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THERMAL STRESS ANALYSIS DUE TO RADIATION, A NON-LINEAR APPROACH

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ABSTRACT

A challenging problem faced by the Engineers and Mathematician is to find solutions of the governing equations representing physical systems. There could be nothing more desirable than to find exact closed form solutions of these equations. However due to non-linearity occurring in most of real life engineering problems one has to adopt numerical techniques to obtain the solution. The present manuscript deals with the finite difference method for the solution of non-linear boundary value problem. The attempt made to analyze heat transfer and thermal stress analysis of thin circular plate subjected to radiation. The thin circular plate is defined in the region $0 \le r \le a, -h \le z \le h$ *under transient temperature distribution. The fixed circular boundary (* $r = a$ *) is kept insulated. The upper surface of the circular plate is subjected to heat transfer due to radiation. The rate of radiation is calculated by Stefan Boltzmann's law. At the lower boundary surface at* $z = -h$ *convection due to dissipation takes place. As a special case, the mathematical model of thermoelastic problem is constructed for aluminum plate. The results for temperature distribution, displacement and thermal stresses are illustrated graphically and interpreted technically.*

Keywords: *Transient heat conduction, Thermal stresses, Non-linear Boundary value problem, Finite difference method.*

AMS Classification: *35K05, 35K61, 65M06, 65N06.*

1. INTRODUCTION

The finite difference approximations for derivatives is one of the convenient and oldest methods to solve differential equations. It was already known by L. Euler 1768, in one dimension of space and was probably extended to dimension two by C. Runge 1908. The advent of finite difference techniques in numerical applications began in the early 1950's and their development was stimulated by the emergence of computers that offered a convenient framework for dealing with complex problems of science and technology. Theoretical results have been obtained during the last five decades regarding the accuracy, stability and convergence of the finite difference method for partial differential equations.

Roy Chaudhari [7] wrote a note on Quasi Static thermal deflection of a thin clamped circular plate due to ramp type heating of a concentric circular region of the upper face. Sheu Tony W. H. *et al.* [9] developed a two-dimensional finite-difference scheme for solving the convection- diffusion equation. Kim Chi-Kyung *et al.* [3] applied finite difference method for Thermoelastic Stress for Rectangular Thin Plate. Wang Haojie *et al.* [11] presented a finite difference method for studying thermal deformation in a thin film exposed to ultra short-pulsed lasers. Malec Marian *et al.* [4] used finite difference method for Nonlinear Parabolic-Elliptic systems of second-order Partial Differential Equations. Chu Hsin-Ping *et al.* [2] applied hybrid differential transformation-finite difference method to analyze nonlinear transient heat conduction problems. Shen Bin *et al.* [8] developed a finite difference model for heat transfer of grinding. Annor-Nyarko M. *et al.* [1] studied control volume finite difference analysis of the transient temperature distributions and associated induced thermal stresses in Ghana Research Reactor-1. Recently Verma Shubha *et al.* [10] did transient heat transfer and thermal stresses analysis in a circular plate due to radiation using finite element method.

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2. FORMULATION OF PROBLEM

Consider a thin circular plate of thickness $2h$ occupying space D defined by $0 \le r \le a$, $-h \le z \le h$. Initially the plate is kept at constant temperature T_i. The upper surface of the circular plate is subjected to heat transfer due to radiation. The rate of radiation is calculated by Stefan Boltzmann's law. At the lower boundary surface at $z = -h$ convection due to dissipation takes place. The fixed circular boundary ($r = a$) is kept insulated. Under these realistic boundary conditions the temperature change, displacement and thermal stresses are required to be determined.

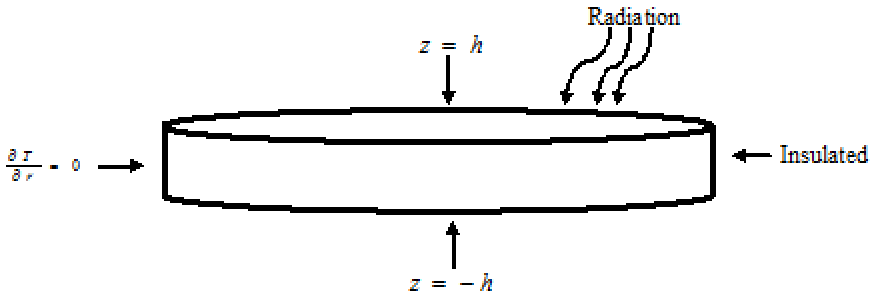

Figure-1: The geometry of heat conduction problem

Following Roy Choudhuri [7], we assume that a circular plate of small thickness *h* is in a plane state of stress. In fact, "the smaller the thickness of the circular disk compared to its diameter, the nearer to a plane state of stress is the actual state". The displacement equations of thermoelasticity have the form,

