International Journal of Mathematical Archive-2(7), July - 2011, Page: 1114-1121
Available online through www.ijma.info ISSN 2229 – 5046

DUALITY FOR MULTIOBJECTIVE PROGRAMMING INVOLVING (Φ, ρ)-UNIVEXITY

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(Received on: 20-06-11; Accepted on: 04-07-11)

The concepts of (Φ, ρ)-invexity have been given by Carsiti, Ferrara and Stefanescu [20]. We consider a dual model associated to a multiobjective programming problem involving support functions and a weak duality result is established under appropriate (Φ, ρ)-univexity conditions.

***AMS 2002 Subject Classification**: 90C29, 90C30, 90C46.*

***Key words**: (Φ, ρ)-(pseudo/quasi)-convexity /univexity, multiobjective programming, duality theorem.*

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1. INTRODUCTION

For nonlinear programming problems, a number of duals have been suggested among which the Wolfe dual [6, 23] is well known. While studying duality under generalized convexity, Mond and Weir [29] proposed a number of deferent duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

For $\Phi(x,a,(y,r)) = F(x,a;y) + rd^2(x,a)$, where $F(x,a;.)$ is sublinear on R^n, the definition of (Φ,ρ)-invexity reduces to the definition of (F,ρ)-convexity introduced by Preda[17], which in turn Jeyakumar[18] generalizes the concepts of F-convexity and ρ-invexity.For more details reader may consult [1,2,3,4,5,7,9,17,18,19,24,27,29].

The more recent literature, Mishra[22], Xu[11], Ojha [12], Ojha and Mukherjee [15] for duality under generalized(F,ρ)-convexity, Liang et al. [13] and Hachimi[14] for optimality criteria and duality involving (F,α,ρ,d)-convexity or generalized $\{F,\alpha,\rho,d\}$-type functions. The (F,ρ)-convexity was recently generalized to (Φ,ρ)-invexity by Caristi, Ferrara and Stefanescu [20], and here we will use this concept to extend some theoretical results of multiobjective programming.

Whenever the objective function and all active restriction functions satisfy simultaneously the same generalized invexity at a Kuhn-Tucker point which is an optimum condition, then all these functions should satisfy the usual invexity, too. This is not the case in multiobjective programming; Ferrara and Stefanescu [16] showed that sufficiency Kuhn-Tucker condition can be proved under (Φ,ρ)-invexity, even if Hanson's invexity is not satisfied, Puglisi [21]. Therefore, the results of this paper are real extensions of the similar results known in the literature.

In Section 2 we define the (Φ, ρ)-univexity. In Section 3 we consider a class of Multiobjective programming problems and for the dual model we prove a weak duality result and strong duality.

2. NOTATIONS AND PRELIMINARIES

We consider $f: R^n \to R^p$, $g: R^n \to R^q$, are differential functions and $X \subset R^n$ is an open set. We define the following multiobjective programming problem:

(P) minimize $f(x) = \left(f_1(x) \ldots\ldots\ldots f_p(x)\right)$

$\quad\quad subject\ to\ g(x) \geq 0x, x \in X$

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Let X_0 be the set of all feasible solutions of (P) that is, $X_0 = \{x \in X \mid g(x) \geq 0\}$.

We quote some definitions and also give some new ones.

Definition 2.1: A vector $a \in X_0$ is said to be an efficient solution of problem (P) if there exit no $x \in X_0$ such that $f(a) - f(x) \in R_+^p \setminus \{0\}$ i.e., $f_i(x) \leq f_i(a)$ for all $i \in \{1,.,.,.,p\}$, and for at least one $j \in \{1,.,.,.,p\}$ we have $f_i(x) < f_i(a)$.

Definition 2.2: A point $a \in X_0$ is said to be a weak efficient solution of problem (VP) if there is no $x \in X$ such that $f(x) < f(a)$.

Definition 2.3: A point $a \in X_0$ is said to be a properly efficient solution of (VP) if it is efficient and there exist a positive constant K such that for each $x \in X_0$ and for each $i \in \{1,2......p\}$ *satisfying* $f_i(x) < f_i(a)$, there exist at least one $i \in \{1,2......p\}$ *such that* $f_j(a) < f_j(x)$ and $f_i(a) - f_i(x) \leq K\left(f_j(x) - f_j(a)\right)$.

Denoting by WE (P), E (P) and PE (P) the sets of all weakly efficient, efficient and properly efficient solutions of (VP), we have PE(P) ⊆ E(P) ⊆ WE(P).

