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NEW SUB-CLASSES OF BI-UNIVALENT FUNCTIONS DEFINED USING THE CONVOLUTION STRUCTURE

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ABSTRACT

I*n the present paper, we introduce and study a new subclass of analytic bi-univalent functions defined in the open unit disc using convolution. We determine estimates of the general Taylor-Maclaurin coefficients of the functions in this class subject to certain gap series as well as providing bounds for coefficients $|a_2|$ and $|a_3|$. For this purpose, we use the Faber polynomial approach. Also connections to earlier well-known results are briefly indicated.*

Mathematics Subject Classification: *30C45.*

Keywords: *Bi-univalent functions; Convolution, Faber polynomials, Taylor-Maclaurin series expansion.*

1. INTRODUCTION AND DEFINITIONS

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc U = $\{z : z \in C \;\; and \; |z| < 1\}$ and satisfy the normalization conditions $f(0) = f'(0) = 1$.

Let S be the class of A consisting of the functions of the form (1.1) which are also univalent in U. It is well known that every function f $\in$ S has an inverse f^{-1}, which is defined by

$f^{-1}(f(z)) = z (z \in U)$ and $f(f^{-1}(w)) = w$ for |w| < 1/4, according to Koebe one quarter theorem[14].

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1).

In 1967, Lewin [12] first investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and clunie [3] conjectured that $|a_2| \le \sqrt{2}$ .However Netanyahu [4] showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. Also, Ali *et.al* [15] remarked that finding the bounds for $|a_n|$ when n ≥ 4 is an open problem. This is because the bi-univalency condition imposed on the functions $f(z) \in A$ makes the behaviour of their coefficients unpredictable.

Recently, several researchers such as ([1, 2, 9, 16, 20]) obtained the coefficients $|a_2|$ and $|a_3|$ of bi-univalent functions for the various subclasses of the function class Σ.

S.G.Hamidi and J.M.Jahangiri [10] used Faber polynomial coefficient for finding the estimates on the coefficient bounds for the classes of bi-univalent functions. These bounds prove to be better than those estimates provided by Srivastava *et al* [9] and Frasin and Aouf [2]. Motivated by their work, we have used Faber polynomial approach to obtain the coefficient estimates of our new subclass of bi-univalent functions.

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The object of the present paper is to introduce a new subclass of the function class Σ and use the Faber polynomial approach to determine estimates for the general coefficient bounds. We also obtain estimates for the first two coefficients $|a_2|$ and $|a_3|$ of these functions.

Definition 1.1: Given a real α ($0 \leq \alpha < 1$), $\lambda \geq 1$ and functions $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ and $\psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$, analytic in U, such that $\phi_n \geq 0$, $\psi_n \geq 0$, we say that $f(z) \in \Sigma$ is in $H_\Sigma(\phi,\psi;\alpha,\lambda)$ if

$$\operatorname{Re}\left\{\frac{(1-\lambda)(f*\phi)(z) + \lambda(f*\psi)(z)}{z}\right\} > \alpha \text{ for all } z \in U.$$

Remark 1.1: The class $H_\Sigma(\phi,\psi;\alpha,\lambda)$, for suitable choices of ϕ and ψ lead to the following known classes of analytic bi-univalent functions studied earlier in the literature.

i) For $\phi(z) = h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $\psi(z) = \frac{z}{(1-z)^2} * h(z)$ we obtain the class $Q_\lambda(h,\alpha)$ defined and studied by R.M.El-Ashwah [16].

ii) For $\phi(z) = z + \sum_{n=2}^{\infty} n^\delta z^n$ and $\psi(z) = z + \sum_{n=2}^{\infty} (n)^{\delta+1} z^n$ we obtain the bi-univalent function class Q(δ,λ,α) studied by Saurabh Porwal, M.Darus[17].

iii) If we choose $\phi(z) = \left(z + \sum_{n=2}^{\infty} n^\delta z^n\right) * \frac{z}{(1-z)^{\delta+1}}$ and $\psi(z) = \left(z + \sum_{n=2}^{\infty} (n)^{\delta+1} z^n\right) * \frac{z}{(1-z)^{\delta+1}}$ we obtain the subclass Q(n,δ,α,λ) studied by A.G.Alamoush and M.Darus[1].

iv) For $\phi(z) = \frac{z}{1-z}$ and $\psi(z) = \frac{z}{(1-z)^2}$ we obtain the bi-univalent function class $Q_\lambda(\alpha)$ introduced by Ding *et al* [18].

The estimates for the coeffіcents $|a_2|$ and $|a_3|$ for this class of functions were obtained by B.A.Frasin and M.K.Aouf [2] employing the techniques used earlier by Srivastava et al [9] and also by Jay.M.Jahangiri and Samaneh G.Hamidi[10] using Faber Polynomial expansions.

