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FUZZY CHROMATIC NUMBER OF MIDDLE, SUBDIVISION AND TOTAL FUZZY GRAPH

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ABSTRACT

Fuzzy coloring of a fuzzy graph G is an assignment of colors to vertices of G. It is said to be proper if no two strong adjacent vertices have the same color. In this paper, fuzzy chromatic number of middle fuzzy graph of fuzzy cycle, fuzzy path, fuzzy star and complete fuzzy graph is given and also we find that the fuzzy chromatic number of subdivision fuzzy graph of any fuzzy graph is two. Fuzzy chromatic number of total fuzzy graph of fuzzy cycle, fuzzy path and fuzzy star is also determined.

Key words: *Strong edges, fuzzy cycle, fuzzy star, fuzzy path, complete fuzzy graph and fuzzy chromatic number.*

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1. INTRODUCTION

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975. It has many applications in real-life situations. In particular, fuzzy coloring plays a vital role in fuzzy graph theory. It is used to solve many problems like Traffic light control problem, Exam scheduling problem, Register allocation problem, Assignment problem etc. In this paper, fuzzy chromatic number of middle fuzzy graph of fuzzy cycle, fuzzy path, fuzzy star and fuzzy complete graph is given and also we find that the fuzzy chromatic number of subdivision fuzzy graph of any fuzzy graph is two. Fuzzy chromatic number of total fuzzy graph of fuzzy cycle, fuzzy path and fuzzy star is also determined.

2. PRELIMINARIES

A *fuzzy graph* G is a pair of functions G= (σ, μ) where σ: V→ [0, 1], V is a node(vertex) set and μ :V×V→ [0, 1], a symmetric fuzzy relation on σ. The *underlying crisp graph* of G = (σ, μ) is G* = (V, E), where E ⊆ V×V. *Strength of a path* in fuzzy graph G is the weight of the weakest arc in that path. A *weakest arc* is an arc of minimum weight in G. A *strongest path* between two nodes u, v is a path corresponding to maximum strength between u and v. The *strength* of the strongest path is denoted by μ^{∞} (u, v). An arc (x, y) is said to be a *strong arc* if $\mu^{\infty}(x, y) = \mu(x, y)$.

A cycle in a fuzzy graph is said to be *fuzzy cycle* if it contains more than one weakest arc. A fuzzy cycle C_n is said to be a strong fuzzy cycle if $\mu(x, y) = \sigma(x) \wedge \sigma(y), \forall (x, y) \in E$. A fuzzy graph G is said to be a *fuzzy path* of n vertices if its underlying crisp graph is a path of n vertices. A fuzzy graph G is said to be a *fuzzy star* $S_{1,n}$ if its underlying crisp graph is a star $K_{1,n}$. A fuzzy graph G is said to be *complete* if $\mu(x, y) = \sigma(x) \wedge \sigma(y), \forall\, x, y \in V$.

If an edge (x, y) is strong then x and y are *strong adjacent*. *Fuzzy coloring* of a fuzzy graph G is an assignment of colors to vertices of G. It is said to be *proper* if no two strong adjacent vertices have the same color. *Fuzzy chromatic number* of G is minimum number of colors needed for proper fuzzy coloring. Two nodes of a fuzzy graph G are said to be *fuzzy independent* if there is no strong arc between them. A subset S of V is said to be *fuzzy independent* of G if any two nodes of S are fuzzy independent. Two edges are fuzzy edge independent if they are strong &have a common vertex. A subset S of E is said to be fuzzy edge independent if any two edges of S are fuzzy edge independent.

