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WEYL TYPE THEOREMS FOR CLASS A(k) OPERATORS

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ABSTRACT

If T is a class $A(k)$ operator where $k \geq 1$ and $\hat{T}$ is its hyponormal transform, then generalized Weyl's theorem is proved for T via $\hat{T}$.

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1. INTRODUCTION:

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For an operator $T \in B(H)$, let $T^*, \sigma(T), \sigma_p(T), \sigma_{ap}(T)$ denote the adjoint, spectrum, point spectrum and approximate point spectrum of T, respectively. We denote by $\alpha(T)$ and $\beta(T)$ the dimension of the kernel $\ker T$ and the codimension of the range $R(T)$, respectively. The operator $T \in B(H)$ is called an upper semi-Fredholm operator if $\alpha(T) < \infty$ and $T(X)$ is closed, while $T \in B(H)$ is called lower semi-Fredholm if $\beta(T) < \infty$. If T is either upper or lower semi- Fredholm then T is called a semi-Fredholm operator, while T is said to be a Fredholm operator if it is both upper and lower semi-Fredholm.

We denote by $\phi_+(H)$ the class of all upper semi-Fredholm operators, by $\phi_-(H)$ the class of all lower semi-Fredholm operators, and by $\phi(H)$ the class of all Fredholm operators. If $T \in B(H)$ is semi-Fredholm, then the index of T is defined by ind$(T) = \alpha(T) - \beta(T)$.

The ascent of T is defined as the smallest non-negative integer $p := p(T)$ such that $N(T^p) = N(T^{p+1})$. If such an integer does not exist we put $p(T) = \infty$. Analogously, the descent of T is defined as the smallest nonnegative integer $q := q(T)$ such that $R(T^q) = R(T^{q+1})$ and if such an integer does not exist we put $q(T) = \infty$.

An operator $T \in B(H)$ is called a Weyl operator if it is a Fredholm operator of index 0, and $T \in B(H)$ is called a Browder if it is a Fredholm operator of finite ascent and descent.

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The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_W(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined as

$$\sigma_e(T) = \{\lambda \in \mathbb{C}\ T-\lambda I \text{ is not Fredholm}\},$$
$$\sigma_W(T) = \{\lambda \in \mathbb{C}\ T-\lambda I \text{ is not Weyl}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C}\ T-\lambda I \text{ is not Browder}\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_W(T) = E_0(T),$$

where $E_0(T)$ is the set of all isolated points of $\sigma(T)$ which are eigen values of finite multiplicity. Let $p_{00}(T) = \sigma(T) \setminus \sigma_b(T)$ denote the set of Riesz points of T. T is said to satisfy Browder's theorem if

$$\sigma(T) \setminus \sigma_W(T) = p_{00}(T).$$

The essential approximate point spectrum is

$$\sigma_{ea}(T) = \bigcap\{\sigma_a(T+K) : K \in K(H)\}$$

and the Browder essential point spectrum is

$$\sigma_{ab}(T) = \bigcap\{\sigma_a(T+K) : TK = KT, K \in K(H)\}.$$

It is well known that $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T-\lambda \notin \phi_+^-(H)\}$, where

$$\phi_+^-(H) = \{T \in \phi_+(H) : ind(T) \le 0\}.$$

We say that a-Weyl's theorem holds for $T \in B(H)$ if $\sigma_a(T) \setminus \sigma_{ea}(T) = E_0^a(T)$, where $E_0^a(T)$ is the set of all eigen values of T of finite multiplicity which are isolated in $\sigma_a(T)$.

For a bounded linear operator T and a nonnegative integer n we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $T_0 = T$). If for some integer n, the range space $R(T^n)$ is closed and T_n is an upper (resp., a lower) semi- Fredholm operator, then T is called an upper (resp., a lower) semi B - Fredholm operator. In this situation, T_m is a semi - Fredholm operator and ind (T_m) = ind(T_n) for each m≥n [8, proposition 2.1].Thus the index of a semi-B-Fredholm operator T is the index of the semi- Fredholm operator T_n where n is any integer such that $R(T^n)$ is closed and T_n is a semi- Fredholm operator. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. A semi - B- Fredholm operator is an upper or a lower semi -B-Fredholm operator.

An operator $T \in B(H)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined as

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C}\ T-\lambda I \text{ is not a B-Weyl operator}\}.$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of isolated eigen values of T ([7], Definition2.13).

Let $SBF_+(H)$ be the class of all upper semi -B-Fredholm operators on H , and $SBF_+^-(H)$ the class of all $T \in SBF_+(H)$ such that ind(T) ≤0. Also let

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C}\ T-\lambda I \text{ is not in } SBF_+^-(H)\}.$$

We say that T obeys generalized a-Weyl's theorem if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$, where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ [7, Definition2.13].

