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Curvature Tensors of Relativistic Significance in a LP-Sasakian Manifold

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ABSTRACT

In this paper after deriving certain important identities for curvature and Ricci tensors in LP-Sasakian manifold V_n, the four curvature tensors ϕ_1, ϕ_2, ϕ_3 and ϕ_4 of relativistic significance defined by Pokhariyal in 1970, 1982 have been considerd and certain useful results have been obtained.

Keywords: *$C^\infty -$ manifold, LP-Sasakian manifold, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 –curvature tensors.*

Mathematics subject classification: *[53].*

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1. INTRODUCTION

We consider a $n-$dimensional C^∞-manifold V_n. Let there exist in V_n, a tensor F of the type (1,1), a vector field U, a $1-$form u and a Riemannian metric g satisfying

$$\bar{\bar{X}} = X + u(X)U, \tag{1.1}$$

$$u(\bar{X}) = 0, \tag{1.2}$$

$$g(\bar{X}, \bar{Y}) = g(X, Y) + u(X)u(Y), \tag{1.3}$$

$$g(X, U) = u(X), \tag{1.4}$$

$$(D_X F)(Y) = \{g(X, Y) + u(X)u(Y)\}U + \{X + u(X)U\}u(Y), \tag{1.5}$$

$$D_X U = \bar{X} \tag{1.6}$$

where

$$F(X) \stackrel{\text{def}}{=} \bar{X}$$

for arbitrary vector fields X, Y. Then V_n satisfying (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) is called a Lorentzian para – Sasakian manifold [2] (in short LP-Sasakian manifold) while the set $\{F, U, u, g\}$ satisfying (1.1) to (1.6) is called a LP-Sasakian structure. It may be noted that D is the Riemannian connexion with respect to the Riemannian metric g.

In a LP-Sasakian manifold, it is easy to calculate that

$$(U) = -1, \tag{1.7a}$$

$$\bar{U} = 0 \tag{1.7b}$$

and

$$rank(F) = n - 1. \tag{1.7c}$$

We define in V_n, a fundamental 2-form $'F$ as

$$'F(X, Y) \stackrel{\text{def}}{=} g(\bar{X}, Y) = g(X, \bar{Y}). \tag{1.8}$$

Then it is easy to calculate that

$$'F(X, Y) = {'F}(Y, X) \tag{1.9}$$

and

$$'F(\bar{X}, \bar{Y}) = {'F}(X, Y). \tag{1.10}$$

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It may be noted that (1.9) and (1.10) imply that $'F$ is symmetric and hybrid.
We also have

$$'F(X,Y) = (D_X u)(Y). \tag{1.11}$$

We know that [1]

