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Some Properties of a Quarter-Symmetric Non-Metric Connexion in a LP- Sasakian Manifold

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ABSTRACT

In this paper I have studied a quarter-symmetric non-metric connexion in a Lorentzian para-Sasakian manifold. Some properties of the curvature tensor and the Ricci tensor of the manifold for quarter-symmetric non-metric connexion have been obtained.

Keywords: *Quarter- symmetric connexion, LP-Sasakian manifold, curvature tensor, Ricci tensor.*

Mathematics subject classification: *[53]*

1. INTRODUCTION

We consider a n −dimensional C^∞-manifold V_n. Let there exist in V_n, a tensor F of the type (1,1), a vector field U, a 1 −form u and a Riemannian metric g such that

$$\bar{\bar{X}} = X + u(X)U, \tag{1.1}$$
$$u(\bar{X}) = 0, \tag{1.2}$$
$$g(\bar{X},\bar{Y}) = g(X,Y) + u(X)u(Y), \tag{1.3}$$
$$g(X,U) = u(X), \tag{1.4}$$
$$(D_X F)(Y) = g(X,Y)U + u(Y)X + 2u(X)u(Y)U, \tag{1.5}$$
$$D_X U = \bar{X}, \tag{1.6}$$

where $F(X) \stackrel{\text{def}}{=} \bar{X}$

for arbitrary vector fields X, Y. Then V_n satisfying (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) is called a Lorentzian para – Sasakian manifold [2] (in short LP-Sasakian manifold) while the set $\{F, U, u, g\}$ satisfying (1.1) to (1.6) is called a LP-Sasakian structure. It may be noted that D is the Riemannian connexion with respect to the Riemannian metric g.

In a LP-Sasakian manifold it is easy to calculate that

$$u(U) = -1, \tag{1.7 a}$$
$$\bar{U} = 0 \tag{1.7 b}$$

and

$$rank(F) = n - 1. \tag{1.7 c}$$

Let us define a fundamental $2 - form$ $'F$ in a LP-Sasakian manifold as below:

$$'F(X,Y) \stackrel{\text{def}}{=} g(\bar{X},Y). \tag{1.8}$$

Barring Y in (1.3) and using (1.1) and (1.2), we get

$$g(\bar{X},Y) = g(X,\bar{Y}) \tag{1.9}$$

From (1.8) and (1.9), we obtain that

$$'F(X,Y) = {'F}(Y,X) \tag{1.10}$$

Which shows that $'F$ is symmetric in a LP-Sasakian manifold.

Barring X and Y both in (1.8) and using (1.1), (1.2), (1.8) and (1.9), we get

$$'F(\bar{X},\bar{Y}) = {'F}(X,Y) \tag{1.11}$$

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which implies that $'F$ is hybrid in a LP-Sasakian manifold.
(1.4) implies

$$u(Y) = g(Y, U).$$

Taking the covariant derivative of above with respect to the connexion D along the vector field X and using (1.4), (1.6) and (1.8), we get

$$'F(X, Y) = (D_X u)(Y). \tag{1.12}$$

The Conformal curvature tensor Q, the Conharmonic curvature tensor L, the Concircular curvature tensor C and the Projective curvature tensor P in V_n are given by [3]

$$Q(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)RX - g(X,Z)RY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \tag{1.13}$$

$$L(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-2}[Ric(Y,Z)X - R(X,Z)Y + g(Y,Z)RX - g(X,Z)RY], \tag{1.14}$$

$$C(X,Y,Z) = K(X,Y,Z) - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.15}$$

and

$$P(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y] \tag{1.16}$$

where K, Ric, R and r are the curvature tensor, Ricci tensor, Ricci tensor of the type (1,1) and scalar curvature in V_n.

Agreement (1.1): A LP-Sasakian manifold will always be denoted by V_n.