$$U_{i,kk} + \left(\frac{1+\nu}{1-\nu}\right) e,_i = 2\left(\frac{1+\nu}{1-\nu}\right) a_t T,_i \tag{1}$$

$$e = U_{k,k}; \quad k, i = 1, 2, \tag{2}$$

where,

$U_i -$ displacement component,

$e -$ dilatation,

$T -$ temperature,

and ν and a_t are respectively, the Poisson's ratio and the linear coefficient of thermal expansion of the circular plate material.

Introducing

$$U_i = \Psi_{,i}, \quad i = 1, 2,$$

One gets

$$\nabla_1^2 \Psi = (1+\nu) a_t T, \tag{3}$$

$$\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

$$\sigma_{ij} = 2\mu\left(\Psi_{,ij} - \delta_{ij}\Psi_{,kk}\right), \quad i, j, k = 1, 2, \tag{4}$$

where μ is the Lamé constant and δ_{ij} is the Kronecker symbol.

In the axially-symmetric case

$$\Psi = \Psi(r,t), \; T = T(r,z,t)$$

and the differential equation governing the displacement potential function $\Psi(r, t)$ is given as

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} = (1+\nu) a_t T, \tag{5}$$

$$\frac{\partial \Psi}{\partial r} = 0 \text{ at } r = a \text{ for all time } t. \tag{6}$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r}\frac{\partial\Psi}{\partial r} \tag{7}$$

$$\sigma_{\theta\theta} = -2\mu\frac{\partial^2\Psi}{\partial r^2} \tag{8}$$

The surfaces of the thin circular plate at $r = a$ are assumed to be traction free. The boundary condition can be taken as

$$\sigma_{rr} = 0 \text{ at } r = a\,. \tag{9}$$

Also in the plane state of stress within the circular plate

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0\,. \tag{10}$$

The one dimensional equilibrium equation in radial direction is given by

$$\frac{\partial\sigma_{rr}}{\partial r} + \left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right) = 0 \tag{11}$$

The governing heat conduction is given by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{12}$$

with the boundary conditions,

$$\frac{\partial T}{\partial r} = 0 \qquad \text{at } r = a\,, t > 0\,, \tag{13}$$

$$k\frac{\partial T}{\partial z} + q_b = \sigma\varepsilon\left(T^4 - \theta^4\right) \qquad \text{at } z = h,\ t > 0\,, \tag{14}$$

$$\frac{\partial T}{\partial z} - h_s T = 0 \qquad \text{at } z = -h,\ t > 0 \tag{15}$$

and the initial condition

$$T = T_i \qquad \text{at } t = 0\,, 0 \le r \le a\,. \tag{16}$$

where

k thermal conductivity ,
α thermal diffusivity ,
θ temperature of surrounding media,
σ Stefans-Boltzmans constant ,
ε emissivity of surface,
q_b heat generation per unit area per time on the boundary surface,
μ Lamé constant,
h_s heat transfer coefficient .

Equations (1) to (16) constitute the mathematical formulation of the problem.

3. SOLUTION OF PROBLEM

3.1 Heat Transfer Analysis

The finite difference method is applied [6] to solve the boundary value problem defined by (12) to (16). One can divide the $r,\ z,\ t$ domain into small intervals $\Delta r,\ \Delta z,\ \Delta t$ such that

$$r = i\Delta r \quad i = 0,1,.....N \ (N\Delta r = a)$$
$$z = j\Delta z \quad j = 0,1,2,........\text{M} \ (M\Delta z = 2h)$$
$$t = n\Delta t \quad n = 0,1,2,........$$