$$'K(X,Y,Z,T) = g(Y,Z)g(X,T) - g(X,Z)g(Y,T) \tag{1.12}$$

and

$$Ric(Y,Z) = (n-1)g(Y,Z). \tag{1.13}$$

Replacing T by U in (1.12) and using (1.4), we get

$$'K(X,Y,Z,U) = g(Y,Z)u(X) - g(X,Z)u(Y). \tag{1.14}$$

Replacing X by U in (1.14) and using (1.4) and (1.7a), we get

$$'K(U,Y,Z,U) = -g(Y,Z) - u(Y)u(Z). \tag{1.15}$$

Replacing X by U in (1.12) and using (1.4), we get

$$'K(U,Y,Z,T) = g(Y,Z)u(T) - u(Z)g(Y,T). \tag{1.16}$$

Putting Y equal to U in (1.12) and using (1.4), we get

$$'K(X,U,Z,T) = u(Z)g(X,T) - u(T)g(X,Z). \tag{1.17}$$

Replacing Y by T and T by U in (1.12) and using (1.14), we get

$$'K(X,T,Z,U) = g(T,Z)u(X) - g(X,Z)u(T). \tag{1.18}$$

Replacing Y by Z, Z by T and T by U in (1.12), we get

$$'K(X,Z,T,U) = g(Z,T)u(X) - g(X,T)u(Z). \tag{1.19}$$

(1.12) implies

$$K(X,Y,Z) = g(Y,Z)X - g(X,Z)Y. \tag{1.20}$$

Replacing Z by U in (1.13) and using (1.4), we get

$$Ric(Y,U) = (n-1)u(Y). \tag{1.21}$$

Replacing X by U in (1.20) and using (1.4), we get

$$K(U,Y,Z) = g(Y,Z)U - u(Z)Y. \tag{1.22}$$

Replacing Z by U in (1.20) and using (1.4), we get

$$K(X,Y,U) = u(Y)X - u(X)Y. \tag{1.23}$$

Barring Y and Z in (1.13) and using (1.3), we get

$$Ric(\bar{Y},\bar{Z}) = (n-1)[g(Y,Z) + u(Y)u(Z)]. \tag{1.24a}$$

Using (1.13) in (1.24a), we get

$$Ric(\bar{Y},\bar{Z}) = Ric(Y,Z) + (n-1)u(Y)u(Z). \tag{1.24b}$$

Replacing Z by $\bar{Z}$ in (1.24) and using (1.1) and (1.2), we get

$$Ric(\bar{Y},\ Z + u(Z)U) = (n-1)\{g(Y,\bar{Z})\}$$

which is equivalent to

$$Ric(\bar{Y},Z) + u(Z)Ric(\bar{Y},U) = (n-1)g(Y,\bar{Z}). \tag{1.25}$$

Barring Y in (1.21) and using (1.2), we get

$$Ric(\bar{Y},U) = 0. \tag{1.26}$$

Barring Z in (1.13), we get

$$Ric(Y,\bar{Z}) = (n-1)g(Y,\bar{Z}). \tag{1.27}$$

From (1.25), (1.26) and (1.27), we get

$$Ric(\bar{Y},Z) = Ric(Y,\bar{Z}). \tag{1.28}$$

Let us consider in V_n the tensors ϕ_1, ϕ_2, ϕ_3 and ϕ_4 of the type (0, 4) [3] [4] defined as follow:

$$\phi_1(X,Y,Z,T) \stackrel{\text{def}}{=} {}'K(X,Y,Z,T) + \tfrac{1}{n-1}[g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)], \tag{1.29}$$

$$\phi_2(X,Y,Z,T) \stackrel{\text{def}}{=} {}'K(X,Y,Z,T) + \tfrac{1}{n-1}[g(X,Z)Ric(Y,T) - g(Y,Z)Ric(X,T)], \tag{1.30}$$

$$\phi_3(X,Y,Z,T) \stackrel{\text{def}}{=} {}'K(X,Y,Z,T) + \tfrac{1}{n-1}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z)], \tag{1.31}$$

$$\phi_4(X,Y,Z,T) \stackrel{\text{def}}{=} {}'K(X,Y,Z,T) + \tfrac{1}{n-1}[g(X,Z)Ric(Y,T) - g(X,Y)Ric(Z,T)]. \tag{1.32}$$

It is obvious that for an empty gravitational field characterized by $Ric(X,Y) = 0$, the above tensors of the type (0, 4) are identical.

2. ϕ_1 − CURVATURE TENSOR

We observe that ϕ_1 is skew-symmetric in X and Y. Therefore breaking $\phi_1$into symmetric and skew-symmetric parts with respect to Z and T, we have

$$\alpha_1(X,Y,Z,T) \stackrel{\text{def}}{=} \tfrac{1}{2}[\phi_1(X,Y,Z,T) + \phi_1(X,Y,T,Z)] \tag{2.1}$$

$$= \frac{1}{2(n-1)}[g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z) + g(X,Z)Ric(Y,T) - g(Y,Z)Ric(X,T)]$$

and

$$\beta_1(X,Y,Z,T) \stackrel{\text{def}}{=} \tfrac{1}{2}[\phi_1(X,Y,Z,T) - \phi_1(X,Y,T,Z)] \tag{2.2}$$

$$= {}'K(X,Y,Z,T) + \frac{1}{2(n-1)}[g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z) - g(X,Z)Ric(Y,T) + g(Y,Z)Ric(X,T)]$$

where α_1 and β_1 are the symmetric and skew-symmetric parts of ϕ_1.

