International Journal of Mathematical Archive-6(12), 2015, 134-138
Available online through www.ijma.info ISSN 2229 – 5046

ORTHOGONALITY OF JORDAN LEFT DERIVATIONS AND JORDAN LEFT BIDERIVATIONS IN SEMIPRIME RINGS

[1]K. SANKARA NAIK*, [2]K. SUVARNA

[2]Department of Mathematics, Sri Krishnadevaraya University,
Anantapuramu-515003, (A.P.), India.

(Received On: 30-11-15; Revised & Accepted On: 25-12-15)

ABSTRACT

This paper gives the notion of orthogonality between the Jordan left derivation and Jordan left biderivation of a semiprime ring. We prove that if R is a 2-tiorsion free semiprime ring, d is a Jordan left derivation and B is a Jordan left biderivation on R, then d and B are orthogonal if and only if any one of the following equivalent conditions holds for every $x, y \in R$:

(i) $B(x,y)d(z)+d(x)B(z,y)=0$

(ii) $d(x)B(x,y)=0$ *or* $d(x)B(y,x)=0$

(iii) $dB=0$ *(iv) dB is a left biderivation.*

***Mathematical Subject Classification**: 16N60, 16W25.*

***Key Words**: Semiprime ring, Derivation, Biderivation, Orthogonal, Jordan derivation, Jordan left derivation, Jordan left biderivation.*

INTRODUCTION

Bresar and Vukman [2], introduced the notion of orthogonality for a pair d and g of derivations on a semiprime ring and they have proved several necessary and sufficient conditions for d and g to be orthogonal. Daif. *et.al.* [4], studied the orthogonality between the derivation and biderivation of a ring and also in terms of a nonzero ideal of a 2-torsion free semiprime ring. In this paper, we give four conditions equivalent to the notion of orthogonality between the Jordan left derivation and Jordan left biderivation of a semiprime ring. It is shown that if *R* is a 2-torsion free semiprime ring, *d* is a Jordan left derivation and *B* is a Jordan left biderivation on *R*, then *d* and *B* are orthogonal if and only if one of the following equivalent conditions holds for every $x, y \in R$:

(i) $B(x,y)d(z)+d(x)B(x,y)=0$

(ii) $d(x)B(x,y)=0$ or $d(x)B(y,x)=0$

(iii) $dB=0$ (iv) *dB* is a left biderivation.

PRELIMINARIES

Throughout this paper *R* will be an associative ring. A ring *R* is said to be 2-torsion-free if $2x = 0$, $x \in R$ implies $x = 0$. *R* is called prime if $xRy = 0$ implies $x = 0$ or $y = 0$, and R is semiprime if $xRx = 0$ implies $x = 0$ for all $x, y \in R$.

We write the usual commutator $[x, y] = xy - yx$ for all $x, y \in R$, and we use the basic commutator identities $[x, yz] = [x, y]z+y[x, z]$ and $[xz, y] = [x, y]z+x[z, y]$.

*Corresponding Author: [1]K. Sankara Naik**
[2]Department of Mathematics, Sri Krishnadevaraya University,
Anantapuramu-515003, A.P., India

An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for every $x, y \in R$. Let R be a semiprime ring, two derivations d and g of R are called orthogonal if $d(x)Rg(y) = 0 = g(y)Rd(x)$ [2]. Following Daif.*et.al.* [4], a biadditive map $B: R\times R \to R$ is called a biderivation of R if $B(xy, z) = B(x, z)y + xB(y, z)$ for all $x,y,z \in R$. For a ring R, a biadditive mapping $B: R\times R \to R$ is called a left biderivation if $B(xy, z) = xB(y, z) + yB(x, z)$ for all $x, y, z \in R$. An additive mapping $d: R \to R$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for every $x \in R$. An additive mapping $d: R \to R$ is called a Jordan left derivation if $d(x^2) = 2xd(x)$ for every $x \in R$. In the same way, an additive mapping $B: R\times R \to R$ is called a Jordan left biderivation if $B(x^2,y) = 2xB(x, y)$ for all $x, y \in R$. A Jordan left derivation d and Jordan left biderivation B of R are called orthogonal if $B(x,y)Rd(z) = 0 = d(z)RB(x,y)$ for all $x, y, z \in R$.