2. CERTAIN PROPERTIES ON V_n

Theorem (2.1): In V_n, we have

$$(D_X{}'F)(Y, U) = g(\bar{X}, \bar{Y}), \tag{2.1}$$

$$(D_X{}'F)(\bar{Y}, Z) + (D_X{}'F)(Y, \bar{Z}) = u(Z)(D_X u)(Y) + u(Y)g(\bar{X}, Z), \tag{2.2}$$

$$(D_X{}'F)(\bar{Y}, \bar{Z}) + (D_X{}'F)(Y, Z) = u(Y)g(\bar{X}, \bar{Z}) - u(Z)g(\bar{X}, \bar{Y}). \tag{2.3}$$

Proof: In view of (1.7 b) and (1.8), we have

$$'F(Y, U) = 0 \tag{2.4}$$

Taking the covariant derivative of (2.4) with respect to the connexion D along the vector field X and using (1.8) and (2.4), we get (2.1).
We know that

$$(D_X{}'F)(Y, Z) = g((D_X F)(Y), Z) \tag{2.5}$$

which implies

$$(D_X{}'F)(\bar{Y}, Z) = g\big((D_X F)(\bar{Y}), Z\big). \tag{2.6}$$

Since

$$F\bar{Y} = F^2 Y.$$

Therefore taking the covariant derivative of above with respect to the connexion D along the vector field X and using (1.1), we get

$$(D_X F)(\bar{Y}) + \overline{(D_X F)(Y)} = u(Y) D_X U + (D_X u)(Y) U.$$

Operating g on both the sides of above and using (1.6) and (2.5), we get (2.2).

Barring Z in (2.2) and using (1.1), (1.2), (1.4), we get (2.3).

Theorem (2.2): In V_n, we have

$$'K(X,Y,Z,T) = g(Y,Z)g(X,T) - g(X,Z)g(Y,T), \tag{2.7}$$

$$Ric(Y, Z) = (n-1)g(Y, Z), \tag{2.8}$$

$$RY = (n-1)Y, \tag{2.9}$$

$$r = n(n-1). \tag{2.10}$$

Proof: From (1.12), we have

$$'F(Y,Z) = (D_Y u)(Z). \tag{2.11}$$

Taking the covariant derivative of above with respect to the connexion D along the vector field X and using (2.11), we get

$$(D_X{}'F)(Y,Z) = (D_X D_Y u)(Z) - \left(D_{D_X Y} u\right)(Z). \tag{2.12}$$

Interchanging X and Y in above, we get

$$(D_Y{}'F)(X,Z) = (D_Y D_X u)(Z) - \left(D_{D_Y X} u\right)(Z). \tag{2.13}$$

Subtracting (2.13) from (2.12), we get

$$(D_X{}'F)(Y,Z) - (D_Y{}'F)(X,Z) = (D_X D_Y u)(Z) - (D_Y D_X u)(Z) - \left(D_{[X,Y]} u\right)(Z). \tag{2.14}$$

From (1.14), we have

$$u(Z) = g(Z,U). \tag{2.15}$$

Taking the covariant derivative of (2.15) with respect to the connexion D along the vector field Y and using (2.15), we get

$$(D_Y u)(Z) = g(Z, D_Y U). \tag{2.16}$$

Taking the covariant derivative of above with respect to the connexion D along the vector field X and using (2.16), we get

$$(D_X D_Y u)(Z) = g(Z, D_X D_Y U). \tag{2.17}$$

Interchanging X and Y in above, we get

$$(D_Y D_X u)(Z) = g(Z, D_Y D_X U). \tag{2.18}$$

Further (2.16) yields

$$\left(D_{[X,Y]} u\right)(Z) = g\left(Z, D_{[X,Y]} U\right). \tag{2.19}$$

Subtracting (2.18) and (2.19) from (2.17) and using (2.14), we get

$$(D_X{}'F)(Y,Z) - (D_Y{}'F)(X,Z) = g\left(Z, K(X,Y,U)\right). \tag{2.20}$$

From (1.5), we have

$$(D_X{}'F)(Y,Z) = g(X,Y)u(Z) + u(Y)g(X,Z) + 2u(X)u(Y)u(Z). \tag{2.21}$$

Using (2.21) in (2.20), we get

$$g\left(Z, K(X,Y,U)\right) = u(Y)g(X,Z) - u(X)g(Y,Z).$$

Which is equivalent to

$$'K(X,Y,U,Z) = u(Y)g(X,Z) - u(X)g(Y,Z) \tag{2.22}$$

where $\quad 'K(X,Y,U,Z) \stackrel{\text{def}}{=} g(K(X,Y,U),Z).$

(2.22) implies

$$'K(X,Y,Z,U) = u(X)g(Y,Z) - u(Y)g(X,Z)$$

which is equivalent to

$$K(X,Y,Z) = g(Y,Z)X - g(X,Z)Y. \tag{2.23}$$

(2.23) is equivalent to (2.7).
Contracting X in (2.23), we get (2.8).