Theorem (2.1): In V_n**,** we have

$$\phi_1(X,Y,Z,U) = 2\big(u(X)g(Y,Z) - u(Y)g(X,Z)\big) = \beta_1(X,Y,Z,U), \tag{2.3a}$$

$$\phi_1(U,Y,Z,T) = 2\big(u(T)g(Y,Z) - u(Z)g(Y,T)\big) = \beta_1(U,Y,Z,T), \tag{2.3b}$$

$$\phi_1(U,Y,Z,U) = -2g(\bar{Y},\bar{Z}) = \beta_1(U,Y,Z,U) \tag{2.3c}$$

and

$$\alpha_1(X,Y,Z,U) = 0 = \beta_1(U,Y,Z,T). \tag{2.3d}$$

Proof: Replacing *T* by *U* in (1.29) and using (1.13), (1.16), we get

$$\phi_1(X,Y,Z,U) = 2\big(u(X)g(Y,Z) - u(Y)g(X,Z)\big). \tag{2.4}$$

Replacing *T* by *U* in (2.2) and using (1. 4), (1.13) and (1.14), we get

$$\beta_1(X,Y,Z,U) = 2\big(u(X)g(Y,Z) - u(Y)g(X,Z)\big). \tag{2.5}$$

From (2.4) and (2.5), we get (2.3a).

Replacing *X* by *U* in (1.29) and using (1. 4), (1.13) and (1.14), we get

$$\phi_1(U,Y,Z,T) = 2\big(g(Y,Z)u(T) - u(Z)g(Y,T)\big). \tag{2.6}$$

Replacing *X* by *U* in (2.2) and using (1.4), (1.13) and (1.16), we get

$$\beta_1(U,Y,Z,T) = 2(u(T)g(Y,Z) - u(Z)g(Y,T)). \tag{2.7}$$

From (2.6) and (2.7), we get (2.3b).

Replacing *X* by *U* in (2.3a) and using (1.3) and (1.7a), we get (2.3c).

Replacing *T* by *U* in (2.1) and using (1.4), (1.13) and (1.21), we get

$$\alpha_1(X,Y,Z,U) = 0. \tag{2.8}$$

Replacing X by U in (2.1) and using (1.4), (1.14) and (1.21), we get
$$\alpha_1(U,Y,Z,T) = 0. \tag{2.9}$$

From (2.8) and (2.9), we get (2.3d).

Theorem (2.2): In V_n, we have
$$\phi_1(X,U,Z,T) = 2\big(u(Z)g(X,T) - u(T)g(X,Z)\big), \tag{2.10a}$$

$$u(Z)\,\phi_1(U,Y,T,U) - u(T)\,\phi_1(U,Y,Z,U) = 2\big(g(Y,Z)u(T) - g(Y,T)u(Z)\big), \tag{2.10b}$$

$$\phi_1(X,U,Z,T) + \phi_1(X,T,Z,U) + \phi_1(X,Z,T,U) = 4\big(g(Z,T)u(X) - g(Z,X)u(T)\big). \tag{2.10c}$$

Proof: Replacing Y by U in (1.29) and using (1.4), we get
$$\phi_1(X,U,Z,T) = \ 'K(X,U,Z,T) + \tfrac{1}{n-1}[g(X,T)Ric(U,Z) - u(T)Ric(X,Z)]. \tag{2.11}$$

Using (1.17), (1.18) and (1.21) in above, we get (2.10a).

Replacing T by U in (2.3b) and using (1.4) and (1.7a), we get
$$\phi_1(U,Y,Z,U) = -2\big(g(Y,Z) + u(Y)u(Z)\big). \tag{2.12}$$

Interchanging T and Z in (2.3b), we get
$$\phi_1(U,Y,T,Z) = 2\big(u(Z)g(Y,T) - u(T)g(Y,Z)\big).$$

Replacing Z by U in above and using (1.4) and (1.7a), we get
$$\phi_1(U,Y,T,U) = -2\big(g(Y,T) + u(T)u(Y)\big). \tag{2.13}$$

Multiply (2.13) by $u(Z)$ and (2.12) by $u(T)$ and then subtracting, we get (2.10b).

Replacing Y by T in (2.3a), we get
$$\phi_1(X,T,Z,U) = 2\big(u(X)g(T,Z) - u(T)g(X,Z)\big). \tag{2.14}$$

Replacing Y by Z and Z by T in (2.3a), we get
$$\phi_1(X,Z,T,U) = 2\big(u(X)g(Z,T) - u(Z)g(X,T)\big). \tag{2.15}$$

Adding (2.10a), (2.14) and (2.15), we get (2.10c).