The operator $T \in B(H)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated as SVEP at $\lambda_0 \in \mathbb{C}$), if for every open disc U of λ_0 the only analytic function $f : U \to H$ which satisfies the equation $(T - \lambda I) f(\lambda) = 0$ for all $\lambda \in U$, is the function $f \equiv 0$.

An operator $T \in B(H)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. It is known that an operator $T \in B(H)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Also every operator T has SVEP at every isolated point of the spectrum. An operator is called isoloid if each $\lambda \in iso\sigma(T)$ is an eigen value of T.

2. GENERALIZED WEYL'S THEOREM FOR CLASS $A(k)$ OPERATORS:

An operator T has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the suitable partial isometry satisfying $\ker U = \ker T = \ker(|T|)$ and $\ker(U^*) = \ker(T^*)$.

Furuta et al. [10] defined a new class of operators, namely class $A(k)$ where $k > 0$. T belongs to class $A(k)$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$ where $k > 0$. A class $A(1)$ operator T is known as a class A operator and satisfies an operator inequality $|T^2| \geq |T|^2$.

Mary and Panayappan [13, 14] studied the properties of $A(k)$ operators using its hyponormal transform $\hat{T}$. Using those properties we prove generalized Weyl's theorem for class $A(k)$ operators via its hyponormal transform $\hat{T}$.

Limit Condition 2.1: [13], For each $\lambda \in \sigma_a(T)$ and a corresponding sequence $\{y_n\}$ of unit vectors, $\hat{T}$ satisfies the condition $\lim_{n\to\infty} \| \, |\hat{T}|^2 y_n \| = |\lambda|^2$ where T is a class $A(k)$ operator, $k > 1$ and $\hat{T}$ is its hyponormal operator transform.

Proposition2.2: [14, Lemma 5]: If T is a class $A(k)$ operator, where $k > 1$ and M is an invariant subspace of T, then $T|_M$ is also a class $A(k)$ operator.

Proposition 2.3: [14, Lemma 6]: Suppose T is a class $A(k)$ operator and $\hat{T}$ its hyponormal transform such that the limit condition is satisfied. Then the Eigen space of T reduces T

Proposition 2.4: [14, Lemma 7]: If T is a class $A(k)$ operator satisfying the Limit condition, then T is isoloid.

Proposition 2.5: [13, Corollary10]: Every quasinilpotent class $A(k)$ operator satisfying limit condition is a zero operator.

Our first result is as follows:

Theorem 2.6: *Let $T \in B(H)$ be a class $A(k)$ operator and $\hat{T}$ its hyponormal operator transform such that limit condition is satisfied. Then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$.*

Proof: Let $f \in H(\sigma(T))$. It suffices to show that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$.

Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T)-\lambda I \in \phi_+(H)$ and $i(f(T)-\lambda I) \leq 0$
and

$$f(T)-\lambda I = c(T-\alpha_1 I)(T-\alpha_2 I)......(T-\alpha_n I)g(T), \quad ....(1)$$

where $c, \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$ and g(T) is invertible .Since the operators on the right hand side of (2.1) commute, therefore $T-\alpha_i I \in \phi_+(H)$ for each $i = 1, 2.....n.$ Since T has SVEP [13, Theorem 11], it follows from [2, Theorem 2.6] that each $T-\alpha_i$ has finite ascent. Therefore by [1, Theorem 3.4] $i(T-\alpha_i) \leq 0$ for each $i = 1, 2.....n.$ It follows that $\lambda \notin f(\sigma_{ea}(T))$.

Theorem 2.7: *Let $T \in B(H)$ be a class $A(k)$ operator and $\hat{T}$ its hyponormal operator transform such that limit condition is satisfied. Then a sufficient condition for $f(T)$ to satisfy a-Weyl's theorem for every $f \in H(\sigma(T))$ is that T^* has SVEP.*

Proof: If T^* has SVEP, then $\sigma(T) = \sigma_a(T)$. Hence $\sigma_{ea}(T) = \sigma_w(T)$ and $E_0^a(T) = E_0(T)$. As we know T satisfies Weyl's theorem [14, Theorem 2], therefore T satisfies a- Weyl's theorem. Since T is isoloid and $\sigma_{ea}(T)$ satisfies Theorem 2.6, therefore $f(T)$ satisfies a-Weyl's theorem.