(2.8) implies

$$g(RY,Z) = (n-1)g(Y,Z)$$

which is equivalent to (2.9).
Contracting Y in (2.9), we get (2.10).

Corollary (2.1): In V_n, we have

$$'K(\bar{X},\bar{Y},Z,T) = {}'K(X,Y,\bar{Z},\bar{T})$$

and

$$'K(\bar{X},\bar{Y},\bar{Z},\bar{T}) = {}'K(X,Y,Z,T) + u(T)(u(X)g(Y,Z) - u(Y)g(X,Z)) + u(Z)\left(u(Y)g(X,T) - u(X)g(Y,Z)\right).$$

The proof is obvious from (1.3), (1.9) and (2.7).

Corollary (2.2): V_n is conformally flat.

Proof: Using equation (2.7), (2.8), (2.9) and (2.10) in (1.13), we get

$$Q(X,Y,Z) = 0 \tag{2.24}$$

which proves the corollary.

Corollary (2.3): In V_n, we have

$$L(X,Y,Z) = \frac{n}{2-n}(g(Y,Z)X - g(X,Z)Y). \tag{2.25}$$

Proof: using equations (2.7), (2.8), (2.9) in (1.14), we get (2.25).

Corollary (2.4): V_n is concircularlly flat.

Proof: Using equations (2.7) and (2.10) in (1.15), we get

$$C(X,Y,Z) = 0 \tag{2.26}$$

which proves the corollary.

Corollary (2.5): V_n is projectively flat.

Proof: Using equations (2.7) and (2.8) in (1.16), we get

$$P(X,Y,Z) = 0 \tag{2.27}$$

which proves the corollary.

3. QUARTER-SYMMETRIC NON-METRIC CONNEXION IN V_n

We consider a quarter-symmetric non-metric connexion E [5] defined by

$$E_X Y = D_X Y + u(Y)\bar{X}. \tag{3.1}$$

Let R and K be the curvature tensor with respect to E and D respectively. Then, it is easy to calculate that

$$R(X,Y,Z) = K(X,Y,Z) + g(\bar{X},Z)\bar{Y} - g(\bar{Y},Z)\bar{X} + u(Z)[u(Y)X - u(X)Y] \tag{3.2}$$

where $R(X,Y,Z) = E_X E_Y Z - E_Y E_X Z - E_{[X,Y]}Z$

and $K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$

Contracting X in (3.2), we get

$$\tilde{R}ic(Y,Z) = Ric(Y,Z) + g(Y,Z) + nu(Y)u(Z) \tag{3.3}$$

where $\tilde{R}ic(Y,Z) = C_1^1 R(X,Y,Z)$

and $Ric(Y,Z) = C_1^1 K(X,Y,Z).$

(3.3) implies

$$g(\tilde{R}(Y),Z) = g(R(Y),Z) + g(Y,Z) + nu(Y)u(Z)$$

where

$$\tilde{R}ic(Y,Z) \stackrel{\text{def}}{=} g(\tilde{R}(Y),Z) \tag{3.4 a}$$

and

$$Ric(Y,Z) \stackrel{\text{def}}{=} g(R(Y),Z). \tag{3.4 b}$$

(3.3) implies

$$\tilde{R}(Y) = R(Y) + Y + nu(Y)U. \tag{3.5}$$

Contracting Y in above, we get

$$\tilde{r} = r \tag{3.6}$$

where $\tilde{r}$ and r are the scalar curvatures with respect to E and D in V_n.

Further (3.6) shows that the scalar curvatures of V_n with respect to E and D are equal.