Corollary (2.1): In V_n**,** we have
$$\phi_1(\bar{X},\bar{Y},Z,U) = 0 = \phi_1(U,Y,\bar{Z},\bar{T}), \tag{2.16a}$$

$$\phi_1(U,\bar{Y},Z,U) - \phi_1(U,Y,\bar{Z},U) = 0, \tag{2.16b}$$

$$\phi_1(U,Y,\bar{Z},U)u(T) + \phi_1(U,Y,\bar{Z},T) = 0. \tag{2.16c}$$

Proof: Barring X and Y in (2.3a) and using (1.2), we get
$$\phi_1(\bar{X},\bar{Y},Z,U) = 0. \tag{2.17}$$

Barring Z and T in (2.3b) and using (1.2), we get
$$\phi_1(U,Y,\bar{Z},\bar{T}) = 0. \tag{2.18}$$

From (2.17) and (2.18), we get (2.16a).

Barring Y in (2.3c) and using (1.1), (1.2), (1.4) and (1.8), we get
$$\phi_1(U,\bar{Y},Z,U) = -2g\ 'F(Y,Z). \tag{2.19}$$

Barring Z in (2.3c) and using (1.1), (1.2), (1.4) and (1.8), we get
$$\phi_1(U,Y,\bar{Z},U) = -2\ 'F(Y,Z). \tag{2.20}$$

Subtracting (2.20) from (2.19), we get (2.16b).

Barring Z in (2.3b) and using (1.2) and (1.8), we get

$$\phi_1(U,Y,\bar{Z},T) = 2u(T)\,'F(Y,Z). \tag{2.21}$$

Multiplying (2.20) by $u(T)$ and then adding (2.21), we get (2.16c).

Theorem (2.3): In V_n, we have

$$\alpha_1(U,Y,Z,U) = 0, \tag{2.22a}$$

$$\alpha_1(U,Y,Z,T) = 0, \tag{2.22b}$$

$$\beta_1(U,Y,\bar{Z},\bar{T}) = 0. \tag{2.22c}$$

Proof: Replacing X and T both by U in (2.1), we get

$$\alpha_1(U,Y,Z,U) = \frac{1}{2(n-1)}[g(U,U)Ric(Y,Z) - g(Y,U)Ric(U,Z) + g(U,Z)Ric(Y,U) - g(Y,Z)R(U,U)].$$

Using (1.4), (1.7a), (1.13) and (1.21) in above, we get (2.22a).

Replacing X by U in (1.29), we get

$$\alpha_1(U,Y,Z,T) = \frac{1}{2(n-1)}[u(T)Ric(Y,Z) - g(Y,T)Ric(U,Z) + g(U,Z)Ric(Y,T) - g(Y,Z)Ric(U,T)].$$

Using (1.4), (1.13) and (1.21) in above, we get (2.22b).

Barring Z and T in (2.7) and using (1.2), we get (2.22c).

Corollary (2.2): In V_n, we have

$$2\alpha_1(X,Y,Z,U) + \phi_1(X,Y,Z,U) = 2\big(u(X)g(Y,Z) - u(Y)g(X,Z)\big), \tag{2.23a}$$

$$2\alpha_1(\bar{X},Y,Z,U) + \phi_1(\bar{X},Y,Z,U) = -2u(Y)g(X,Z), \tag{2.23b}$$

$$2\alpha_1(X,\bar{Y},Z,U) + \phi_1(X,\bar{Y},Z,U) = 2u(X)g(Y,Z). \tag{2.23c}$$

Proof: From (2.3a) and (2.3d), we get (2.23a).

Barring X in (2.23a) and using (1.2), we get (2.23b).

Further barring Y in (2.23a) and using (1.2), we get (2.23c).

3. ϕ_2-CURVATURE TENSOR

We observe that $\phi_2(X,Y,Z,T)$ is skew-symmetric in (X,Y). Therefore breaking ϕ_2 into symmetric and skew-symmetric parts with respect to Z and T, we get

$$\alpha_2(X,Y,Z,T) \overset{\text{def}}{=} \frac{1}{2}[\phi_2(X,Y,Z,T) + \phi_2(X,Y,T,Z)] \tag{3.1}$$

$$= \frac{1}{2(n-1)}[g(X,Z)Ric(Y,T) - g(Y,Z)Ric(X,T) + g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)]$$

and

$$\beta_2(X,Y,Z,T) \overset{\text{def}}{=} \frac{1}{2}[\phi_2(X,Y,Z,T) - \phi_2(X,Y,T,Z)] \tag{3.2}$$

$$= {}'K(X,Y,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z) + g(Y,T)Ric(X,Z)]$$

where α_2 and β_2 are the symmetric and the skew-symmetric parts of ϕ_2.