We denote by $\nabla f(a)$ the gradient vector at a of a differentiable function $f : R^p \to R$, and by $\nabla^2 f(a)$ the Hessian matrix of f at a. For a real valued twice differentiable function $\psi(x, y)$ defined on an ope (a,b) n set in $R^p \times R^q$, we denote by $\nabla_x \psi(a,b)$ the gradient vector of ψ with respect to x at, and by $\nabla_{xx}\psi(a,b)$ the Hessian matrix with respect to x at (a,b). Similarly, we may define, $\nabla_{xy}\psi(a,b)$ and $\nabla_{yy}\psi(a,b)$.

For convenience, let us write the definitio $\nabla_y \psi(a,b)$ ns of (Φ,ρ)-univexity from [1], Let $\varphi : X_0 \to R$ be a differentiable function $(X_0 \subseteq R^n)$, $X \subseteq X_0$, and $a \in X_0$. An element of all (n+1)- dimensional Euclidean Space R^{n+1} is represented as the ordered pair (z, r) with $z \in R^n$ *and* $r \in R, \rho$ is a real number and Φ is real valued function defined on $X_0 \times X_0 \times R^{n+1}$, *such that* $\Phi(x,a,.)$ *is* convex on R^{n+1} *and* $\Phi(x,a,(0,r)) \geq 0$, for every $(x,a) \in X_0 \times X_0$ and $r \in R_+$. $b_0, b_1 : X \times X \times [0,1] \to R_+$ $b(x,a) = \lim_{\lambda \to 0} b(x,a,\lambda) \geq 0$, and b does not depend upon λ if the corresponding functions are differentiable. $\psi_0, \psi_1 : R \to R$ is an n-dimensional vector- valued function.
We assume that $\psi_0, \psi_1 : R \to R$ satisfying $u \leq 0 \Rightarrow \psi_0(u) \leq 0$ *and* $u \leqq 0 \Rightarrow \psi_1(u) \leqq 0$, *and* $b_0(x,a) > 0$ *and* $b_1(x,a) \geqq 0$. and $\psi_0(\alpha) = -\psi_0(\alpha)$ and $\psi_1(-\alpha) = -\psi_1(\alpha)$.

Example 2.1:

$$\min f(x) = x - 1$$
$$g(x) = -x - 1 \leq 0, x \in X_0 \in [1, \infty)$$

$$\Phi(x, a; (y, r)) = 2(2^r - 1)|x - a| + \langle y, x - a \rangle$$

for $\psi_0(x) = x$, $\psi_1(x) = -x$, $\rho_1 = \frac{1}{2}$ (for f) and $\rho = 1$ (for g), then this is (ϕ, ρ)-univex but it is not (ϕ, ρ)-invex .

Definition 2.4: A real-valued twice differentiable function $f(.,y) : X \times X \to R$ is said to be (Φ, ρ)-univex at $u \in X$ with respect to $p \in R^n$, if for all $b : X \times X \to R_+$, $\Phi : X \times X \times R^{n+1} \to R$, ρ is a real number, we have

$$b(x,u)[\psi\{f_i(x,y) - f_i(u,y)\}] \geqq \Phi(x,u;(\nabla f_i(u,y), \rho_i)) \tag{2.1}$$

Definition 2.5:
A real-valued twice differentiable function $f(.,y): X\times X \to R$ is said to be (Φ,ρ) -pseudounivex at $a\in X$ with respect to $p\in R^n$,if for all $b: X\times X \to R_+$, $\Phi: X\times X\times R^{n+1} \to R$, ρ is a real number, we have

$$\Phi(x,u;(\nabla f_i(u,y),\rho_i))\geqq 0 \Rightarrow b(x,u)[\psi\{f_i(x,y)-f_i(u,y)\}]\geqq 0 \quad (2.2)$$

Definition 2.6:
A real-valued twice differentiable function $f(.,y): X\times X \to R$ is said to be (Φ,ρ) -quasiunivex at $a\in X$ with respect to $p\in R^n$, if for all $b: X\times X \to R_+$, $\Phi: X\times X\times R^{n+1} \to R$, ρ is a real number, we have

$$b(x,u)[\psi\{f_i(x,y)-f_i(u,y)\}]\leqq 0 \Rightarrow \Phi(x,u;(\nabla f_i(u,y),\rho_i))\leqq 0 \quad (2.3)$$

Remark 2.1:
(i) If we consider the case $b=1,\psi\equiv I$ $\Phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u))$ (with F is sublinear in third argument, and then the above definition reduce to F-convexity.