We now consider some well known results that will be needed in the subsequent results.

Lemma 1: [[2], Lemma 1] Let R be a 2-torsion free semiprime ring and $a, b \in R$.Then the following are equivalent :

- $axb = 0$ for all $x \in R$
- $bxa = 0$ for all $x \in R$
- $axb + bxa = 0$ for all $x \in R$

If one of the above conditions is fulfilled, then $ab = ba = 0,$ too.

Lemma 2: [[4], Lemma 2.2] Let R be a semiprime ring. Suppose that an additive mapping h on R and a biadditve mapping $f : R\times R \to R$ satisfy $f(x, y)Rh(x) = (0)$, then $f(x, y)Rh(z) = (0)$ for all $x, y, z \in R$.

Lemma 3: Let d be a Jordan left derivation and B a Jordan left biderivation of a semiprime ring R. The following identity holds, for all $x, y, z \in R$.

$$(dB)(xy, z) = y(dB)(x, z) + x(dB)B(y, z) + B(y, z)d(x) + B(x, z)d(y).$$

Proof: Let d and B such that $(dB)(xy, z) = d(B(xy, z))$, for all $x, y, z \in R$.

$(dB)(xy, z) = d(xB(y, z) + y(B(y, z))$, for all $x, y, z \in R$ .we get

$(dB)(xy, z) = B(y, z)d(x) + x(dB)(y, z) + y(dB)(x, y) + B(x, z)d(y)$, for all $x, y, z \in R$. Thus

$(dB)(xy, z) = y(dB)(x, z) + x(dB)B(y, z) + B(y, z)d(x) + B(x, z)d(y)$, for all $x, y, z \in R$.

MAIN RESULTS

In this section we prove the main results. The above lemmas are useful to prove the following theorem.

Theorem 1: Let R be a 2-torsion free semiprime ring. A Jordan left derivation d and a Jordan left biderivation B are orthogonal if and only if $B(x, y)d(z) + d(x)B(z, y) = 0$, for all $x, y, \in R$.

Proof: Suppose d and B are such that $B(x, y)d(z) + d(x)B(z, y) = 0$, for all $x, y, z \in R$. By taking $z = zx$ in this equation, we get

$B(x, y)d(zx) + d(x)B(zx, y) = 0$. Then

$B(x, y)zd(x) + B(x, y)xd(z) + d(x)zB(x, y) + d(x)xB(z, y) = 0$, for all $x, y, z \in R$.

Then $d(x)zB(x, y) + d(x)xB(z, y) = 0$, according to lemma 2.

In particular $d(x)zB(x, y) = -d(x)xB(z, y) = 0$, for all $x, y, z \in R$.

By left multiplying this equation with $d(x)zB(x, y)$, we have

$d(x)zB(x, y)Rd(x)zB(x, y) = -d(x)zB(x, y)Rd(x)xB(z, y)$, then

$d(x)zB(x, y)Rd(x)zB(x, y) = 0$.

Since R is semiprime, we have

$d(x)zB(x, y) = 0$, for all $x, y, z \in R$.

$d(x)RB(x, y) = 0$, for all $x, y, z \in R$.

Hence by lemma 2, we get
$d(x)RB(z,y)=0$, for all $x,y,z\in R$. Using again lemma 2 in the last equation, we get $d(x)RB(z,y)=(0)=B(z,y)Rd(x)$. So *d* and *B* are orthogonal. If *d* and *B* are orthogonal then $d(x)B(z,y)=0=B(x,y)d(z)$, by lemma 2.

Thus $d(x)B(z,y)+B(x,y)d(z)=0$.

Theorem 2: Let *R* be a 2-torsion free semiprime ring. A Jordan left derivation *d* and a Jordan left biderivation *B* are orthogonal if and only if $d(x)B(x,y)=0$ or $d(x)B(y,x)=0$ for all $x,y,\in R$.

Proof: We assume *d* and *B*, such that
$d(x)B(x,y)=0$ for all $x,y,\in R$. (1)

A linearization of *x* , gives
$d(x+z)B(x+z,y)=0$. for all $x,y,z\in R$.We have
$(d(x)+d(z))B(x+z,y)=0$.Then
$d(x)B(x,y)+d(z)B(x,y)+d(x)B(z,y)+d(z)B(z,y)=0$.