Theorem (3.1): In V_n, we have

$$\phi_2(X,Y,Z,U) = 0, \tag{3.3a}$$

$$\phi_2(X,U,Z,T) + \phi_2(X,T,Z,U) + \phi_2(X,Z,T,U) = 0. \tag{3.3b}$$

Proof: Replacing T by U in (1.30), we get

$$\phi_2(X,Y,Z,U) = {}'K(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(Y,Z)Ric(X,U)].$$

Using equations (1.14) and (1.21) in above, we get (3.3a).

Replacing Y by U in (1.30), we get

$$\phi_2(X,U,Z,T) = {}'K(X,U,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(U,T) - g(U,Z)Ric(X,T)].$$

Using (1.4), (1.17) and (1.21) in above, we get

$$\phi_2(X,U,Z,T) = 0. \tag{3.4}$$

Replacing Y by T and T by U in (1.30), we get

$$\phi_2(X,T,Z,U) = {}'K(X,T,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(T,U) - g(T,Z)Ric(X,U)].$$

Using (1.18) and (1.21) in above, we get

$$\phi_2(X,T,Z,U) = 0. \tag{3.5}$$

Replacing Y by Z, Z by T and T by U in (1.30), we get

$$\phi_2(X,Z,T,U) = {}'K(X,Z,T,U) + \frac{1}{n-1}[g(X,T)Ric(Z,U) - g(Z,T)Ric(X,U)].$$

Using (1.19) and (1.21) in above, we get

$$\phi_2(X,Z,T,U) = 0. \tag{3.6}$$

Adding (3.4), (3.5) and (3.6), we get (3.3b).

Theorem (3.2): In V_n, we have

$$\alpha_2(X,Y,Z,U) = 0, \tag{3.7a}$$

$$\alpha_2(X,Y,Z,U) + \phi_2(X,Y,Z,U) = 0 \tag{3.7b}$$

and

$$\beta_2(X,Y,Z,U) = u(Y)g(X,Z) - u(X)g(Y,Z). \tag{3.7c}$$

Proof: Replacing T by U in (3.1), we get

$$\alpha_2(X,Y,Z,U) = \frac{1}{2(n-1)}[g(X,Z)Ric(Y,U) - g(Y,Z)Ric(X,U) + g(X,U)Ric(Y,Z) - g(Y,U)Ric(X,Z)].$$

Using (1.4), (1.13) and (1.21) in above, we get (3.7a).

From (3.3a) and (3.7a), we get (3.7b).

Replacing T by U in (3.2), we get

$$\beta_2(X,Y,Z,U) = {}'K(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(Y,Z)Ric(X,U) - g(X,U)Ric(Y,Z) + g(Y,U)Ric(X,Z)].$$

Using (1.4), (1.13), (1.14) and (1.21) in above, we get (3.7c).

Corollary (3.1): In V_n, we have

$$\beta_2(\bar{X},Y,Z,U) = u(Y)g(X,Z), \tag{3.8a}$$

$$\beta_2(X,\bar{Y},Z,U) = -u(X)g(Y,Z), \tag{3.8b}$$

$$\beta_2(\bar{X},\bar{Y},Z,U) = 0, \tag{3.8c}$$

$$\beta_2(\bar{X}, Y, \bar{Z}, U) = u(Y)\ 'F(X, Z), \tag{3.8d}$$

$$\beta_2(X, \bar{Y}, \bar{Z}, U) = -u(X)\ 'F(Y, Z) \tag{3.8e}$$

and

$$\beta_2(\bar{X}, Y, Z, U) + \beta_2(X, \bar{Y}, Z, U) = -'K(X, Y, Z, U). \tag{3.8f}$$

Proof: Replacing X by $\bar{X}$ in (3.7c) and using (1.2), we get (3.8a).

