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SYMMETRIC DUALITY FOR MULTIOBJECTIVE OPTIMIZATION UNDER CONVEXITY ASSUMPTION USING THE CONCEPT OF PROPER EFFICIENCY

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ABSTRACT

In this paper, a new class of symmetric duality for multiobjective optimization and their scalar parametric problems are formulated and the duality results are established under convexity assumption using the concept of proper efficiency. Also the duality relations with saddle point theory are presented.

Kew words: *symmetric duality, proper efficiency, saddle point, convex function.*

INTRODUCTION

The importance of convex function is well known in optimization theory. But for many mathematical model used in decision sciences, applied mathematics and engineering, the notion of convexity does no longer suffice. So it is possible to generalize the notion of convexity and to extend the validity of results to larger class of optimization problems. Consequently, various generalization of convex function has been introduced in the literature. The field of multiobjective programming, also known as vector programming has grown remarkable in different direction in the setting of optimality condition and duality theory since the 1980. It has been enriched by the application of various types of generalization of convex theory with and without differentiable assumption and in the frame work of continuous time programming, fractional programming, inverse vector optimization saddle point theory, symmetric duality, variational problem etc.

Symmetric duality in nonlinear programming problem was first introduced by Dorn [1] who defined a mathematical programming problem and it's dual to be symmetric if the dual is the primal problems. Later Dantzing *et al.* [2] and Mond [3] formulated a pair of symmetric dual programs for scalar function f (x, y) that is convex in the first variable and that is concave in the second variable respectively.

Bazaars [1] and Dantzing *et al.* [2] studied on symmetric duality in nonlinear programming. Devi [4] studied symmetric duality for nonlinear programming problem involving η-bonvex functions. Dorn [5] formulated symmetric dual theorem for quadratic programs. Egudo established efficiency and generalized convex duality for multiobjective programs.

Geoffrion [6] gave the idea of proper efficiency and the theory of vector maximization. Gulati *et al.* [7] established second order symmetric duality with cone constraints where as Gulati and Geeta [8] established Mond-Weir type second order symmetric duality in multiobjective programming over cones. Gupta and Danger [9] duality for second order symmetric multiobjective programming with cone constraint. Kassem [10] established multiobjective nonlinear symmetric duality involving generalized pseudo convexity. Khurana [11] established symmetric duality in multiobjective programming involving generalized cone-invex functions. Kim *et al.* [12] established Multiobjective symmetric duality with cone constraint. Preda [13] studied on efficiency and duality for multiobjective programs. Suneja *et al.* [14] established multiobjective symmetric duality involving cones.

In this paper, we introduced a new class of symmetric duality for multiobjective optimization under convexity assumption using the concept of proper efficiency. Also the duality relations with saddle point theory are presented.

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NOTATION AND DEFINITION

Let C_1 and C_2 denote closed convex cones with nonempty interiors in R^n and R^m respectively.

Let C_i^* (i=1,2,) be the polar of C_i ,i.e. $C_i^* = \{z : x^T z \leq 0 \; for \forall x \in C_i\}$, where x^T denote the transpose of x. Let f(x, y) be a vector valued function defined on an open set in R^{n+m}.

Definition 1: A real-valued function ϕ is said to be convex if

$$\phi(x) - \phi(\bar{x}) \geq (x - \bar{x})^T \nabla \phi(\bar{x}) \text{ for all x, } \bar{x} \in C_1.$$

Definition 2: A real-valued function $f(x, y) = (f_1(x, y), f_2(x, y), .\ .\ f_k(x, y))^T$ is said to be convex in the first variable if each $f_i(., \bar{y})$ is convex for fixed $\bar{y}$, i =1, 2, 3,……k.

Similarly f is said to be convex in second variable if $f_i(\bar{x}, .)$ is convex for fixed $\bar{x}, i = 1,2,3,...k$.

Primal (Pv):

$$\text{Min } F(x, y) = \{ f_1(x, y) - y^T \nabla_y f_1(x, y), .\ .\ f_k(x, y) - y^T \nabla_y f_k(x, y)\}^T$$

$$\text{Subject to } (x, y) \in C_1 \times C_2, \nabla_y f_i(x, y) \in C_2^*, C_1 \subset R^m, C_2 \subset R^n.$$

Dual (D):

$$\text{Max } G(x, y) = \{ f_1(x, y) - x^T \nabla_x f_1(x, y), .\ .\ f_k(x, y) - x^T \nabla_x f_k(x, y)\}^T$$

$$\text{Subject to } (x, y) \in C_1 \times C_2, -\nabla x f_i(x, y) \in C_1^*, C_1 \subset R^m, C_2 \subset R^n.$$

Now we introduced the following scalar parametric problem:

