $(r, s, ((a, b), x, y)z) = - [r, s, ((x, y, z) + (z, y, x))] (a, b)=0$, using a linearization of (i).

But then (4) implies $(r, s, z) ((a, b), x, y) = 0$.

So that $((a, b), x, y) \subset T$. Hence using Lemma 2 we obtain $((a, b), x, y) = 0$.

This proves (7).

Using the linearization of (i), we obtain
$(r, s, ((a, b, a), x, y) = - (r, s, (y, x, (a, b, a)) = 0$.

Thus $((a, b, a), x, y) \subset N_r$. Also (1) implies
$((a, b, a) x, y, z) – ((a, b, a), xy, z) + ((a, b, a), x, yz) = (a, b, a) (x, y, z) + ((a, b, a), x, y)z$.

By forming the associators both sides we get
$(r, s, (a, b, a), x, y)z) = (r, s, ((a, b, a) x, y, z)) – (r, s, (a, b, a) (x, y, z))$.

Since $(a, b, a) \subset N_r$ and using (4) we get
$- (r, s, (a, b, a) (x, y, z)) = - (r, s, (x, y, z)) (a, b, a)$ also

$$
\begin{aligned}
(r, s, (a, b, a) x, y, z) &= - (r, s, (z, y, (a, b, a)x)) \\
&= - (r, s, (z, y, x) (a, b, a)) \\
&= - (r, s, (z, y, x)) (a, b, a)
\end{aligned}
$$

Hence $(r, s, ((a, b, a), x, y) z) = - (r, s, (x, y, z) + (z, y, x)) (a, b, a) = 0$, using (i).

Then using (4) implies $(r, s, z) ((a, b, a), x, y) = 0$

So that $((a, b, a), x, y) \subset T$. Thus using Lemma 2

We obtain $((a, b, a), x, y) = 0$

This proves (8).

Theorem 1: If R is a simple ring satisfying (i) and (ii), then (a, b, a) and all commutators are in the center.

Proof: Using (5), we see that
$(x, y, z) ((a, b), c) = - (x, y, (a, b)) (z, c) = 0$, because of (ii)

This implies that $((a, b), c) \subset T$. Then by using Lemma 2 we have $((a, b), c) = 0$.
i.e. $((a, b), R) = 0$ (9)

We know that the following identity is valid in any ring:
$(xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) - (x, z, y)$. (10)

By putting z equal to a commutator (a, b), we get
$(xy, (a, b)) = x(y, (a, b) + (x, (a, b))y + (x, y, (a, b)) + ((a, b), x, y) - (x, (a, b), y)$

Using (9), (ii) and Lemma3, we get
$(x, (a, b), y) = 0$. (11)

Similarly by substituting $z = (a, b, a)$ in (10), we obtain
$(x, (a, b, a), y) = 0$. (12)

From (i), (8), (12) and (6), it follows that (a, b, a) is in the center of R.

Hence from (ii), (7), (11) and (9), it follows that all commutators are in the center.

Theorem 2: If R is a simple ring satisfying (i) and (ii), then R is commutative.

Proof: We write

$$\begin{aligned}((R, R)R, R) &= (R, R)R.R - R.(R, R)R \\ &= (R, R)R.R - R(R, R).R, \text{ using (11)} \\ &= (R, R)R.R - (R, R)R.R, \text{ using (9)} \\ &= (0).\end{aligned}$$

Thus $((R, R)R, R) = (0)$. (13)

Now

$$\begin{aligned}(R, R, R(R, R)) &= (R R).R(R, R) - R. (R.R(R, R)) \\ &= (RR).R(R, R) - R.((R R).(R, R)), \text{ using (ii)} \\ &= (RR).R(R, R) - R.((R, R).(R R)), \text{ using (9)} \\ &= (RR).R(R, R) - R(R, R).(R R), \text{ using (11)} \\ &= (RR).R(R, R) - (RR).R(R, R), \text{ by (13)} \\ &= (0).\end{aligned}$$

So $(R, R, R(R, R)) = 0$.

Since $(R, R) \subset N_r$ and using (4), we get
$(R, R, R) (R, R) = 0$

Thus $(R, R) \subset T$. Then using lemma2 we get $(R, R) = (0)$.

Hence R is commutative.

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