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3. FUZZY CHROMATIC NUMBER OF MIDDLE FUZZY GRAPH

Definition 3.1: Let G = (σ, μ) be a fuzzy graph and its underlying crisp graph is G* = (V, E), where V is the vertex set and E is edge set. Then the *Middle fuzzy graph* of G is M (G) = (σ_M, μ_M) where σ_M is a fuzzy set on

'$V \cup E$ & is defined as σ_M (u) = σ (u), if u∈V
= μ (u), if u∈E
= 0 otherwise and

μ_M is a fuzzy relation & is defined as
$\mu_M (e_i, e_j) = \mu (e_i) \wedge \mu (e_j)$, if $e_i, e_j \in E$ & are adjacent in G*.
= 0 otherwise

$\mu_M (v_i, v_j) = 0$ if $v_i, v_j \in V$

$\mu_M (v_i, e_j) = \mu (e_j)$ if $v_i \in V$ and it lies on e_j.
= 0 otherwise

Theorem 3.2: If C_n is a fuzzy cycle of length n≥3 then

$$\chi_f\left(M\left(C_n\right)\right) = \begin{cases} 3, & if \quad n \quad is \quad even \\ 4, & if \quad n \quad is \quad odd \end{cases}$$

Proof: Let C_n be a fuzzy cycle of length n. Assume that $v_1, v_2 \ldots v_n$ are vertices and $e_1, e_2 \ldots e_n$ are edges of C_n. Since C_n is the fuzzy cycle, each v_i is strong adjacent to v_{i-1} & v_{i+1} (1≤i≤n, $v_0 = v_n$ and $v_{n+1} = v_1$). By the definition of middle fuzzy graph, each edge in M (C_n) is strong. The vertices of M (C_n) are $v_1, v_2 \ldots v_n, e_1, e_2 \ldots e_n$ and two vertices in M (C_n) are adjacent if any one of the following condition is satisfied.

i) The two vertices are in V.
ii) One of the vertex u is in V and another vertex e is in E and u lies on e.

Now we have to determine the fuzzy chromatic number of M (C_n) which is the minimum number of colors to color all the vertices of M (C_n) such that two strong adjacent vertices will receive different color. To find χ_f(M (C_n)), it is enough to find the minimum number of fuzzy independent sets whose union is $V \cup E$ & intersection is empty.

Since the vertex set of C_n {$v_1, v_2 \ldots v_n$} form a fuzzy independent set of M (C_n), assign color 1 to all vertices of this set. The remaining uncolored vertices of M (C_n) are $e_1, e_2 \ldots e_n$. This set can be partitioned in to two fuzzy independent sets, if n is even. They are {$e_1, e_3 \ldots e_{n-1}$} & {$e_2, e_4 \ldots e_n$}. So assign color 2 to the vertices of first fuzzy independent set and color 3 to vertices second fuzzy independent set. Since we cannot partition the set $V \cup E$ as the number of fuzzy independent set less than 3, χ_f(M (C_n)) = 3.

If n is odd, E can be partitioned in to three fuzzy independent sets. They are {$e_1, e_3 \ldots e_{n-2}$} & {$e_2, e_4 \ldots e_{n-1}$} & {e_n}. Hence χ_f(M (C_n)) = 4.

Theorem 3.3: Let P_n be a fuzzy path of n≥3 vertices. Then $\chi_f\left(M\left(P_n\right)\right) = 3.$

Proof: Let P_n be a fuzzy path with vertices $v_1, v_2 \ldots v_n$ and edges $e_1, e_2 \ldots e_n$. In M (P_n), each edge becomes a vertex and so M (P_n) has 2n–1 vertices. The membership value of each vertex is followed from the definition of middle fuzzy graph. If e = (x, y) is an edge in M (P_n) either if x & y are in V or x is in V& y is in E and x lies on e. The vertex set V of G form a fuzzy independent set of M (P_n). To find the remaining fuzzy independent, we have two cases.

Case-I: If n is odd, then the edge set E is partitioned in to two fuzzy independent sets. They are {$e_1, e_3 \ldots e_{n-2}$} & {$e_2, e_4 \ldots e_{n-1}$}.

Case-II: If n is even, then the two fuzzy independent sets are {$e_1, e_3 \ldots e_{n-1}$} & {$e_2, e_4 \ldots e_{n-2}$}.

Thus in two cases, we have three fuzzy independent sets of M (P_n). Hence χ_f(M (P_n)) = 3.