Replacing Y by $\bar{Y}$ in (3.8a) and using (1.2), we get (3.8b).

Barring X in (3.8b) (or Y in (3.8a)) and using (1.7b), we get (3.8c).

Barring Z in (3.8a) and using (1.8), we get (3.8d).

Barring Z in (3.8b) and using (1.8), we get (3.8e).

Adding (3.8a) and (3.8b) and using (1.14), we get (3.8f).

4. ϕ_3_CURVATURE TENSOR

Breaking ϕ_3 into the symmetric and skew-symmetric parts with respect to X and Y, we get

$$\alpha_3(X, Y, Z, T) \overset{\text{def}}{=} \frac{1}{2}[\phi_3(X, Y, Z, T) + \phi_3(Y, X, Z, T)] \tag{4.1}$$

$$= \frac{1}{2(n-1)}[g(X, Z)Ric(Y, T) - g(Y, T)Ric(X, Z) + g(Y, Z)Ric(X, T) - g(X, T)Ric(Y, Z)]$$

$$\beta_3(X, Y, Z, T) \overset{\text{def}}{=} \frac{1}{2}[\phi_3(X, Y, Z, T) - \phi_3(Y, X, Z, T)] \tag{4.2}$$

$$= 'K(X, Y, Z, T) + \frac{1}{2(n-1)}[g(Y, Z)Ric(X, T) - g(Y, T)Ric(X, Z) - g(X, Z)Ric(Y, T) + g(X, T)Ric(Y, Z)]$$

where α_3 and β_3 are the symmetric and skew-symmetric parts of ϕ_3 with respect to X, Y.

Theorem (4.1): In V_n, we have

$$\phi_3(X, Y, Z, T) = \phi_1(X, Y, Z, T). \tag{4.3}$$

Proof: Using (1.12), (1.13) in (1.29), we get

$$\phi_1(X, Y, Z, T) = 2\big(g(Y, Z)g(X, T) - g(X, Z)g(Y, T)\big). \tag{4.4}$$

Using (1.12), (1.13) in (1.31), we get

$$\phi_3(X, Y, Z, T) = 2\big(g(Y, Z)g(X, T) - g(X, Z)g(Y, T)\big). \tag{4.5}$$

From (4.4) and (4.5), we get (4.3).

Remark (4.1): In view of (4.3) the study of curvature tensors ϕ_1 and ϕ_3 are identical in V_n.

5. ϕ_4_CURVATURE TENSOR

Breaking ϕ_4 into symmetric and skew-symmetric parts with respect to X and Y, we get

$$\alpha_4(X, Y, Z, T) \overset{\text{def}}{=} [\phi_4(X, Y, Z, T) + \phi_4(Y, X, Z, T)] \tag{5.1}$$

$$= \frac{1}{2(n-1)}[g(X, Z)Ric(Y, T) + g(Y, Z)Ric(X, T) - 2g(X, Y)Ric(Z, T)]$$

and

$$\beta_4(X, Y, Z, T) \overset{\text{def}}{=} \frac{1}{2}[\phi_4(X, Y, Z, T) - \phi_4(Y, X, Z, T)]. \tag{5.2}$$

$$= 'K(X, Y, Z, T) + \frac{1}{2(n-1)}[g(X, Z)Ric(Y, T) - g(Y, Z)Ric(X, T)].$$

Theorem (5.1): In V_n, we have

$$\phi_4(X, Y, Z, U) = u(X)g(Y, Z) - g(X, Y)u(Z), \tag{5.3a}$$

$$\phi_4(U,Y,Z,T) = g(Y,Z)u(T) - u(Y)g(Z,T), \tag{5.3b}$$

$$\phi_4(U,Y,Z,U) = -g(Y,Z) - u(Y)u(Z). \tag{5.3c}$$

Proof: Replacing T by U in (1.32), we get

$$\phi_4(X,Y,Z,U) = {}'K(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(X,Y)Ric(Z,U)].$$

Using (1.14) and (1.21) in above, we get (5.3a).

Replacing X by U in (5.3a) and using (1.4) and (1.7a), we get (5.3c).

Replacing X by U in (1.32), we get

$$\phi_4(U,Y,Z,T) = {}'K(U,Y,Z,T) + \frac{1}{n-1}[g(U,Z)Ric(Y,T) - g(U,Y)Ric(Z,T)].$$

Using (1.3), (1.4) and (1.16) in above, we get (5.3b).