Primal (P_λ): Min $\lambda f(x, y) - y^T \nabla_y \lambda f(x, y)$,

$$\text{Subject to } (x, y) \in C_1 \times C_2, \nabla_y f_i(x, y) \in C_2^*, C_1 \subset R^m, C_2 \subset R^n,$$

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)^T \in R^n$$

Dual (D_λ)**:** Max $\lambda G(x, y)$

$$\text{Subject to } (x, y) \in C_1 \times C_2, -\nabla_y f_i(x, y) \in C_1^*, C_1 \subset R^m, C_2 \subset R^n.$$

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)^T \in R^n$$

We denote the set of feasible solution of P by S i.e

$$S = \{(\bar{x}, \bar{y}) \in C_1 \times C_2 \mid \nabla_y f_i(x, y) \in C_2^*$$

Definition 3: A feasible point $(\bar{x}, \bar{y})$ is said to be an efficient solution of (P_v) if

$$f_i(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y f_i(\bar{x}, \bar{y}) \geq f_i(x, \bar{y}) - \bar{y}^T \nabla_y f_i(x, \bar{y})$$

$$\Rightarrow f_i(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y f_i(\bar{x}, \bar{y}) = f_i(\bar{x}, y) - \bar{y}^T \nabla_y f_i(x, \bar{y}) \text{ for all i=1,2,3, ……k.}$$

and

$$f_i(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y f_i(\bar{x}, \bar{y}) \leq f_i(\bar{x}, y) - \bar{y}^T \nabla_y f_i(\bar{x}, y)$$

$$\Rightarrow f_i(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y f_i(\bar{x}, \bar{y}) = f_i(\bar{x}, y) - \bar{y}^T \nabla_y f_i(\bar{x}, y)$$

Definition 4: A feasible point $(\bar{x},\bar{y})$ is said to be properly efficient if it is efficient for P_v and if there exist scalars $M_1>0$ and $M_2>0$, such that

$$\frac{(f_i(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y})-(f(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,\bar{y})}{f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y})-f_j(\bar{x},\bar{y})-y^T\nabla_y f_j(\bar{x},\bar{y})}\le M_1 \text{ for some j,}$$

$f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y})> f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y})$ and

$f_i(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,\bar{y})< f_i(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y})$ whenever $(x,\bar{y})$ is a feasible for (P_v).

and
$$\frac{(f_i(\bar{x},y)-\bar{y}^T\nabla_y f_i(\bar{x},y)-(f(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,\bar{y})}{f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y})-f_j(\bar{x},y)-y^T\nabla_y f_j(\bar{x},y)}\le M_2 \text{ for some j,}$$

$f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y})> f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y})$

and $f_i(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y})< f_i(\bar{x},y)-\bar{y}^T\nabla_y f_i(\bar{x},y).$ whenever $(\bar{x},y)$ is feasible for P_v.

Lemma 3.1: If $(\bar{x},\bar{y})$ is a feasible solution of (P_λ) then $(\bar{x},\bar{y})$ is properly efficient solution of (P_v) where $\lambda_i = 1,2,...k$

Proof: Since $(\bar{x},\bar{y})$ is feasible solution of P_v then obviously it is efficient in P_v .Now we have to show that $(\bar{x},\bar{y})$ is properly efficient solution in P_v. If it is not properly efficient solution then

$$(f_i(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y}))-(f_i(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,\bar{y}))>$$
$$M_{\mathbf{1}}[f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y})> f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y})]$$

Let us consider $M_{\mathbf{1}}=(k-\mathbf{1})\max\limits_{i,j}(\lambda_j/\lambda_i)$ where $k\ge 2$ for all j such that

$$f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y})<(f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}).$$

Hence $(f_i(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y}))-(f_i(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,\bar{y}))>$

$$\text{(k-1)}\ \frac{\lambda_j}{\lambda_i}[(f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))-(f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y}))]$$

for all $j\ne i$ Multiplying both sides by $\frac{\lambda_i}{(k-1)}$ then we get

$$\frac{\lambda_i}{(k-\mathbf{1})}[(f_i(\bar{x},\bar{y}-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y})-(f_i(x,\bar{y})-\bar{y}^T\nabla_y f(x,\bar{y}))]$$
$$>\lambda_j[(f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))-(f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y}))] \text{ for all } j\ne 1.$$

By summing over

$$j\ne i,\ \sum_{j\ne i}\lambda_i[(f_i(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,\bar{y}))-(f_i(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y}))]$$
$$>\sum_{j\ne i}\lambda_j[(f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))-(f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y}))]$$

which contradicts to feasible solution of P_λ .

Lemma 3.2: Let $(\bar{x},\bar{y})$ is a is properly efficient solution of (P_v) where $\lambda_i = 1,2,...k$. If each f_i is convex in the first variable and $-f_i$ is convex in second variable then $(\bar{x},\bar{y})$ is a feasible solution of P_λ .