(ii)When $\Phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u)) = \eta(x,u)^T \nabla f(u)$, where $\eta: X\times X \to R^n, b=1,\psi\equiv I$ the above definition reduce to η - (pseudo/quasi)-convexity.

A real valued twice differentiable function f is (Φ,ρ) -pseudoconcave if $-f$ is (Φ,ρ) -pseudoconvex.

3. TWO WOLFE TYPE SYMMETRIC DUALITY

We consider here the following pair of multiobjective mathematical programs and establish weak, strong duality theorems.

(MP)
$Minimize\, f_i(x,y)$
subject to

$$\sum_{i=1}^{r}\lambda_i \nabla_y f_i(x,y)\leqq 0 \quad (3.1)$$

$$\lambda>0,\ \lambda^T e=1 \quad (3.2)$$

$$x\geqq 0 \quad (3.3)$$

(MD)
$Maximize\, f_i(u,v)$
subject to

$$\sum_{i=1}^{r}\lambda_i \nabla_u f_i(u,v)\geqq 0 \quad (3.4)$$

$$\lambda>0\ ,\ \lambda^T e=1 \quad (3.5)$$

$$v\geqq 0 \quad (3.6)$$

Here, $e(1,1,1,.,.,1)^T \in R$, $\lambda_i\in R, i=1,2,.,.,r$ and $f_i, i=1,2,.,.,r$ are twice differentiable function from $R^n\times R^n \to R$, and $b_i = R^n\times R^m\times R^n\times R^m \to R_+$

Theorem 3.1 (Weak duality):
Let $(x,y,\lambda,z_1,z_2,.,.,z_r)$ be a feasible solution of (MP) and $(u,v,\lambda,w_1,w_2,.,.,w_r)$ a feasible solution of (MD) and

(i) $\sum_{i=1}^{r}\lambda_i f_i(.,v)$ is (Φ_0,ρ) -univex at u for fixed v,

(ii) $\sum_{i=1}^{r}\lambda_i f_i(x,.)$ is (Φ_1,ρ) -unicave at y for fixed x,

(iii) $\Phi_0(x,u;(\xi,\rho))+u^T\xi \geqq 0$, where $\xi=\sum_{i=1}^{r}\lambda_i\nabla_u f_i(u,v)$,and

(iv) $\Phi_1(v,y;(\zeta,\rho))+y^T\zeta \leqq 0$, for $\zeta=\sum_{i=1}^{r}\lambda_i\nabla_y f_i(x,y)$,

Then, $f_i(x,y) \not\leq f_i(u,v)$.

Proof: Since $\sum_{i=1}^{r}\lambda_i f_i(.,v)$ be (Φ_0,ρ) -univex at u for fixed v, for $\lambda>0$,

$$\sum_{i=1}^{r}\lambda_i[b_0(x,y,u,v)\psi\{f_i(x,v) - f_i(u,v)\}] \geqq \Phi_0(x,u;(\nabla_u f_i(u,v),\rho_i))$$

We get facilitate,with the help of hypothesis (iii) and (3.4),with property of b_0 and ψ

$$\sum_{i=1}^{r}\lambda_i f_i(x,v) \geqq \sum_{i=1}^{r}\lambda_i(f_i(u,v) \tag{3.7}$$

Now, $f_i(x,.)$ be (Φ_1,ρ) pseudounicavity assumption at y for fixed x, for $\lambda>0$, we have,

we have, $\sum_{i=1}^{r}\lambda_i b_1(x,y,u,v)\{\psi[f_i(x,v) - f_i(x,y)]\} \leqq \Phi_1(x,y;(\nabla_y f_i(x,y),\rho_i))$

and hypothesis (iv),(3.1), with property of b_0 and ψ ,it implies that

$$\sum_{i=1}^{r}\lambda_i f_i(x,v) \leqq \sum_{i=1}^{r}\lambda_i f_i(x,y) \tag{3.8}$$

Combining (3.7) and (3.8), we get

$$\sum_{i=1}^{r}\lambda_i f_i(x,y) \geqq \sum_{i=1}^{r}\lambda_i f_i(u,v)$$

3.1 MOND-WEIR TYPE SYMMETRIC DUALITY

We consider here the following pair of multiobjective mathematical programs and establish weak, strong and converse duality theorems.