Theorem 3.4: If $S_{1,n}$ be a fuzzy star for $n\geq 3$, then $\chi_f(M(S_{1,n})) = n+1$.

Proof: Consider the fuzzy star $S_{1,n}$ whose middle vertex is u and remaining vertices are $u_1, u_2 \dots u_n$. Construct M ($S_{1,n}$) by using the definition of middle fuzzy graph. In M ($S_{1,n}$), there are 2n+1 vertices. They are u, $u_1, u_2 \dots u_n, e_1, e_2 \dots e_n$, where e_i's are edges of $S_{1,n}$. The sub graph of $S_{1,n}$ with vertices $e_1, e_2 \dots e_n$ form a complete fuzzy graph of n vertices. Since each vertex is strong adjacent to each of the remaining vertices in the complete fuzzy graph, we have to assign a unique color to these vertices so that n colors are needed to color the vertices $e_1, e_2 \dots e_n$. The vertex set V of $S_{1,n}$ form a fuzzy independent set of M ($S_{1,n}$). Hence $\chi_f(M(S_{1,n})) = n+1$.

Theorem 3.5: If G is a complete fuzzy graph on n vertices then $\chi_f\left(M(G)\right)=\begin{cases} n, & if \quad n \quad is \quad even \\ n+1, & if \quad n \quad is \quad odd \end{cases}$.

Proof: Let G be a complete fuzzy graph with n vertices $v_1, v_2 \dots v_n$ & $\frac{n(n-1)}{2}$ strong edges $e_1, e_2 \dots e_{\frac{n(n-1)}{2}}$. To find the fuzzy chromatic number of M (G), we have to construct it using the definition of middle fuzzy graph. Since each edge of G becomes a vertex in M (G), there are $\frac{n(n-1)}{2}+n=\frac{n^2-n+2n}{2}=\frac{n^2+n}{2}=\frac{n(n+1)}{2}$ vertices in M (G). Join two vertices of M (G) either if one vertex $u\in V$ & another vertex $e\in E$ and u lies on e or both vertices belongs to E &they are adjacent in G. The membership values of vertices and edges are given by using the definition of middle fuzzy graph. Since no two vertices of G are strong adjacent in M (G), the vertex set V of G form a fuzzy independent set of M (G).

Observe that, two vertices e_i & e_j (which are edges in G) are strong adjacent in M (G) iff they are strong edges of G and are strong adjacent edges in G and so each fuzzy edge independent set of G gives a fuzzy independent set of M (G). Now we have two cases.

Case-I: If the number of vertices in G is even, there are n–1 fuzzy edge independent set in G and hence M (G) have n–1+1 = n fuzzy independent sets. Since the fuzzy chromatic number of M (G) is the minimum number of fuzzy independent set of M (G), $\chi_f\left(M(G)\right)=n.$

Case-II: If n is odd, G have n fuzzy edge independent set. Thus M(G) has (n+ 1) fuzzy independent sets. Hence $\chi_f\left(M(G)\right)=n+1.$

4. FUZZY CHROMATIC NUMBER OF SUBDIVISION GRAPH

Definition 4.1: Let G = (σ,μ) be a fuzzy graph with its underlying crisp graph G* = (V, E). Then the *Subdivision fuzzy graph* of G is Sd (G) = (σ_{sd}, μ_{sd}),
where

$$\sigma_{sd}(u) = \sigma(u), \text{ if } u\in V$$
$$= \mu(u), \text{ if } u\in E$$
$$= 0 \text{ otherwise.}$$

$$\mu_{sd}(u, e) = \sigma(u) \wedge \sigma(e), \text{ if } v \in V, e\in E \text{ \& and u lies on e.}$$
$$= 0 \text{ otherwise}$$

Thus the vertex set of Sd (G) is $V\cup E$. Clearly Sd(G) is a fuzzy graph

Theorem 4.2: For any fuzzy graph G, $\chi_f\left(Sd(G)\right)=2.$.