Proof: Since f_i is convex in the first variable, from the properties of convexity, we obtain

$$\lambda_i^i(f_i(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,y))+\sum_{j\neq i}\lambda_j^i\left[(f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))+M_1(f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))\right]$$

$$\geq\lambda_j^i(f(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y}))+\sum_{j\neq i}\lambda_j^i\left[(f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))-(f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y}))\right]$$

Since $\lambda_j^i \geq 0,\ j=1,2,3,...k$ and $\sum_{j=1}^{k}\lambda_j^i = 1$, from above inequation, we get

$$(f_i(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,\bar{y}))+M_1\sum_{j\neq i}\lambda_j^i((f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))$$

$$\geq\lambda_j^i(f(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y}))+\sum_{j\neq i}\lambda_j^i\left[(f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))-(f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y}))\right]$$

summing over j yields after some rearrangement,

$$\sum_{j=1}^{k}(1+M_1\sum_{i\neq j}\lambda_j^i)((f_j(x,\bar{y})-\bar{y}^T\nabla_y f_j(x,\bar{y}))\geq\sum_{j=1}^{k}(1+M_1\sum_{i=j}\lambda_j^i)((f_j(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_j(\bar{x},\bar{y}))$$

$$\Rightarrow(f_i(x,\bar{y})-\bar{y}^T\nabla_y f_i(x,\bar{y}))\geq(f_i(\bar{x},\bar{y})-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y}))$$

$$\Rightarrow F(x,\bar{y})\geq F(\bar{x},\bar{y}).$$

Similarly we can prove $F(\bar{x},y)\leq F(\bar{x},\bar{y})$ by taking $-f_i$ as convex in second variable.

Hence $(\bar{x},\bar{y})$ is a saddle point of F.

Now $F(\bar{x},\bar{y})=\min_{x\geq0,y\geq0}\max F(\bar{x},\bar{y})=\max_{y\geq0,x\geq0}\min F(\bar{x},\bar{y})$

This shows that the component x of a saddle point $(\bar{x},\bar{y})$ solves the minimization problem P_λ .

Lemma 3.3: Let $(\bar{x},\bar{y})$ be a saddle point of $\lambda^T f$ in P_λ , where $\lambda_i > 0, i=1,2.3....k,$ and $\sum_{j=1}^{k}\lambda_i = 1$.

If each f_i is convex in first variable and $-f_i$ is convex in second variable then it satisfies

$\nabla_x f_i(\bar{x},\bar{y})\in C_1^*, \nabla_x f_i(\bar{x},\bar{y})\bar{x}=0, \bar{x}\geq 0$ and $\nabla_y f_i(\bar{x},\bar{y})\in C_2^*, \nabla_y f_i(\bar{x},\bar{y})\bar{y}=0, \bar{y}\geq 0$

Lemma 3.4: Suppose the feasible point $(\bar{x},\bar{y})$ satisfies

$\nabla_x f_i(\bar{x},\bar{y})\in C_1^*, \nabla_x f_i(\bar{x},\bar{y})\bar{x}=0, \bar{x}\geq 0$ and $\nabla_y f_i(\bar{x},\bar{y})\in C_2^*, \nabla_y f_i(\bar{x},\bar{y})\bar{y}=0, \bar{y}\geq 0$, $i=1,2,3,...,k.$.

If f_i is convex in first variable and $-f_i$ is convex in second variable then it $(\bar{x},\bar{y})$ is a saddle point of $\lambda^T f$ in P_λ for all $\lambda_i > 0, i=1,2.3....k,$ and $\sum_{j=1}^{k}\lambda_i = 1$.

Theorem 3.1 (Weak duality): Let (x, y) be feasible for P_v and (u, v) be feasible for (D). If each f_i is convex in first variable and $-f_i$ is convex in second variable then $F(x, y) \geq G(u, v)$.

Proof: Since each f_i is convex in first variable and $-f_i$ is convex in second variable then

$$\sum_{i=1}^{k} \lambda_i (f_i(x,v) - f_i(u,v)) \geq (x-u)^T \sum_{i=1}^{k} \lambda_i \nabla_x f_i(u,v) \quad (1)$$

and
$$\sum_{i=1}^{k} \lambda_i (f_i(x,v) - f_i(x,y)) \leq (v-y)^T \sum_{i=1}^{k} \lambda_i \nabla_y f_i(u,v) \quad (2)$$

For all $\lambda_i > 0, i = 1, 2.3....k, and \quad \sum_{j=1}^{k} \lambda_i = 1.$

Multiplying-1 in inequality (2) and adding to inequality (1) we get

$$\sum_{i=1}^{k} \lambda_i (f_i(x,y) - \sum_{i=1}^{k} \lambda_i \nabla_x f_i(u,v) \geq (x-u)^T \sum_{i=1}^{k} \lambda_i \nabla_x f_i(u,v) - (v-y)^T \sum_{i=1}^{k} \lambda_i \nabla_y f_i(x,y)$$

Since $(x-u) \in C_1$ and $(v-y) \in C_2$, therefore $F(x, y) \geq G(u, v)$.