(MP)

$Minimize\, f_i(x,y)$

subject to

$$\sum_{i=1}^{r}\lambda_i\nabla_y f_i(x,y) \leqq 0 \tag{3.9}$$

$$y^T\sum_{i=1}^{r}\lambda_i[\nabla_y f_i(x,y)] \geqq 0 \tag{3.10}$$

$$\lambda>0,\ \ \lambda^T e=1 \tag{3.11}$$

$$x \geqq 0 \tag{3.12}$$

(MD)

$Maximize\, f_i(u,v)$

subject to

$$\sum_{i=1}^{r}\lambda_i\nabla_u f_i(u,v) \geqq 0 \tag{3.13}$$

$$u^T\sum_{i=1}^{r}\lambda_i\nabla_u f_i(u,v) \leqq 0 \tag{3.14}$$

$$\lambda > 0 \quad , \ \lambda^T e = 1 \tag{3.15}$$

$$v \geqq 0 \tag{3.16}$$

Here, $e(1,1,1,.,.,1)^T \in R$, $\lambda_i \in R, i = 1,2,.,.,r$ and $f_i, i = 1,2,.,.,r$ are twice differentiable function from $R^n \times R^n \to R$, and $b_i = R^n \times R^m \times R^n \times R^m \to R_+$

Theorem 3.1 (Weak duality):
Let $(x, y, \lambda, z_1, z_2,.,.,z_r)$ be a feasible solution of (MP) and $(u, v, \lambda, w_1, w_2,.,.,w_r)$ a feasible solution of (MD) and

(i) $\sum_{i=1}^{r} \lambda_i f_i(.,v)$ is (Φ_0, ρ)-pseudounivex at u for fixed v,

(ii) $\sum_{i=1}^{r} \lambda_i f_i(x,.)$ is (Φ_1, ρ)-pseudounicave at y for fixed x,

(iii) $\Phi_0(x,u;(\xi,\rho)) + u^T\xi \geqq 0$, where $\xi = \sum_{i=1}^{r} \lambda_i \nabla_u f_i(u,v)$, and

(iv) $\Phi_1(v, y;(\zeta,\rho)) + y^T\zeta \leqq 0$, for $\zeta = \sum_{i=1}^{r} \lambda_i \nabla_y f_i(x, y)$,

Then, $f_i(x, y) \not\leqq f_i(u,v)$.

Proof: Since $\sum_{i=1}^{r} \lambda_i f_i(.,v)$ be (Φ_0, ρ)-pseudounivex at u for fixed v, for $\lambda > 0$, we get facilitate

with the help of hypothesis (iii) and (3.14) with property of b_0 and ψ,

$$\sum_{i=1}^{r} \lambda_i f_i(x,v) - \sum_{i=1}^{r} \lambda_i [f_i(u,v) \geqq 0 \text{ for all } i = 1,2,.,.,r \tag{3.17}$$

$$\sum_{i=1}^{r} \lambda_i f_i(x,v) \geqq \sum_{i=1}^{r} \lambda_i f_i(u,v) \tag{3.18}$$

Now, $f_i(x,.)$ be (Φ_1, ρ) pseudounicavity assumption at y for fixed x, for $\lambda > 0$, we have,

we have, hypothesis (iv)and (3.10), with property of b_0 and ψ,it implies that

$$\sum_{i=1}^{r} \lambda_i f_i(x,v) - \sum_{i=1}^{r} \lambda_i [f_i(x, y) \leqq 0 \tag{3.19}$$

$$\sum_{i=1}^{r} \lambda_i f_i(x,v) \leqq \sum_{i=1}^{r} \lambda_i (f_i(x, y) \tag{3.20}$$

Combining (3.18) and (3.20), we get

$$\sum_{i=1}^{r} \lambda_i f_i(x, y) \geqq \sum_{i=1}^{r} \lambda_i f_i(u,v)$$

Theorem 3.2(Strong duality):
Let (x', y', λ') be a weak efficient solution for (MP) for fixed $\lambda = \lambda'$ in(MD), assume that (i) The set $\sum_{i=1}^{r} \lambda_i [\nabla_{yy} f_i]$ is $+$ ive or $-$ ive definite for all $i = 1,2,.,.,r$;

(ii) and the set $[\nabla_y f_1, \nabla_y f_2,.,.,\nabla_y f_r]$ for all $i = 1,2,.,.,r$; is linearly independent, such that (x', y', λ') is a feasible solution of (MD), $b_i(x', y', u', v') > 0, i = 1,2,.,.,r$, and the two objectives have the same values. Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (MP) and (MD), then (x', y', λ') is an efficient solution for (MD).

