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AN EXTENSION OF BANACH CONTRACTION PRINCIPLE THROUGH RATIONAL EXPRESSION

GEETA MODI[1], PRIYANKA TYAGI*[2]

[1]Head of the Department of Mathematics, Govt. MVM, Bhopal – (M.P.), India.

[2] Research Scholar, Barkatullah University, Bhopal – (M.P.), India.

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ABSTRACT

In This Paper we prove An Extension of Banach contraction principle through rational expression satisfying Three continuous mappings. Some result with S. banach (1922). Our Result include the well known result of R.Kannan(1968), L.B.Ciric (1971), B.Fisher (1971),K.Isoki, S.S.Rajput and P.L.sharma(1982) and include the result of An Extension of Banach contraction principle (1988) as special case with a different and constructive method.

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MAIN RESULT

Theorem 1.1: Let E, F and T are three continuous mappings of a Banach space satisfying the following conditions:

ET = TE, FT = TF, E(X) ⊂ T(X) and F(X) ⊂ T(X). (1.1)

$$\|(Ex - Fy)\| \le \alpha \frac{\|Ty-Fy\|\,[1+\|Tx-Ex\|]}{1+\|Tx-Ty\|} + \beta\,[\|Tx - Ex\| + \|Ty - Fy\|] + \gamma \begin{bmatrix} \|Tx - Fy\| \\ +\|Ty - Ex\| \end{bmatrix} + \delta\|Tx - Ty\| \qquad (1.2)$$

For all x, y in X where $\alpha, \beta, \gamma, \delta \ge 0, \alpha + 2\beta + 2\gamma + \delta < 1.$ then E, F and T have a common fixed point in X.

Proof: Let x_0 be an arbitrary element of X and let $\{Tx_n\}$ be defined as

$Tx_{2n+1} = Ex_{2n}$, $Tx_{2n+2} = Fx_{2n+1}$ for n- 1, 2, 3, 4…… (1.3)

We can do this since E(X) ⊂ T(X) and F(X) ⊂ T(X).

From (1.2) we have

$\|Tx_{2n+1} - Tx_{2n+2}\| = \|Ex_{2n} - Fx_{2n+1}\|$

$$\le \alpha \frac{\|Tx_{2n+1}-Tx_{2n+2}\|[1+\|Tx_{2n}-Tx_{2n+1}\|]}{[1+\|Tx_{2n}-Tx_{2n+1}\|]} + \beta[\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] + \gamma[\|Tx_{2n} - Tx_{2n+2}\| + \|Tx_{2n+1} - Tx_{2n+1}\|] + \delta\|Tx_{2n} - Tx_{2n+1}\|$$

$$\le \alpha \frac{\|Tx_{2n+1}-Tx_{2n+2}\|[1+\|Tx_{2n}-Tx_{2n+1}\|]}{[1+\|Tx_{2n}-Tx_{2n+1}\|]} + \beta[\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] + \gamma[\|Tx_{2n} - Tx_{2n+2}\| + 0] + \delta\|Tx_{2n} - Tx_{2n+1}\|$$

$$\le \alpha \frac{\|Tx_{2n+1}-Tx_{2n+2}\|[1+\|Tx_{2n}-Tx_{2n+1}\|]}{[1+\|Tx_{2n}-Tx_{2n+1}\|]} + \beta[\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] + \gamma[\|Tx_{2n} - Tx_{2n+2}\| + \|Tx_{2n+1} - Tx_{2n+1}\|] + \delta\|Tx_{2n} - Tx_{2n+1}\|$$

$$\le \alpha \frac{\|Tx_{2n+1}-Tx_{2n+2}\|[1+\|Tx_{2n}-Tx_{2n+1}\|]}{[1+\|Tx_{2n}-Tx_{2n+1}\|]} + \beta[\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] + \gamma[\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] + \delta\|Tx_{2n} - Tx_{2n+1}\|$$

[using triangle inequality]

Corresponding Author: Priyanka Tyagi[2]*

$$\|Tx_{2n+1} - Tx_{2n+2}\| \le \alpha\|Tx_{2n+1} - Tx_{2n+2}\| + \beta\|Tx_{2n} - Tx_{2n+1}\| + \beta\|Tx_{2n+1} - Tx_{2n+2}\| + \gamma\|Tx_{2n} - Tx_{2n+1}\| + \gamma\|Tx_{2n+1} - Tx_{2n+2}\| + \delta\|Tx_{2n} - Tx_{2n+1}\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}\|(1-\alpha-\beta-\gamma) \le (\beta+\gamma+\delta)\|Tx_{2n} - Tx_{2n+1}\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}\| \le \frac{(\beta+\gamma+\delta)}{(1-\alpha-\beta-\gamma)}\|Tx_{2n} - Tx_{2n+1}\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}\| \le h\,\|Tx_{2n} - Tx_{2n+1}\|$$