The temperature at the nodal point $(i\Delta r,\ \text{j}\Delta z)$ at the time $n.\Delta t$ is denoted by

$$T(i.\Delta r,\ j.\Delta z) = T_{i,\,j}^{n}$$

Using the forward difference in time domain, time derivative of temperature is given as

$$\frac{\partial T}{\partial t} = \frac{T_{i,\,j}^{n+1} - T_{i,\,j}^{n}}{\Delta t}$$

Finite difference expressions for the partial derivatives with respective to the space variables are given as,

$$\frac{\partial T}{\partial r} = \frac{T_{i+1,\,j}^{n} - T_{i-1,\,j}^{n}}{2\Delta r}$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{T_{i-1,\,j}^{n} + T_{i+1,\,j}^{n} - 2T_{i,\,j}^{n}}{(\Delta r)^2}$$

and

$$\frac{\partial^2 T}{\partial z^2} = \frac{T_{i,\,j-1}^{n} + T_{i,\,j+1}^{n} - 2T_{i,\,j}^{n}}{(\Delta z)^2}$$

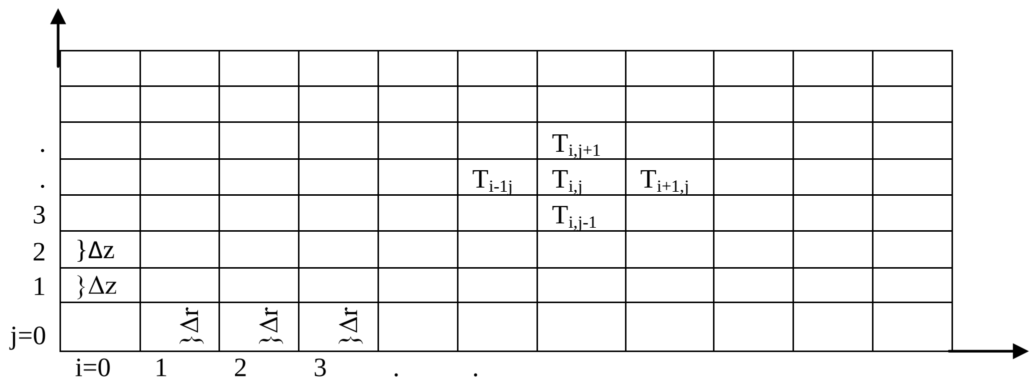

Figure-2: A (r, z) network in the cylindrical coordinate system

For square grids $\Delta r = \Delta z$, substituting these values in the two dimensional heat equation, one gets the recursive relation as,

$$T_{i,\,j}^{n+1} = (1-4\lambda)T_{i,\,j}^{n} + \lambda(1+\frac{1}{2i})T_{i+1,\,j}^{n} + \lambda(1+\frac{1}{2i})T_{i-1,\,j}^{n} + \lambda T_{i,\,j-1}^{n} + \lambda T_{i,\,j+1}^{n}$$

$$i = 1,2,....., \ j = 1,2,..........., n = 0,1,2,..... \qquad (17)$$

where, $\lambda = \dfrac{\alpha\Delta t}{(\Delta r)^2}$.

At $r = 0$, (12) becomes

$$\lim_{r\to 0}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) = \left(2\frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2}\right)_{(0,\,j)}$$

One get the scheme (17) for $r = 0$,

$$T_{0,j}^{n+1} = (1-6\lambda)T_{0,j}^{n} + 4\lambda T_{1,j}^{n} + \lambda T_{0,j-1}^{n} + \lambda T_{0,j+1}^{n} \tag{18}$$

Initially at $t = 0$

$$T_{i,j}^{0} = T_i \tag{19}$$

at lower boundary surface ($z = -h$)

$$T_{i,j-1}^{n} = T_{i,j+1}^{n} - 2\lambda \Delta z T_{i,j}^{n} \tag{20}$$

at upper boundary surface ($z = h$) the boundary condition is not linear function in temperature, so this condition offers some difficulty during the solution of the equation. If the difference between the temperature T and θ is not large, it may be approximated to linear form. Thus

$$k\frac{\partial T}{\partial z} + q_b = \sigma\varepsilon\left(T^4 - \theta^4\right)$$

is approximated to

$$k\frac{\partial T}{\partial z} + q_b = 4\sigma\varepsilon\theta^3\left(T - \theta\right)$$

The finite difference approximation is

$$T_{i,j+1}^{n} = T_{i,j-1}^{n} - \frac{2\Delta z q_b}{k} + \frac{2\Delta z \varepsilon\sigma\theta^3}{k}\left(T_{i,j}^{n} - \theta\right) \tag{21}$$

at the fixed circular boundary ($r = a$)

$$T_{i,j}^{n} = T_{i,j-1}^{n} \tag{22}$$

Equation (19) gives the initial temperature at each grid point of the plate (at $t = 0$).