Proof: Let (x', y', λ') is a weak efficient solution for (MP), then it is weakly efficient solution. Hence, there exist $\alpha \in R^r$, $\beta \in R^r$, $\gamma \in R^r$, $\mu \in R^r$ and $n \in R$ not all zero, $i = 1, 2, ., ., r$ such that the following Fritz john

optimality condition (28) are satisfied at (x', y', λ').

$$\alpha_i \nabla_x f_i + (\beta - ny')^T \lambda_i (\nabla_{yy} f_i) = s \tag{3.21}$$

$$\sum_{i=1}^{r} (\alpha_i - n\lambda_i) \nabla_y f_i + (\beta - ny')^T \lambda' (\nabla_{yy} f_i) = 0 \text{, for all } i = 1, 2, ., ., r. \tag{3.22}$$

$$(\beta - ny')^T \lambda' \nabla_y f_i - \mu_i = 0 \text{, for all } i = 1, 2, ., ., r. \tag{3.23}$$

$$\beta^T \sum_{i=1}^{r} \lambda' \nabla_y f_i = 0 \tag{3.24}$$

$$\Rightarrow ny'^T \sum_{i=1}^{r} \lambda' \nabla_y f_i = 0 \tag{3.25}$$

$$\mu^T \lambda' = 0 \tag{3.26}$$

$$(\alpha, \beta, s, \lambda, \mu, n) \geq 0, \tag{3.27}$$

$$(\alpha, \beta, s, \lambda, \mu, n) \neq 0, \lambda' > 0 \tag{3.28}$$

and $\mu \geq 0$,(3.26) implies $\mu = 0$. Consequently, (3.23) gives

$$(\beta - ny')^T \lambda' \nabla_y f_i = 0 \tag{3.29}$$

$$(\beta - ny')^T \lambda' [\nabla_{yy} f_i](\beta - ny') = 0 \tag{3.30}$$

hence, in the view of (i), $\beta = ny'$, (3.31)

From (3.22) and (3.31), $\sum_{i=1}^{r} (\alpha_i - n\lambda_i) \nabla_y f_i = 0$. (3.32)

assumption (ii) and (3.32), gives

$$\alpha_i = n\lambda_i \text{ for all } i = 1, 2, ., ., r. \tag{3.33}$$

If, $n = 0, \Rightarrow \alpha_i = 0, \beta = 0, \mu_i = 0, s = 0,$ for all $i = 1, 2, ., ., r$,

$$(\alpha, \beta, s, \lambda, \mu, n) = 0, n > 0.$$

Then we obtain $(\alpha, \beta, s, \lambda, \mu, n) = 0$, which contradicts(3.28), hence $n > 0$.

from (3.33) $\lambda' > 0$ we have $\alpha_i > 0$, $i = 1, 2, ., ., r$. From (3.21), (3.31)and (3.33)we get ,

$$\alpha_i \nabla_x f_i = s / n \geq 0 \tag{3.34}$$

By (3.27) and (3.31)

Since, $\eta > 0$, we have,

$$y' = \beta / n \geq 0 \tag{3.35}$$

From (3.34) ,it follows that $\alpha_i \nabla_x f_i = 0$ (3.36)

From ((3.34)- (3.36)),we know that (x', y', λ') is feasible for (MD).

$$f_i(x', y') = f_i(x', y') \tag{3.37}$$

and the objective values of (MD) and (MP) are equal.

We claim that (x', y', λ') is an efficient solution for (MD) for if it is not true , then ,there would exist

(u,v,λ') feasible for (MD) such that $f_i(u,v) \not\leq f_i(x',y')$, $i = 1,2,.,.,r$ (3.38)

Using equality(3.38) a contradiction(Weak Duality Theorem3.2)is obtained.

If (x',y',λ') is improperly efficient, then for every scalar M > 0, there exist a feasible solution (u',v',λ') in (MD) and an index i such that

$$f_i(u,v) - f_i(x',y') > M[f_j(x',y') - f_j(u,v)] \text{ for all j satisfying} \quad (3.39)$$

$$f_j(x',y') > f_j(u,v) \quad (3.40)$$

Whenever $f_i(u,v) > f_i(x',y')$ (3.41)

It implies that

$f_i(u,v) > f_i(x',y')$ can be made arbitrarily large and hence for λ' with $\lambda_i' > 0$, we have

$$\sum_{i=1}^{r} \lambda_i' f_i(u,v) > \sum_{i=1}^{r} \lambda_i' f_i(x',y') \quad (3.42)$$

Which contradicts the weak duality theorem 3.2.

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