Where $h = \frac{(\beta+\gamma+\delta)}{(1-\alpha-\beta-\gamma)} < 1$

Similarly we can see
$\|Tx_{2n} - Tx_{2n+1}\| \le h\,\|Tx_{2n-1} - Tx_{2n}\|$

Proceeding in t is way, we have

$$\begin{aligned}\|Tx_{2n+1} - Tx_{2n+2}\| &\le h\,\|Tx_{2n} - Tx_{2n+1}\| \\ &\le h^2\,\|Tx_{2n-1} - Tx_{2n}\| \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\le h^{2n+1}\,\|Tx_0 - Tx_1\|\end{aligned}$$

By routine calculation the following inequalities hold for k>n

$$\begin{aligned}\|Tx_n - Tx_{n+k}\| &\le \sum_{i=1}^{k}\|Tx_{n+1-1} - Tx_{n+1}\| \\ &\le \textstyle\sum_{i=1}^{k} h^{n+1-1}\|Tx_0 - Tx_1\| \\ &\le \frac{h^n}{1-h}\,\|Tx_0 - Tx_1\| \to 0 \; as \; n \to \infty.\end{aligned}$$

Hence $\{Tx_n\}$ is a Cauchy sequence by the completeness of X. $\{Tx_n\}$ converges to a point u in X. It follows from (1.3) that $\{Ex_{2n}\}$ and $\{Fx_{2n+1}\}$ also converges to u. since E, F and T are continuous we have

$E(Tx_{2n}) \to Eu$,$F(Tx_{2n+1}) \to Fu$ (1.4)

From (1.1) t commutes with E and F therefore
$E(Tx_{2n}) = T(Ex_{2n})$, $F(Tx_{2n+1}) = T(Fx_{2n+1})$ for all n= 0,1,2,3…………

Taking $n \to \infty$ we have

Eu=Tu=Fu (1.5)
and
T(Tu= T(Eu)) = E(Tu) = E(Fu) = F(Eu) =T(Fu) = F(Tu) = F(Eu) = F(Fu). (1.6)

By (1.2) , (1.5) and (1.6). if Eu $\neq$ F(Eu) we have

$$\begin{aligned}\|Eu - F(Eu)\| &\le \alpha\frac{\|T(Eu) - F(Eu)\|\,[1 + \|Tu - Eu\|]}{[1 + \|Tu - T(Eu)\|]} + \beta[\|Tu - Eu\| + \|T(Eu) - F(Eu)\|] \\ &\quad + \gamma[\|Tu - F(Eu)\| + \|T(Eu) - Eu\|] + \delta\|Tu - T(Eu)\| \\ &\le (2\gamma+\delta)\|Eu - F(Eu)\| \\ &< \|Eu - F(Eu)\| \quad [\because (2\gamma+\delta) < 1]\end{aligned} \tag{1.7}$$

Leading to a contradiction. Hence

Eu = F(Eu). Using (1.6) and (1.7) we get

Eu = F(Eu) = T(Eu) =E (Eu)

Which shows that Eu is the common fixed of E, F and T.

Let z and w (z$\neq$ w) be two points in X such that

Ez = Fz =Tz= z and Ew = Fw= Tw = w. Then by (1.2) we have

$$
\begin{aligned}
\|z-w\| &= \|Ez-Fw\| \\
&\le \alpha \frac{\|Tw-Fw\|[1+\|Tz-Ez\|]}{[1+\|Tz-Fw\|]} + \beta[\|Tz-Ez\| + \|Tw-Fw\|] + \gamma[\|Tz-Fw\| + \|Tw-Ez\|] + \delta\|Tz-Tw\| \\
&\le \alpha.0 + \beta.0 + \gamma[\|Tz-Fw\| + \|Tw-Ez\|] + \delta\|Tz-Tw\| \\
&\le (2\gamma+\delta)\,\|Ez-Fw\| \\
&< \|Ez-Fw\| \qquad [\because (2\gamma+\delta) < 1]
\end{aligned}
$$

Leading to a contraction. Hence z = w. This implies the uniqueness of common fixed point for E, F and T. This completes the proof of the theorem.

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