Theorem (5.2): In V_n, we have

$$\phi_4(X,U,Z,T) = u(Z)g(X,T) - u(X)g(Z,T), \tag{5.4a}$$

$$u(Z)\,\phi_4(U,Y,T,U) - u(T)\,\phi_4(U,Y,Z,U) = u(T)g(Y,Z) - u(Z)g(Y,T), \tag{5.4b}$$

$$\phi_4(X,U,Z,T) + \phi_4(X,T,Z,U) + \phi_4(X,Z,T,U) = u(X)g(Z,T) - u(T)g(X,Z). \tag{5.4c}$$

Proof: Replacing Y by U in (1.32), we get

$$\phi_4(X,U,Z,T) = {}'K(X,U,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(U,T) - g(X,U)Ric(Z,T)].$$

Using (1.4), (1.17) and (1.21) in above, we get (5.4a).

Replacing Z by T in (5.3c), we get

$$\phi_4(U,Y,T,U) = -g(Y,T) - u(Y)u(T).$$

Multiplying by $u(Z)$ in above, we get

$$u(Z)\,\phi_4(U,Y,T,U) = -u(Z)\big(g(Y,T) + u(Y)u(T)\big). \tag{5.5}$$

Multiplying by $u(T)$ in (5.3c), we get

$$u(T)\,\phi_4(U,Y,Z,U) = -u(T)\big(g(Y,Z) + u(Y)u(Z)\big). \tag{5.6}$$

Subtracting (5.6) from (5.5), we get (5.4b).

Replacing Y by T and T by U in (1.32), we get

$$\phi_4(X,T,Z,U) = {}'K(X,T,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(T,U) - g(X,T)Ric(Z,U)].$$

Using (1.18) and (1.21) in above, we get

$$\phi_4(X,T,Z,U) = u(X)g(T,Z) - u(Z)g(X,T). \tag{5.7}$$

Replacing Y by Z, Z by T and T by U in (1.32), we get

$$\phi_4(X,Z,T,U) = {}'K(X,Z,T,U) + \frac{1}{n-1}[g(X,T)Ric(Z,U) - g(X,Z)Ric(T,U)].$$

Using (1.19) and (1.21) in above, we get

$$\phi_4(X,Z,T,U) = g(Z,T)u(X) - g(X,Z)u(T). \tag{5.8}$$

Adding (5.4a), (5.7) in above, we get (5.4c).

Theorem (5.3): In V_n, we have

$$\alpha_4(U,Y,Z,U) = -\frac{1}{2}[g(Y,Z) + u(Y)u(Z)], \tag{5.9a}$$

$$\alpha_4(X,Y,Z,U) = \frac{1}{2}[g(X,Z)u(Y) - 2g(X,Y)u(Z) + g(Y,Z)u(X)] \tag{5.9b}$$

and

$$\alpha_4(U,Y,Z,T) = \frac{1}{2}[u(Z)g(Y,T) - 2u(Y)g(Z,T) + g(Y,Z)u(T)]. \tag{5.9c}$$

Proof: Replacing X and T both by U in (5.1) and using (1.4), (1.7a) and (1.21), we get (5.9a).

Replacing T by U in (5.1) and using (1.21), we get (5.9b).

Further replacing X by U in (5.1) and using (1.4) and (1.21), we get (5.9c).

Theorem (5.4): In V_n, we have

$$\beta_4(X,Y,Z,U) = \frac{1}{2}[g(Y,Z)u(X) - u(Y)g(X,Z)], \tag{5.10a}$$

$$\beta_4(U,Y,Z,T) = \frac{1}{2}[g(Y,Z)u(T) - u(Z)g(Y,T)], \tag{5.10b}$$

$$\beta_4(U,Y,Z,U) = -\frac{1}{2}[g(Y,Z) + u(Y)u(Z)]. \tag{5.10c}$$

Proof: Replacing T by U in (5.2) and using (1.14) and (1.21), we get (5.10a).

Replacing X by U in (5.2) and using (1.4), (1.13), (1.16) and (1.21), we get (5.10b).

Replacing X by U in (5.10a) and using (1.7a), we get (5.10c).

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