Equation (18) gives the temperature at $r = 0$, Equation (17) gives the temperature at each internal node and equations (20) to (22) gives the temperature at all the boundary points.

The MATLAB Programming has been used to determine the temperature at all the nodal points for different time intervals.

3.2 Convergence and Stability Analysis

Theorem 1: According to Thomas J.W. [12], the finite difference scheme equation (17) and equation (18) are stable for

$$0 \le \lambda = \frac{\alpha \Delta t}{(\Delta r)^2} \le 1/6$$

and

Theorem 2: Due to Lax [12], a consistent difference scheme for a well posed linear initial boundary value problem is convergent if and only if it is stable.

The above theorems have been used for the stability and convergence of solution obtained.

3.3 Thermal Stress Analysis

The finite difference scheme for equation (5) and (6) is given by

$$(1+\frac{1}{2i})\Psi_{i+1,j} + (1-\frac{1}{2i})\Psi_{i-1,j} - 2\Psi_{i,j} = (1+\nu)a_t T_{i,j}^{n}$$

$$i = 1, 2, \ldots.., \; j = 1, 2, \ldots\ldots\ldots \tag{23}$$

At $r = 0$, (6) becomes

$$\lim_{r \to 0}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\right) = \left(2\frac{\partial^2\Psi}{\partial r^2}\right)_{(0,\, j)}$$

One get the difference scheme (16) for $r = 0$,

$$4\frac{\Psi_{1,\, j} - \Psi_{0,\, j}}{\Delta r^2} = (1+\nu)a_t T^n_{0,\, j} \tag{24}$$

at the circular boundary ($r = a$)

$$\Psi_{i+1,\, j} = \Psi_{i-1,\, j} \tag{25}$$

The set of simultaneous linear equations formed by (23) to (25) gives the displacement at all the nodal points. The coefficient matrix for these simultaneous linear equations is irreducible and diagonally dominant for at least one row hence it is invertible, one gets unique solution.

The finite difference scheme for equation (8) and (9) is given by

$$\left(\sigma_{rr}\right)_{i,\, j} = \frac{-\mu}{i\Delta r^2}\left(\Psi_{i+1,\, j} - \Psi_{i-1,\, j}\right) \tag{26}$$

and

$$\left(\sigma_{\theta\theta}\right)_{i,\, j} = \frac{-2\mu}{i\Delta r^2}\left(\Psi_{i+1,\, j} + \Psi_{i-1,\, j} - 2\Psi_{i,\, j}\right) \tag{27}$$

At $r = 0$

$$\left(\sigma_{rr}\right)_{0,\, j} = -4\mu\frac{\Psi_{1,\, j} - \Psi_{0,\, j}}{\Delta r^2} \tag{28}$$

The finite difference scheme (26) to (28) gives the thermal stresses at all the nodal points.

4. NUMERICAL CALCULATIONS

The plate is divided in equal grids of $\Delta r = \Delta z$ =0.025 meters. Fifty iterations has been performed for each time step of Δt =1.11368 seconds. For convergence and stability we have used λ=0.15.

The simultaneous equations formed by equation (23) to (25) are solved by using MATLAB programming up to 20 iterations.

4.1 Dimensions

Radius of a thin circular plate $a = 1m,$

Height of circular plate $2h = 0.1m$.

4.2 Material properties

The numerical calculation has been carried out for an Aluminum (Pure) circular disc with the material properties as,
Thermal conductivity k =204.2 *W/mK*,

Thermal diffusivity $\alpha = 84.18 m^2 / s$,

Poisson ratio $\nu = 0.35$,

Coefficient of linear thermal expansion $a_t = 22.2 \times 10^{-6}\frac{1}{K}$,

Lamé constant $\mu = 26.67$,

Heat generation per unit area per time on the boundary surface $q_b = \sigma\varepsilon T^4 = 353.6379 W / m^2 s$

Temperature of surrounding media $\theta = 300K$

Stefans-Boltzman's constant $\sigma = 5.67\text{x}10^{-8} W / m^2 K^4$,

Emissivity of surface $\varepsilon = 0.77$.