Theorem (3.1): In V_n, we have

$$(E_X F)Y = g(\bar{X},\bar{Y})U, \tag{3.7}$$

$$E_X U = 0, \tag{3.8}$$

$$(E_X u)Y = g(\bar{X},Y), \tag{3.9}$$

$$'R(X,Y,Z,T) = {}'K(X,Y,Z,T) + g(\bar{X},Z)g(\bar{Y},T) - g(\bar{Y},Z)g(\bar{X},T) + u(Z)[u(Y)g(X,T) - u(X)g(Y,T)]. \tag{3.10}$$

Proof: We known that

$$(E_X F)Y = E_X\bar{Y} - \overline{E_X Y}.$$

Using (1.3) and (3.1) in above, we get (3.7).

Replacing Y by U in (3.1), we get

$$E_X U = D_X U + u(U)\bar{X}.$$

Using (1.6) and (1.7a) in above, we get (3.8).

We know that

$$(E_X u)Y = E_X\big(u(Y)\big) - u(E_X Y).$$

Using (1.8) and (3.1) in above, we get (3.9).

Operating g on both the sides of (3.2) and using

$$'R(X,Y,Z,T) \overset{\text{def}}{=} g(R(X,Y,Z),T)$$

and

$$'K(X,Y,Z,T) \overset{\text{def}}{=} g(K(X,Y,Z),T)$$

we get (3.10).

Theorem (3.2): In V_n, we have

$$R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0 \tag{3.11}$$

Proof*:* Using equation (3.2) and Bianchi first identity with respect to Levi-Civita connexion D, we get the result.

Theorem (3.3): In V_n, the conformal curvature tensor $\tilde{Q}$ with respect to the quarter-symmetric non-metric connexion E is given by

$$\tilde{Q}(X,Y,Z) = g(\bar{X},Z)\bar{Y} - g(\bar{Y},Z)\bar{X} - \frac{2}{n-2}u(Z)[u(Y)X - u(X)Y] \\ -\frac{2}{n-2}[g(Y,Z)X - g(X,Z)Y] - \frac{n}{n-2}[g(Y,Z)u(X) - g(X,Z)u(Y)]U. \tag{3.12}$$

Proof: In view of (1.13) $\tilde{Q}$ in V_n is given by

$$\tilde{Q}(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-2}[\tilde{R}ic(Y,Z)X - \tilde{R}ic(X,Z)Y + g(Y,Z)\tilde{R}X - g(X,Z)\tilde{R}Y] \\ +\frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \tag{3.13}$$

Which is equivalent to

$$'\tilde{Q}(X,Y,Z,T) = {}'R(X,Y,Z,T) - \frac{1}{n-2}[\tilde{R}ic(Y,Z)g(X,T) - \tilde{R}ic(X,Z)g(Y,T) \\ +g(Y,Z)\tilde{R}ic(X,T) - g(X,Z)\tilde{R}ic(Y,T)] + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \tag{3.14}$$

where

$$'\tilde{Q}(X,Y,Z,T) = g\big(\tilde{Q}(X,Y,Z),T\big).$$

Now using equation (1.13), (3.2), (3.3) and (3.6) in the above equation, we get

$$'\tilde{Q}(X,Y,Z,T) = {}'Q(X,Y,Z,T) + g(\bar{X},Z)g(\bar{Y},T) - g(\bar{Y},Z)g(\bar{X},T) - \frac{2}{n-2}u(Z)[u(Y)g(X,T) - u(X)g(Y,T)] \\ -\frac{2}{n-2}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)] - \frac{n}{n-2}[g(Y,Z)u(X) - g(X,Z)u(Y)]u(T).$$

Using (2.24) in above, we get (3.12).