Theorem 3.2 (Strong Duality): Let $(\bar{x}, \bar{y})$ be said to be properly efficient solution for P_v assume that each f_i is convex in first variable and $-f_i$ is convex in second variable . Then $(\bar{x}, \bar{y})$ be said to be properly efficient solution for (D) and the objective values of P_v and (D) are equal.

Proof: Since $(\bar{x}, \bar{y})$ is properly efficient solution for P_v, Therefore from Lemma 3.2 and Lemma 2.3 it follows that

$\nabla_x f_i(\bar{x}, \bar{y}) \in C_1^*, \nabla_x f_i(\bar{x}, \bar{y})\bar{x} = \mathbf{0}, \bar{x} \geq \mathbf{0}$ and

$\nabla_y f_i(\bar{x}, \bar{y}) \in C_2^*, \nabla_y f_i(\bar{x}, \bar{y})\bar{x} = \mathbf{0}, \bar{y} \geq \mathbf{0}$ for $i = \mathbf{1}, \mathbf{2}, \mathbf{3}, ..., k$

Thus $(\bar{x}, \bar{y}) \in (D)$

Now $F(\bar{x}, \bar{y}) = \lambda^T (f(\bar{x}, \bar{y}) - \nabla_y f(\bar{x}, \bar{y})\bar{y} = \lambda^T (f(\bar{x}, \bar{y})$

and $G(\bar{x}, \bar{y}) = \lambda^T (f(\bar{x}, \bar{y}) - \nabla_y f(\bar{x}, \bar{y})\bar{y} = \lambda^T (f(\bar{x}, \bar{y})$

Hence $F(\bar{x}, \bar{y}) = G(\bar{x}, \bar{y})$.

CONCLUSION

In this paper, we present a new class of symmetric duality for multiobjective optimization and their scalar parametric problems and established the duality results under convexity assumption using the concept of proper efficiency. Also we discussed the duality relations with saddle point theory.

REFERENCE

1. Bazaars, M.S., Goode, J. J., On symmetric duality in nonlinear programming, *Oper. Res.*, 21, (1973), 1-9.
2. Dantzing, G. B., Eisenberg, E. and Cottle, R. W., Symmetric dual nonlinear programs, *Pacific J. Math.*, 15, (1965), 809-812.
3. Devi, G., Symmetric duality for nonlinear programming problem involving η-bonvex function, *European J. Oper. Res.*, 104, (1998), 615-621.
4. Dorn, W. S., A symmetric dual theorem for quadratic programs. *J. Oper. Res. Soc. Japan*, 2, (1960), 93-97.
5. Egudo, R. R., Efficiency and generalized convex duality for multiobjective programs, *J. Math. Anal. Appl.*, 138, (1989) 84-94.
6. Geoffrion, A. M., Proper efficiency and the theory of vector maximization. *J. Math. Anal. Appl.* 22, (1968) 618-630.

7. Gulati, T. R., Gupta, S.K. and Ahmad, A., Second order symmetric duality with cone constraints, *J.Comput. App. Math.*, 220, (2008), 347-357.
8. Gulati, T. R. and Geeta, Mond-Weir type second order symmetric duality in multiobjective programming over cones, *Appl. Math. Lett.* 23, (2010), 466-471.
9. Gupta, S.K. and Danger, D., Duality for second order symmetric multiobjective programming with cone constraint. *Int. J. Math. Oper. Res.*, 4(2), (2012), 128-151.
10. Kassem, M., Multiobjective nonlinear symmetric duality involving generalized pseudo convexity, *Appl. Math.*, 2(10), (2011), 1236-1242.
11. Khurana, S., Symmetric duality in multiobjective programming involving generalized cone-invex function, *European J. Oper. Res.*, 188, (2005), 652-661.
12. Kim, D. S., Yun, Y. B. and Lee, W.J., Multiobjective symmetric duality with cone constraint, *European J. Oper. Res.*, 107, (1998), 686-691.
13. Preda, V., On efficiency and duality for multiobjective programs, *J. Math. Ana. Appl.*, 166, (1992), 365-377.
14. Suneja, S. K., Agarwal, S.K. and Davar, S., Multiobjective symmetric duality involving cones, *European J. Oper. Res.*, 141, (2002), 471-479.

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