Proof: Let G be a fuzzy graph with underlying crisp graph G* = (V, E). In Sd(G), each edge of G is subdivided and a new vertex and two vertices u, e are adjacent in Sd (G) if u is in V & e is in E and u lies on e. By using the definition of subdivision fuzzy graph, assign the membership values to both vertices and edges of Sd (G). Since any two vertices of G are fuzzy independent in Sd (G) and any two edges of G are fuzzy independent in Sd (G), vertex set V & edge set E form two fuzzy independent sets of Sd (G) and so vertex set $V\bigcup E$ of Sd (G) is partitioned in to two fuzzy independent set and hence Sd (G) is fuzzy bipartite graph. Therefore $\chi_f\left(Sd(G)\right)=2.$.

Theorem 4.3: For any connected fuzzy graph G, $\chi_f(G) \geq \chi_f(Sd(G))$..

Proof: Since the fuzzy chromatic number of a connected fuzzy graph is greater than or equal to 2, $\chi_f((G) \geq 2$. By theorem 4.2, $\chi_f(Sd(G)) = 2.$ Thus $\chi_f((G)) \geq 2 = \chi_f(Sd(G)).$

5. FUZZY CHROMATIC NUMBER OF TOTAL FUZZY GRAPH

Definition 5.1: Let G = (σ,μ) be a fuzzy graph and its underlying crisp graph is G* = (V, E), where V is the vertex set and E is edge set. Then the *Total fuzzy graph* of G is T (G) = (σ_T, μ_T) where σ_T is a fuzzy set on $V \cup E$ & is defined as

σ_T (u) = σ (u), if u∈V
= μ (u), if u∈E and
μ_T is a fuzzy relation & is defined as
μ_T (u, v) = μ (u, v) if u, v ∈V

μ_M (u, e) = σ (u)∧σ (e) if u ∈V and it lies on e∈E.
= 0 otherwise

μ_T (e_i, e_j) = μ (e_i) ∧ μ (e_j), if e_i, e_j ∈E & have a common vertex in G.
= 0 otherwise

Theorem 5.2: If C_3 is a fuzzy cycle of 3 vertices with $\mu(x,y) = \sigma(x) \Lambda \sigma(y), \ \forall (x,y) \in E_\tau$ then $\chi_f(T(C_3)) = 3.$

Proof: Let C_3be a fuzzy cycle of length 3 such that $\mu(x,y) = \sigma(x) \Lambda \sigma(y)$ for all edges of C_3. Suppose that v_1, v_2 & v_3 are vertices and e_1, e_2, e_3 are edges of C_3. Then T (C_3) has vertex set {v_1, v_2, v_3, e_1, e_2, e_3} and two vertices are adjacent if any one of the following conditions are hold.

i. Both vertices are in V
ii. One vertex u is in V & another vertex e is in E and u lies on e.
iii. Both vertices are in E

The membership values are assigned from the definition of total fuzzy graph. Since $\mu(x,y) = \sigma(x) \Lambda \sigma(y), \ \forall (x,y) \in E_\tau$, each edge of T ($C_3$) must be strong. The fuzzy independent sets of T (C_3) are {v_1, e_2}, {v_2, e_3} and {v_3, e_1}. So T (C_3) has proper 3-fuzzy coloring. Since we cannot use lesser then 3 colors to color all the vertices of T (C_3) in such a way that no two strong adjacent vertices have same color, $\chi_f(T(C_3)) = 3.$

Theorem 5.3: $\chi_f(T(C_n)) = 4.$, for n≥4 where C_n is a strong fuzzy cycle of length n.