Temperature, Displacement and Stresses in radial and axial direction at grid points equally spaced with $\Delta r = \Delta z$ =0.025 meters and at time t = 50x1.11368=50.6842 seconds.

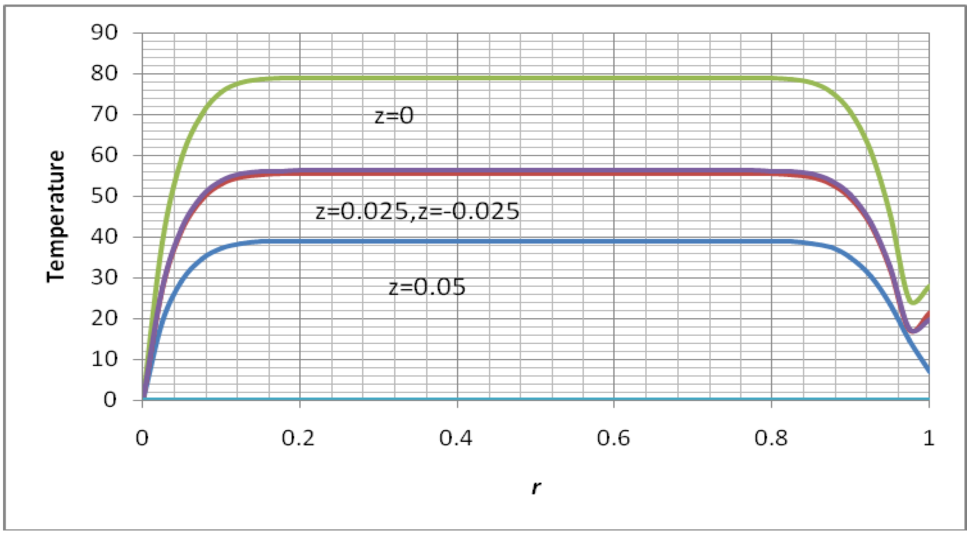

Figure-3: Temperature T disrtibuion in radial direction

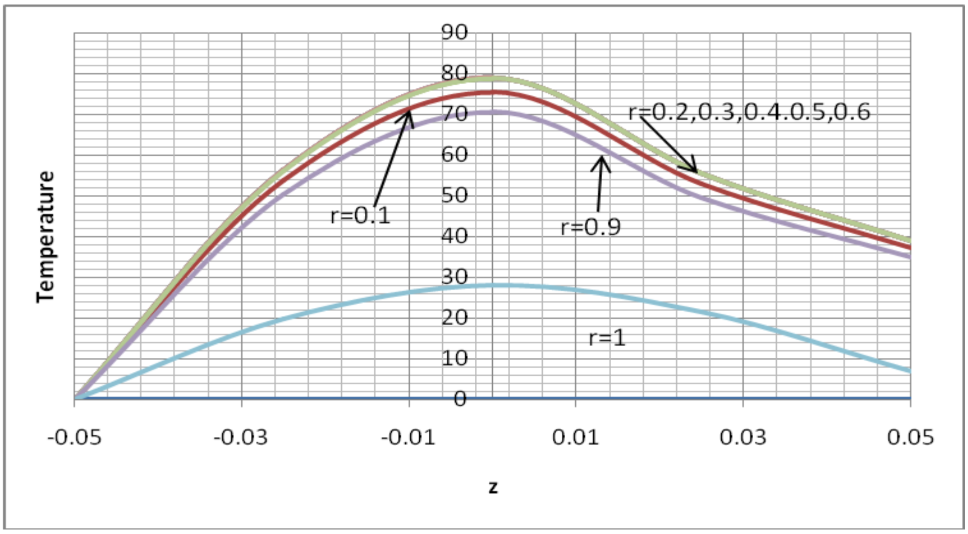

Figure-4: Temperature T disrtibuion in axial direction

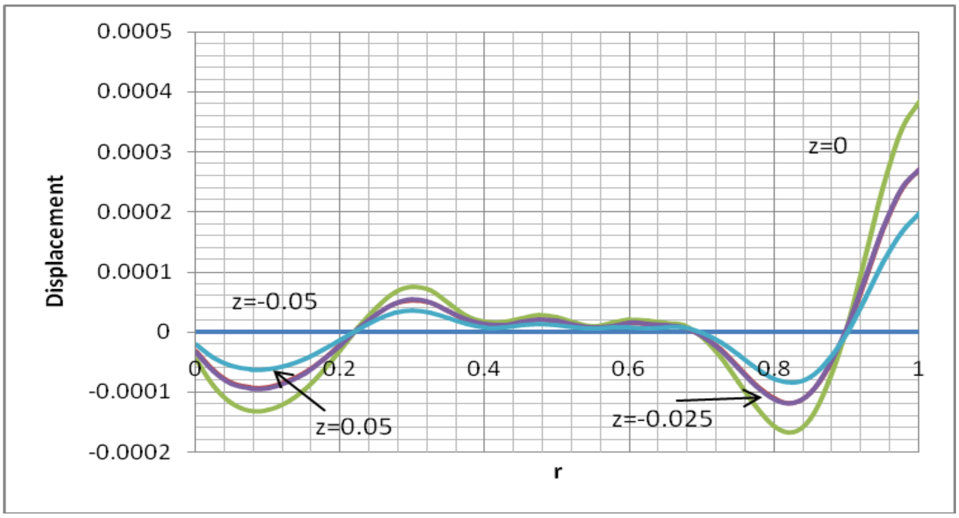

Figure-5: Displacement Ψ in radial direction

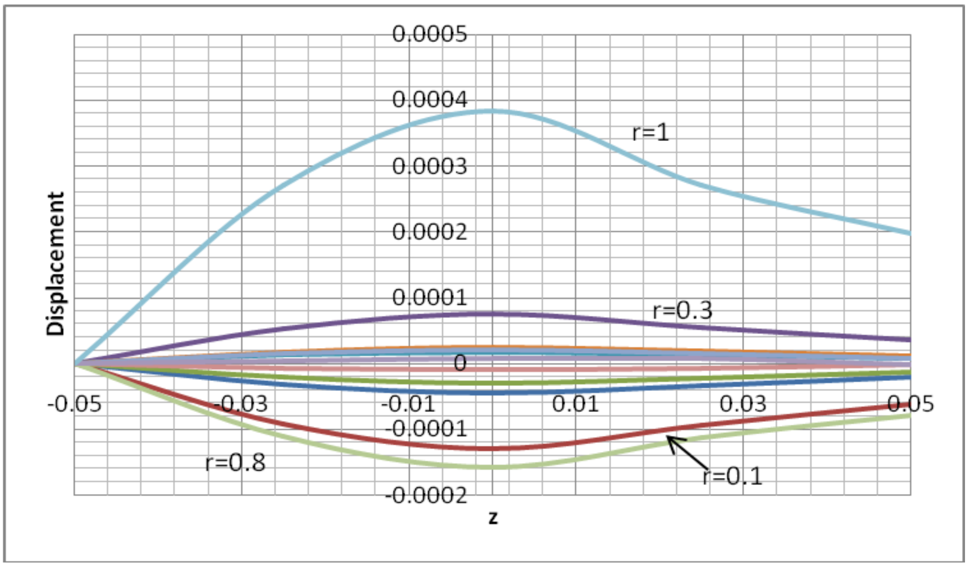

Figure-6: Displacement Ψ in axial direction

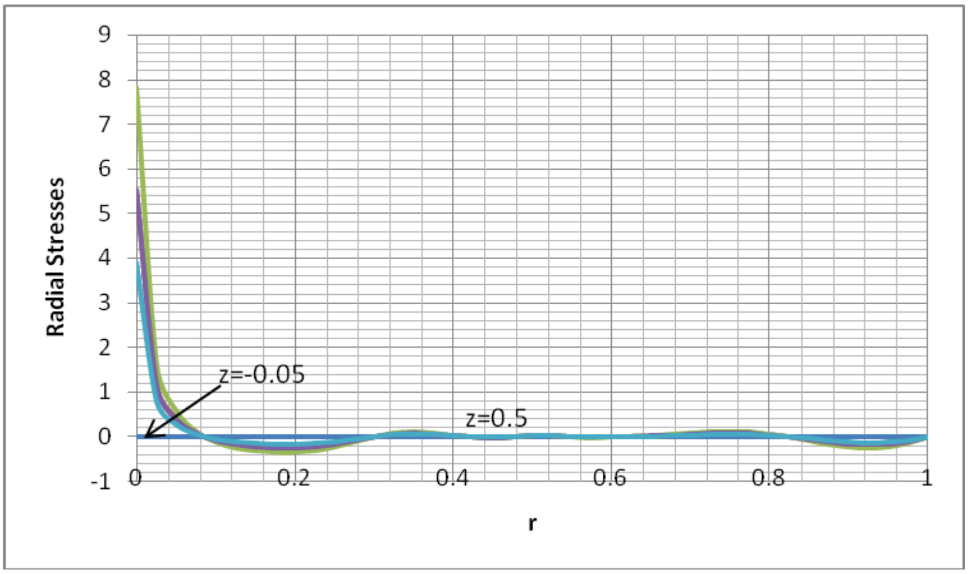

Figure-7: Thermal stresses σ_{rr} in radial direction

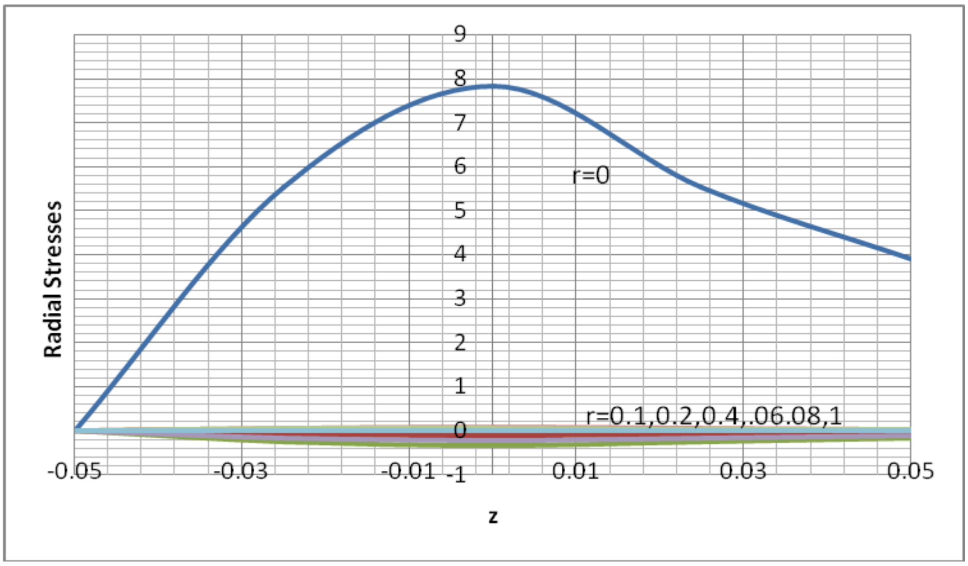

Figure-8: Thermal stresses σ_{rr} in axial direction

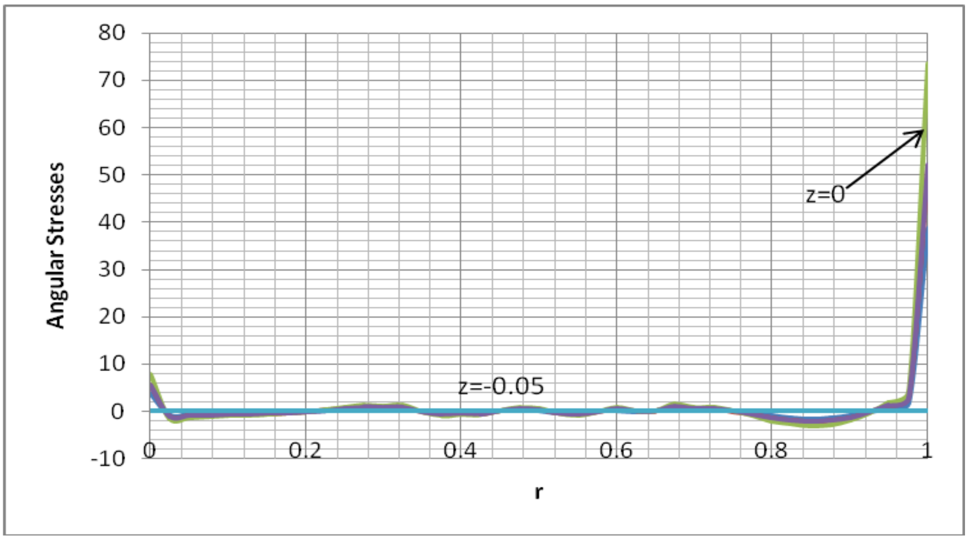