2. COEFFICIENT BOUNDS FOR THE CLASS $H_\Sigma(\phi,\psi;\alpha,\lambda)$

Using the Faber Polynomial expansion of functions $f(z) \in A$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as [5],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)w^n$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4$$
$$+ \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6}\left[a_6 + (-2n+5)a_3 a_4\right] + \sum_{i\geq 7} a_2^{n-i} V_i,$$

such that V_i with $7 \leq i \leq n$ is a homogenous polynomial in the variables $a_2, a_3, ... a_n$[6].

In particular, the first three terms of K_{n-1}^{-n} are [see, 5]

$$\frac{1}{2} K_1^{-2} = -a_2$$
$$\frac{1}{3} K_2^{-3} = 2a_2^2 - a_3$$
$$\frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4)$$

In general, for any p $\in$ N, an expansion of K_n^p is as, [5, page183]

$$K_n^p = pa_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + ... + \frac{p!}{(p-n)!n!} D_n^n$$

where

$D_n^p = D_n^p(a_2, a_3, ...)$

and by [13] or [8],

$$D_n^m(a_1, a_2, ... a_n) = \sum_{m=1}^{\infty} \frac{m!(a_1)^{\mu_1} ... (a_n)^{\mu_n}}{\mu_1! ... \mu_n!}$$

while a_1=1, and the sum is taken over all non negative integers $\mu_1, ..., \mu_n$ satisfying

$\mu_1 + \mu_2 + ... + \mu_n = m$,

$\mu_1 + 2\mu_2 + ... + n\mu_n = n$

It is clear that

$D_n^n(a_1, a_2, ... a_n) = a_1^n$ [7].

Theorem 2.1: For $(0 \le \alpha < 1)$ and $\lambda \ge 1$ let $f(z) \in H_\Sigma(\phi, \psi; \alpha, \lambda)$) and $g(z) \in H_\Sigma(\phi, \psi; \alpha, \lambda)$

If $a_k = 0$; $2 \le k \le n-1$, then $|a_n| \le \frac{2(1-\alpha)}{(1-\lambda)\phi_n + \lambda\psi_n}$; $n \ge 4$ (2.1)

Proof: For the function $f(z) \in H_\Sigma(\phi, \psi, \alpha, \lambda)$ of the form (1.1) we have

$$\frac{(1-\lambda)(f*\phi)(z) + \lambda(f*\psi)(z)}{z} = 1 + \sum_{n=2}^{\infty} [(1-\lambda)\phi_n + \lambda\psi_n] a_n z^{n-1} \tag{2.2}$$

and for its inverse map, g = f^{-1}, we have

$$\frac{(1-\lambda)(g*\phi)(w) + \lambda(g*\psi)(w)}{w} = 1 + \sum_{n=2}^{\infty} [(1-\lambda)\phi_n + \lambda\psi_n] b_n w^{n-1}$$

$$= 1 + \sum_{n=2}^{\infty} [(1-\lambda)\phi_n + \lambda\psi_n] \times \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n) w^{n-1} \tag{2.3}$$

On the other hand, since $f(z) \in H_\Sigma(\phi, \psi; \alpha, \lambda)$ and $g(z) = f^{-1}(z) \in H_\Sigma(\phi, \psi; \alpha, \lambda)$, by definition, there exist two positive real part functions

p(z) = $1 + \sum_{n=1}^{\infty} c_n z^n$

and q(w) = $1 + \sum_{n=1}^{\infty} d_n w^n$

where Re{p(z)} > 0 and Re{q(w)} > 0 in U so that

$$\frac{(1-\lambda)(f*\phi)(z) + \lambda(f*\psi)(z)}{z} = 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(c_1, c_2, ..., c_n) z^n \tag{2.4}$$

$$\frac{(1-\lambda)(f*\phi)(w) + \lambda(f*\psi)(w)}{w} = 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(d_1, d_2, ..., d_n) w^n \tag{2.5}$$

Comparing the corresponding coefficients of (2.2) and (2.4) yields

$$((1-\lambda)\phi_n + \lambda\psi_n) a_n = (1-\alpha) K_{n-1}^1(c_1, c_2, ..., c_{n-1}) \tag{2.6}$$

and similarly, from (2.3) and (2.5) we obtain

$$\frac{1}{n}((1-\lambda)\phi_n + \lambda\psi_n) K_{n-1}^{-n}(b_0, b_1, ..., b_n) = (1-\alpha) K_{n-1}^1(d_1, d_2, ..., d_{n-1}) \tag{2.7}$$

Note that for $a_k = 0$; $2 \le k \le n-1$ we have $b_n = -a_n$ and so

$[(1-\lambda)\phi_n + \lambda\psi_n] a_n = (1-\alpha) c_{n-1}$,

$-[(1-\lambda)\phi_n + \lambda\psi_n] a_n = (1-\alpha) d_{n-1}$. (2.8)