By equation (1), we get
$d(z)B(x,y)+d(x)B(z,y)=0$, for all $x,y,z\in R$. (2)

Taking $z=zs$ in equation (2), give
$d(zs)B(x,y)+d(x)B(zs,y)=0$ for all $x,y,z,s\in R$.
$d(x)zB(s,y)+d(x)sB(z,y)+zd(s)B(x,y)+sd(z)B(x,y)=0$, $\forall$ $x,y,z,s\in R$. (3)

Let $d(x)sB(z,y)=-d(z)sB(x,y)$ and $d(x)zB(s,y)=-d(s)zB(x,y)$.

So equation (3) becomes
$d(x)zB(s,y)-zd(x)B(s,y)-d(z)sB(x,y)+sd(z)B(x,y)=0$ $\forall$ $x,y,z,s\in R$. (4)

We replace *z* by *d(x)* in equation (4). Then
$d^2(x)B(s,y)-d^2(x)B(s,y)-d^2(x)sB(x,y)+sd^2(x)B(x,y)=0$ for all $x,y,z,s\in R$ (5)

Then we have $d^2(x)sB(x,y)=0$. (6)

By right multiplying (6) with *w*, we have

$d^2(x)sB(x,y)w=0$, for all $x,y,s,w\in R$. (7)

By taking $s=sw$ in (6) we get
$d^2(x)swB(x,y)=0$, for all $x,y,s,w\in R$ (8)

From equations (7) and (8) we have
$d^2(x)sB(x,y)w-d^2(x)swB(x,y)=0$, for all $x,y,s,w\in R$.

Then $d^2(x)s[w,B(x,y)]=0$, for all $x,y,s,w\in R$.

So $d^2(x)R[w,B(m,y)]=0$, for all $x,y,m,w\in R$. (9)

Put $x = xu$ in equation (9), we get

$d^2(xu)R[w, B(m, y)] = 0$ for all $x, y, m, w, u \in R$.

$(ud^2(x) + 2d(x)d(u) + xd^2(u))R[w, B(m, y)] = 0$, then

$2d(x)d(u)R[w, B(m, y)] = 0$ for all $x, y, m, w, u \in R$.

Since *R* is 2-torsion free semiprime, we have

$d(x)d(u)R[w, B(m, y)] = 0$ for all $x, y, m, w, u \in R$. (10)

Let $d(u) = zd(u)$ in equation (10), we get

$d(x)zd(u)R[w, B(m, y)] = 0$ for all $x, y, m, w, u \in R$.

$d(x)Rd(u)R[w, B(m, y)] = 0$ for all $x, y, m, w, u \in R$.

In particular $d(x)R[w, B(m, y)]Rd(x)R[w, B(m, y)] = 0$.

Since *R* is semiprime ring, it implies that $d(x)R[w, B(m, y)] = 0$, for all $x, y, m, w, \in R$.

But $[d(x), B(m, y)]R[d(x), B(m, y) = 0$ for all $x, y, m, \in R$.

$[d(x), B(m, y)] = 0$ for all $x, y, m, \in R$.

Hence $d(x)B(m, y) = B(m, y)d(x)$ for each $x, y, m, \in R$.

Therefore equation (2) can be written as

$B(m, y)d(x) + d(m)B(x, y) = 0$ for all $x, y, m, \in R$. Thus, using theorem 1, gives the required result. Similarly, we can prove that if $d(x)B(y, x) = 0$, then *d* and *B* are orthogonal. If *d* and *B* are orthogonal, then $d(x)RB(x, y) = (0)$ for all $x, y, \in R$, therefore $d(x)B(x, y) = (0)$. Similarly $d(x)B(y, x) = 0$.