Figure-9: Thermal stresses $\sigma_{\theta\theta}$ in radial direction

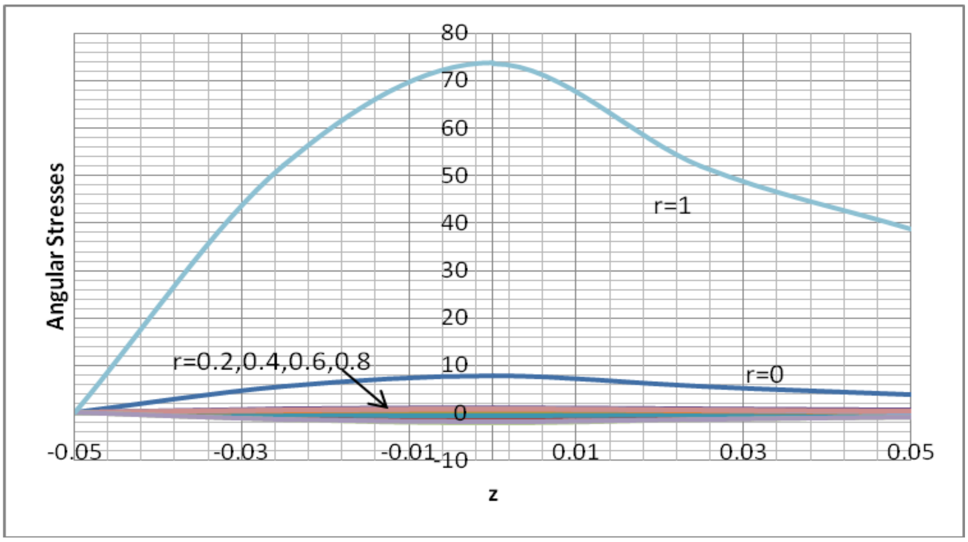

Figure-10: Thermal stresses $\sigma_{\theta\theta}$ in axial direction

5. CONCLUDING REMARKS

In this manuscript a thin circular plate is considered which is free from traction. The thermoelastic behavior due to radiation under transient temperature conditions is discussed. As a special case mathematical model is constructed for thin aluminum plate and performed numerical calculations for temperature change, displacement and thermal stresses in radial direction. In plane state the thermal stresses along axial direction is zero.

For the solution non-linear boundary value problem in thermoelsaticity, the finite difference method has been used. For better accuracy of results the more number of iterations are performed.