Proof: Let C_n be a strong fuzzy cycle of length n≥4. By using the definition of middle fuzzy graph, construct T (C_n) and assign the membership values to both vertices and edges. Each vertex v_i (of G) is strong adjacent to v_{i-1}, v_{i+1}, e_{i-1} & e_i in T (C_n) and each edge e_i (of G) is strong adjacent to e_{i-1}, e_{i+1}, v_i & v_{i+1}, where v_0 (e_0) represents v_n (e_n) & v_{n+1} (e_{n+1}) represents v_1 (e_1). This implies that number of strong edges in all vertices is 4. Now we have two cases

Case-I: Let n be even. Then the fuzzy independent sets of C_n are {v_1, v_3 … v_{n-1}} & {v_2, v_4 … v_n}. These two sets are also the fuzzy independent sets of T (C_n), since no two vertices of the sets are strong adjacent. Similarly the edge set E of G is partitioned in to two fuzzy independent sets and hence T (C_n) has 4 fuzzy independent sets. Thus $\chi_f(T(C_n)) = 4.$

Case-II: If n is odd, we can find the following 4 fuzzy independent sets of T (C_n).

i. {v_1, v_3, v_5 … v_{n-2}, e_{n-1}}
ii. {v_2, v_4, v_6 … v_{n-1}, e_n}
iii. {v_n, e_1, e_3 … e_{n-4}, e_{n-2}}
iv. {e_2, e_4, e_6 … e_{n-5}, e_{n-3}}

Thus vertex set $V \cup E$ of T (C_n) is portioned into 4 fuzzy independent sets. Hence $\chi_f\left(T\left(C_n\right)\right)=4.$.

Theorem 5.4: If G is a fuzzy path of n vertices with $\mu(x,y)=\sigma(x)\Lambda\sigma(y),\ \forall(x,y)\in E_\tau$ then $\chi_f\left(T\left(G\right)\right)=3.$.

Proof: Let $v_1, v_2 \ldots v_n$ be the vertices and $e_1, e_2 \ldots e_{n-1}$ be the edges of G. In T (G), there are three fuzzy independent sets. We generalise the fuzzy independent sets in three different cases.

Case-I: n ≡ 0(mod 3) In this case, the three fuzzy independent sets of T (G) are

1. $\{v_1, v_4 \ldots v_{n-2}, e_2, e_5 \ldots e_{n-1}\}$
2. $\{v_2, v_5 \ldots v_{n-1}, e_3, e_6 \ldots e_{n-3}\}$
3. $\{v_3, v_6 \ldots v_n, e_1, e_4 \ldots e_{n-2}\}$.

Case-II: n ≡ 1(mod 3) If n≡1 (mod 3), the fuzzy independent sets are

1. $\{v_1, v_4 \ldots v_n, e_2, e_5 \ldots e_{n-2}\}$
2. $\{v_2, v_5 \ldots v_{n-2}, e_3, e_6 \ldots e_{n-1}\}$
3. $\{v_3, v_6 \ldots v_{n-3}, e_1, e_4 \ldots e_{n-3}\}$.

Case-III: n ≡2 (mod 3) In this case, we have the following three fuzzy independent sets.

1. $\{v_1, v_4 \ldots v_{n-1}, e_2, e_5 \ldots e_{n-3}\}$
2. $\{v_2, v_5 \ldots v_n, e_3, e_6 \ldots e_{n-2}\}$
3. $\{v_3, v_6 \ldots v_{n-2}, e_1, e_4 \ldots e_{n-1}\}$.

Since we cannot find the number of fuzzy independent sets lesser then 3 in T (G), $\chi_f\left(T\left(G\right)\right)=3.$.

Theorem 5.5: Let $S_{1,n}$ be a fuzzy star with $\mu(x,y)=\sigma(x)\Lambda\sigma(y),\ \forall(x,y)\in E$ then $\chi_f\left(T\left(S_{1,n}\right)\right)=n+2.$.

Proof: Let u be the middle vertex and $u_1, u_2 \ldots u_n$ be the remaining vertices of $S_{1,n}$. Then the vertex set of $S_{1,n}$ is partitioned in to two fuzzy independent sets of T ($S_{1,n}$). They are {u} & $\{u_1, u_2 \ldots u_n\}$. Since the edge set of $S_{1,n}$ form a complete fuzzy graph of n vertices in T ($S_{1,n}$), each edge of G will receive a unique color and hence there are n+2 fuzzy independent sets in T ($S_{1,n}$). Thus $\chi_f\left(T\left(S_{1,n}\right)\right)=n+2.$.

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