Theorem 3: Let *R* be a 2-torsion free semiprime ring. A Jordan left derivation *d* and a Jordan left biderivation *B* are orthogonal if and only if *dB=0.*

Proof: We assume *B* and *d*, such that $dB = 0$. By lemma 3, we have

$(dB)(xy, z) = y(dB)(x, z) + x(dB)B(y, z) + B(y, z)d(x) + B(x, z)d(y)$, we get

$B(y, z)d(x) + B(x, z)d(y) = 0$. Now put $y = x$ in the above equation. Then $2B(x, z)d(x) = 0$. Since R is a 2-torsion free semiprime ring,

$B(x, z)d(x) = 0$ for all $x, z \in R$ (11)

Let $d(x) = yd(x)$ in the equation (11) Then we get

$B(x, z)yd(x) = 0$ for all $x, y, z \in R$. (12)

By multiplying left side with $d(x)$ and right side with $B(x, z)$ in the above relation, we have

$d(x)B(x, z)yd(x)B(x, z) = 0$, for all $x, y, z \in R$.

$d(x)B(x, z)Rd(x)B(x, z) = (0)$, for all $x, z, \in R$. (13)

Since R is a semiprime ring, then $d(x)B(x, z) = 0$, for all $x, z \in R$. (14)

Hence by theorem 2, *d* and *B* are orthogonal.

If d and B are orthogonal then $d(x)sB(y, z) = 0$, for all $x, y, s, z, \in R$. Hence

$d(d(x)sB(y, z)) = d(d(x))sB(y, z) + d(x)d(s)B(y, z) + d(x)s(dB)(y, z) = 0$.

The sum of the first two terms is zero. So we have
$d(x)s(dB)(y,z)=0$, for all $x,y,s,z,\in R$. (15)

Let $x=B(y,z)$ and we substitute in equation (15). Then we get
$(dB)(y,z)R(dB)(y,z)=(0)$, for all $y,z\in R$.

Since R is a semiprime ring, $(dB)(y,z)=0$ for all $y,z\in R$,
Hence $dB=0$.

Theorem 4: Let *R* be a 2-torsion free semiprime ring. A Jordan left derivation *d* and a Jordan left biderivation *B* are orthogonal if and only if dB is a left biderivation.

Proof: Let *B* and *d* be such that dB is a biderivation.
Then $(dB)(xy,z)=y(dB)(x,z)+x(dB)(y,z)$ for all $x,y,z\in R$. (16)

But by lemma 3, we have
$(dB)(xy,z)=y(dB)(x,z)+x(dB)B(y,z)+B(y,z)d(x)+B(x,z)d(y)=0$,
for all $x,y,z\in R$. (17)

From equation (16) and (17), we get
$B(y,z)d(x)+B(x,z)d(y)=0$ for all $x,y,z\in R$. (18)
So by the proof of the first part of theorem 3, we have that *d* and *B* are orthogonal.

Conversely, let *d* and *B* are orthogonal. Theorem 2 implies that
$d(x)B(x,z)=0$ for $x,y,z\in R$. (19)

Again, by lemma 3, we get
$(dB)(xy,z)=y(dB)(x,z)+x(dB)B(y,z)=0$ for each $x,y,z\in R$.

It is clear now that dB is a left biderivation.

Theorem 5: Assume that R is a 2-torsion free semiprime ring. A Jordan left derivation *d* and a Jordan left biderivation *B* on *R*. Then *d* and *B* are orthogonal if and only if the following conditions are equivalent:
(i) $B(x,y)d(z)+d(x)B(x,y)=0$. For all $x,y,z\in R$.
(ii) $d(x)B(x,y)=0$ or $d(x)B(y,x)=0$, for all $x,y,\in R$.
(iii) $dB=0$
(iv) dB is a left biderivation.

Proof: It follows easily from, theorem 1, 2, 3 and 4.

REFERENCES

1. Asharaf, M. and Rehman. N., "On lie ideals and Jordan left derivations of prime rings". Archivum mathematicum, vol.36 (2000), No.3, 201-206.
2. Bresar, M. and Vukman.J., 1989, "Orthogonal derivations and an extension of a theorem of posner", Radovi Mathematicki,5, pp.237-246.
3. Daif, M.N., El-Sayiad, M.S.T. and Haetinger, C., "Reverse, Jordan and Left Biderivations", Oriental Journal Of Mathematics 2(2) (2010), pp. 65-81.
4. Daif, M.N., Tammam, M.S., El-Sayiad, M.S.T. and Haetinger,C., "Orthogonal derivations and biderivations" JMI International Journal of Mathematical Sciences, Vol.1, No.1, January-June 2010,pp.23-34.

Source of support: Nil, Conflict of interest: None Declared