From figure 3 and 4, the temperature variation can be seen along radial and axial direction of circular plate. The increase in the temperature can be observed from centre $r = 0$ to the outer circular boundary $r = 1$, whereas dissipation due to convection takes place at $z = -0.05$.

From figure 5 and 6, it appears that the radial displacement function u_r increases from upper surface to lower surface in axial direction where as in radial direction it increases from centre $r = 0$ to outer circular boundary $r = 1$.

From figure 7 and 8, it can be seen that the radial stress function σ_{rr} develops tensile stresses around the center $r = 0$ and it decreases towards the outer circular boundary $r = 1$. On the traction free surface $r = 1$ the radial stress function vanishes. In the axial direction, σ_{rr} increases from upper surface $z = 0.05$ to lower surface $z = -0.05$. .

From figure 9 and 10, the stress function $\sigma_{\theta\theta}$ decreases from lower surface to upper surface where as it develops tensile stresses at outer circular boundary $r=1.$.

One observed that, the process of heat transfer due to radiation takes place from upper surface to lower surface of circular plate. The displacement and stress components occur on the radiated surface. Radial stresses develop tensile stresses at the centre where as angular stress develops tensile stresses at outer circular boundary. From figures of radial and axial displacements, it may be concluded that due to radiation applied on upper surface under transient temperature conditions, the circular plate expands in the axial direction.

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