Theorem 2.8: *Let T be a class $A(k)$ operator and $\hat{T}$ its hyponormal operator transform such that for each $\lambda \in \sigma_a(T)$ and a corresponding sequence $\{y_n\}$ of unit vectors , $\hat{T}$ satisfies the condition $\lim_{n\to\infty} \| \, |\hat{T}|^2 y_n \| = |\lambda|^2$ Then generalized Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof: Since T is isoloid and has SVEP [13 ,Theorem11], therefore it suffices to prove that generalized Weyl's theorem holds for T.

Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T-\lambda I$ is a B-Fredholm operator of index zero. Hence it follows from [5, Lemma4.1] that there exist two closed linear spaces M and N of H such that $H = M \oplus N$ and $T-\lambda I = U \oplus V$ with $U = (T-\lambda I)|_M$ a Fredholm operator of index zero and $V = (T-\lambda I)|_N$ a nilpotent operator. Let $S = T|_M$ and $I_M = I|_M$.

Since T is class $A(k)$ operator, then by Proposition 2.2, S is also class $A(k)$ operator and $S-\lambda I_M = U$ is a Fredholm operator of index zero.

If $\lambda \in \sigma(S)$, then by [14, Theorem2], we have $\sigma_w(S) = \sigma(S) \setminus E_0(S)$. As $S-\lambda I_M$ is Fredholm operator of index zero, we have $\lambda \in E_0(S)$. In particular λ is isolated in $\sigma(S)$. 0 is isolated in $\sigma(S-\lambda I_M) = \sigma(U)$.

Since $T-\lambda I = U \oplus V = (S-\lambda I_M) \oplus V$, and V is a nilpotent operator then $\sigma(U) \setminus \{0\} = \sigma(T-\lambda I) \setminus \{0\}$.

Therefore 0 is isolated in $\sigma(T-\lambda I)$ or equivalently λ is isolated in $\sigma(T)$. As $\lambda \in E_0(S)$, then $\lambda \in E(T)$.

If $\lambda \notin \sigma(S)$, then we also deduce from $T-\lambda I = (S-\lambda I_M) \oplus V$, that λ is isolated in $\sigma(T)$. Since $T-\lambda I$ is not invertible and hence $\lambda \in E(T)$.

Conversely, if $\lambda \in E(T)$, then λ is isolated in $\sigma(T)$. From [12, Theorem7.1] we have $H = M \oplus N$ where M and N are closed linear subspace of H, $U = (T-\lambda I)|_M$ is an invertible operator and $V = (T-\lambda I)|_N$ is a quasinilpotent operator. Since T is class $A(k)$ operator, then V is also class $A(k)$ operator. As V is quasinilpotent operator, from Proposition 2.5, we have $V = 0$. Since V is invertible it follows from [5, Lemma4.1] $T - \lambda I$ is a B-Fredholm operator of index zero.

Remark 2.9: The inclusion $\sigma(T) \setminus \sigma_{BW}(T) \subset E(T)$ in the above result can also be proved as follows:

Proof: Assume $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda I$ is B-Weyl and not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume to the contrary that λ is an interior point of $\sigma(T)$. Then there exists a neighbourhood U of λ such that $\dim N(T - \mu I) > 0$ for all $\mu \in U$. It follows from [9, Theorem 10] that T does not have SVEP. On the other hand, since T is class $A(k)$ operator, it follows from [13, Theorem11] that T has SVEP, which is a contradiction. Therefore $\lambda \in \partial\sigma(T) \setminus \sigma_{BW}(T)$ and it follows from the punctured neighbourhood theorem that $\lambda \in E(T)$.

Corollary 2.10: Let $T \in B(H)$ be a class $A(k)$ operator and $\hat{T}$ its hyponormal operator transform such that limit condition is satisfied. If $\sigma(T)$ has no isolated points then T^* satisfies generalized Weyl's theorem.

Proof: We know $\sigma(T^*) = \overline{\sigma(T)}$, $\sigma_{BW}(T^*) = \overline{\sigma_{BW}(T)}$ and $E_0(T^*) = \overline{E_0(T)} = \phi$. and from Theorem 2.8 generalized Weyl's theorem holds for T. Therefore we have that $\sigma(T^*) \setminus \sigma_{BW}(T^*) = E_0(T^*)$.

The following corollary is an immediate consequence of the above Theorem and [6, Proposition 3.6]:

Corollary 2.11: Let $T \in B(H)$ be a class $A(k)$ operator and $\hat{T}$ its hyponormal operator transform such that limit condition is satisfied. Let F be a finite rank nilpotent operator commuting with T. Then generalized Weyl's theorem holds for $f(T) + F$ for every $f \in H(\sigma(T))$.

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