Theorem (3.4): The conharmonic curvature tensors $\tilde{L}$ with respect to quarter-symmetric non-metric connexion E in V_n is given by

$$\tilde{L}(X,Y,Z) = g(\bar{X},Z)\bar{Y} - g(\bar{Y},Z)\bar{X} - \frac{2}{n-2}u(Z)[u(Y)X - u(X)Y] \\ -\frac{n+2}{n-2}[g(Y,Z)X - g(X,Z)Y] - \frac{n}{n-2}[g(Y,Z)u(X) - g(X,Z)u(Y)]U. \tag{3.15}$$

Proof: In view of (1.14), we have

$$\tilde{L}(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-2}[\tilde{R}ic(Y,Z)X - \tilde{R}ic(X,Z)Y + g(Y,Z)\tilde{R}X - g(X,Z)\tilde{R}Y] \tag{3.16}$$

which implies

$$'\tilde{L}(X,Y,Z,T) = {}'R(X,Y,Z,T) - \frac{1}{n-2}[\tilde{R}ic(Y,Z)g(X,T) - \tilde{R}ic(X,Z)g(Y,T) \\ +g(Y,Z)\tilde{R}ic(X,T) - g(X,Z)\tilde{R}ic(Y,T)] \tag{3.17}$$

where

$$'\tilde{L}(X,Y,Z,T) \overset{\text{def}}{=} g\big(\tilde{L}(X,Y,Z),T\big).$$

Now using equation (1.14), (3.2) and (3.3) in (3.17), we get

$$'\tilde{L}(X,Y,Z,T) = {}'L(X,Y,Z,T) + g(\bar{X},Z)g(\bar{Y},T) - g(\bar{Y},Z)g(\bar{X},T) - \frac{2}{n-2}u(Z)[u(Y)g(X,T) - u(X)g(Y,T)] \\ -\frac{2}{n-2}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)] - \frac{n}{n-2}[g(Y,Z)u(X) - g(X,Z)u(Y)]u(T).$$

Using (2.25) in above, we get (3.15).

Theorem (3.5): The concircular curvature tensor $\tilde{C}$ with respect to the quarter-symmetric non-metric connexion E is given by

$$\tilde{C}(X,Y,Z) = g(\bar{X},Z)\bar{Y} - g(\bar{Y},Z)\bar{X} + u(Z)[u(Y)X - u(X)Y]. \tag{3.18}$$

Proof: In view of (1.15), $\tilde{C}$ is given by

$$\tilde{C}(X,Y,Z) = R(X,Y,Z) - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{3.19}$$

Which is equivalent to

$$'\tilde{C}(X,Y,Z,T) = 'R(X,Y,Z,T) - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)] \tag{3.20}$$

where $$'\tilde{C}\,(X,Y,Z,T) = g\big(\tilde{C}(X,Y,Z),T\big).$$

Now using equation (1.15), (2.26), (3.2) and (3.6) in the above equation, we get (3.18).

Theorem (3.6): The projective curvature tensor $\tilde{P}$ in V_n with respect to quarter-symmetric non-metric connexion E is given by

$$\tilde{P}(X,Y,Z) = g(\bar{X},Z)\bar{Y} - g(\bar{Y},Z)\bar{X} - \frac{1}{n-1}[g(Y,Z)X - g(X,Z)Y] - \frac{1}{n-1}[u(Y)X - u(X)Y]u(Z). \tag{3.21}$$

Proof: In view of (1.16), the projective curvature tensor in V_n with respect to E is given by

$$\tilde{P}(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[\tilde{R}ic(Y,Z)X - \tilde{R}ic(X,Z)Y]$$

which is equivalent to

$$'\tilde{P}(X,Y,Z,T) = 'R(X,Y,Z,T) - \frac{1}{n-1}[\,\tilde{R}ic(Y,Z)g(X,T) - \tilde{R}ic(X,Z)g(Y,T)] \tag{3.22}$$

where $$'\tilde{P}(X,Y,Z,T) = g\big(\tilde{P}(X,Y,Z),T\big).$$

Using equations (1.16), (3.2), (3.3) in (3.22), we get

$$'\tilde{P}(X,Y,Z,T) = 'P(X,Y,Z,T) + g(\bar{X},Z)g(\bar{Y},T) - g(\bar{Y},Z)g(\bar{X},T)$$
$$- \frac{1}{n-1}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)] - \frac{1}{n-1}[u(Y)g(X,T) - u(X)g(Y,T)]u(Z). \tag{3.23}$$

Using (2.27) in above, we get (3.21).

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