Taking the absolute values of the above equalities, we obtain

$$|a_n| = \frac{(1-\alpha)|c_{n-1}|}{|(1-\lambda)\phi_n + \lambda\psi_n|} = \frac{(1-\alpha)|d_{n-1}|}{|(1-\lambda)\phi_{n|} + \lambda\psi_n|}$$

By applying the Caratheodory Lemma [14], (n $\in$ N) we have,

$$|a_n| \le \frac{2(1-\alpha)}{(1-\lambda)\phi_n + \lambda\psi_n}. \tag{2.9}$$

Theorem 2.2: For (0 ≤ α < 1) and λ ≥ 1 let $f(z) \in H_\Sigma(\phi,\psi,\alpha,\lambda)$ and $g(z) \in H_\Sigma(\phi,\psi,\alpha,\lambda)$. Then one has the following

i) $|a_2| \le \min\left\{\sqrt{\frac{2(1-\alpha)}{(1-\lambda)\phi_3 + \lambda\psi_3}}, \frac{2(1-\alpha)}{(1-\lambda)\phi_2 + \lambda\psi_2}\right\}$

ii) $|a_3| \le \frac{2(1-\alpha)}{(1-\lambda)\phi_3 + \lambda\psi_3}$

iii) $\left|a_3 - 2a_2^2\right| \le \frac{2(1-\alpha)}{(1-\lambda)\phi_3 + \lambda\psi_3}$ (2.10)

Proof: If we set n = 2 and n = 3 in (3.6) and (3.7) respectively, we get

$$[(1-\lambda)\phi_2 + \lambda\psi_2]a_2 = (1-\alpha)c_1, \tag{2.11}$$

$$[(1-\lambda)\phi_3 + \lambda\psi_3]a_3 = (1-\alpha)c_2, \tag{2.12}$$

$$-[(1-\lambda)\phi_2 + \lambda\psi_2]a_2 = (1-\alpha)d_1, \tag{2.13}$$

$$[(1-\lambda)\phi_3 + \lambda\psi_3](2a_2^2 - a_3) = (1-\alpha)d_2. \tag{2.14}$$

Dividing (2.11) or (2.13) by $[(1-\lambda)\phi_2 + \lambda\psi_2]$,taking their absolute values and applying the Cartheodory lemma [14], we have

$$|a_2| = \frac{(1-\alpha)|c_1|}{(1-\lambda)\phi_2 + \lambda\psi_2} = \frac{(1-\alpha)|d_1|}{(1-\lambda)\phi_2 + \lambda\psi_2}$$

$$\le \frac{2(1-\alpha)}{(1-\lambda)\phi_2 + \lambda\psi_2} \tag{2.15}$$

Adding (2.12) to (2.14) implies

$$[(1-\lambda)\varphi_3 + \lambda\psi_3](2a_2^2) = (1-\alpha)(c_2 + d_2)$$

$$a_2^2 = \frac{(1-\alpha)(c_2 + d_2)}{2[(1-\lambda)\phi_3 + \lambda\psi_3]} \tag{2.16}$$

Using the caratheodory lemma [14], followed by taking the square roots yields

$$|a_2| = \sqrt{\frac{(1-\alpha)(|c_2| + |d_2|)}{2[(1-\lambda)\phi_3 + \lambda\psi_3]}} \le \sqrt{\frac{2(1-\alpha)}{(1-\lambda)\phi_3 + \lambda\psi_3}}$$

and combining this with the inequality (2.15) we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (2.10). Dividing (2.12) by $[(1-\lambda)\phi_3 + \lambda\psi_3]$, taking the absolute value on both sides and applying the caratheodory lemma [14] yield

$$|a_3| = \frac{(1-\alpha)|c_2|}{(1-\lambda)\phi_3 + \lambda\psi_3} \le \frac{2(1-\alpha)}{(1-\lambda)\phi_3 + \lambda\psi_3}$$

Dividing (2.14) by $(1-\lambda)\phi_3 + \lambda\psi_3$, taking the absolute values on both sides and applying the caratheodory lemma[14], we obtain

$$\left|a_3 - 2a_2^2\right| \le \frac{2(1-\alpha)}{(1-\lambda)\phi_3 + \lambda\psi_3}.$$

Remark 2.1: By taking special values in the above theorems for the functions $\phi(z)$ and $\psi(z)$, as mentioned in Remark 1.1, we obtain the results due to R.M.El-Ashwah[16], Saurath Porwal and M.Darus[17], A.G.Alamoush and M.Darus[1], B.A.Frasin and M.K.Aouf[2] , J.M.Jahangiri and G.Hamidi[10].

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