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CONTROLLABILITY OF NONLINEAR BOUNDARY VALUE CONTROL SYSTEMS IN UNIFORMLY CONVEX BANACH SPACE USING KIRK FIXED POINT THEOREM

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ABSTRACT

In this paper, sufficient conditions for controllability of nonlinear boundary value control system in uniformly convex Banach spaces are established. The results are obtained by using semigroup theory "C_0-semigroup" and some techniques of nonlinear functional analysis, such as, Kirk fixed point theorem. Moreover, example is provided to illustrate the theory.

KeyWords: *Controllability, Uniformly convex Banach space, Semigroup theory, Kirk fixed point theorem.*

1. INTRODUCTION

The theory of semigroup of linear operators lends a convenient setting and offers many advantages for applications. Control theory in infinite-dimensional spaces is a relatively new field and started blooming only after well-developed semigroup theory was at hand. Many scientific and engineering problems can be modeled by partial differential equations, integral equations, or coupled ordinary and partial differential equations that can be described as differential equations in infinite–dimensional spaces using semigroups. Nonlinear equations, with and without delays, serve as an abstract formulation for many partial equations which arise in problems connected with heat flow in materials with memory, viscoelasticity, and other physical phenomena. So, the study of controllability results for such systems in infinite–dimensional spaces is important. For the motivation of abstract system and controllability of linear system, one can refer to the [1].

Now, Let $S=L_P(J,X)$, for $1 < p < \infty$ with X a real Banach space, be an uniformly convex Banach space and U be a Banach space, with norms $\|.\|_p$ and $|.|$, respectively. Let A be a linear closed and densely defined operator with $D(A) \subseteq S$, $\|A\|_P \leq C_1$, C_1 is a positive constant, and let τ be a linear operator with $D(\tau) \subseteq S$, and $R(\tau) \subseteq E$, where E is Banach space.

In this paper we discuss the controllability of mild solution of the following nonlinear boundary value control problem in uniformly convex Banach space..

$$\left.\begin{aligned} &\dot{z}(t) = Az(t) + (Bu)(t) + f(t,z(t)) + Q(t,K(t,z(t))), \text{ a.e in } J=[0,b] \\ &\tau z(t) = B_1 u(t), \\ &z(0) = z_0, \end{aligned}\right\} \qquad (1.1)$$

where $B_1 : U \rightarrow E$ is a linear continuous operator, the control function $u \in L^2(J,U)$, a uniformly convex Banach space of admissible control functions. The nonlinear operators,

$$f \in C(J \times S, S),\ K \in C(J \times S, S) \text{ and } Q \in C(J \times S, S)$$

are all satisfy Lipschitz condition on the seconed argument. Where the state $z(.)$ takes values in the uniformly convex Banach space $S=L_P(J,X)$, for $1 < p < \infty$ with X a real Banach space, and the control function $u(.)$ is given in $L^2(J,U)$, a uniformly convex Banach space of admissible control functions, with U a Banach space. Here, the linear operator A generates a strongly continuous semigroup (C_0–semigroup) $T(t)$, $t \geq 0$, on an uniformly convex Banach space S with norm $\|.\|_p$, and B is a bounded linear operator from U into S. Let $S_0=\{z : z \in C(J,X) \subset S,\ z(0)= z_0, \|z(t)\|_P \leq r$, for $t \in J\}$, where r is a positive constant.

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Controllability of the above system in any Banach spaces with different conditions has been studied by several authors. Balakrishnan in [2] showed that the solution of a parabolic boundary control equation with L^2 controls can be expressed as a mild solution to an operator equation. Fattorini in [3] discussed the general theory of boundary control systems. Han and Park in [4] derived a set of sufficient conditions for the boundary controllability of a semilinear system with a nonlocal condition. Al-Moosawy in [5] discussed the controllability of the mild solution for the problem (1.1) by using Banach fixed point theorem, where $f \equiv 0$, A generates a strongly continuous semigroup (C0-semigroup) and the operators K, Q are satisfying Lipschitz condition on the second argument. Al-Jawari and Amna in [6] extended the work in [5] by studying the controllability in quasi-Banach spaces of kind L^p, $0 < p < 1$, using a quasi-Banach contraction principle theorem. Al-Jawari and Imad in [7] studied The controllability of the system (1.1), where $T(t)$, $t > 0$ is a compact semigroup on a Banach space and the operators f, K and Q are all uniformly bounded continuous in (1.1), by using Schauder fixed point theorem. From all the above and since every uniformly convex Banach space is a Banach space but the converse, in general, not true. And since a nonexpansive mapping on a nonempty, closed, bounded and convex subset of a Banach space has no fixed point in general (see, Example 2.2), we find a reasonable justification to accomplish the study of this paper. The purpose of this paper is to study the controllability of nonlinear boundary value control system (1.1) in uniformly convex Banach spaces by using Kirk fixed point theorem.

2. FIXED POINT THEOREMS AND SEMIGROUP THEORY

Fixed point throrems are the basic mathematical tools used in solving nonlinear equations. In this section we present the basic fixed point results and some definitions of one - parameter semigroup of operators.

Definition 2.1[8]: let (X, $\|\ .\ \|$) be a normed space. A map $T: X \to X$ is said to be ***Lipschitz continuous*** if there is $\lambda \geq 0$ such that

$$\|T(x_1) - T(x_2)\| \leq \lambda\ \|x_1 - x_2\|, \text{ for all } x_1, x_2 \in X.$$

The smallest λ for which the above inequality holds is the *Lipschitz constant* of T. If $\lambda \leq 1$, T is said to be ***nonexpansive**, if* $\lambda < 1$, T is said to be *a* ***contraction*** .

Note that each contraction is nonexpansive, while an isometry is nonexpansive but not contractive .

Theorem 2.1[8] (Banach Theorem): Eevry contraction mapping of a Banach space into itself has a unique fixed point .

Theorem 2.2[8] (Schauder Theorem): Every continuous operator that maps a nonempty convex subset of a Banach space into a compact subset of itself has at least one fixed point.

Definition 2.2[9]: A normed space X is said to be ***uniformly convex*** if for every $\epsilon > 0$, there exists some $\delta > 0$ such that for all x and y in X with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \epsilon$, we have $\|x + y\| \leq 2(1 - \delta)$.

Definition 2.3[9]: Let $0 < p < \infty$, then the collection of all measurable function f for which $|f|^p$ is integrable will be denoted by $L_p(\mu)$. For each $f \in L_p(\mu)$, let $\|f\|_p = (\int |f|^p\, d\mu)^{1/p}$, the number $\|f\|_p$ is called the L_P**-norm** of f .

Examle 2.1[9]: Every Hilbert space is uniformly convex. The spaces ℓ^p of p – summable scalar seqences and the space L^p of p- integrable functions are uniformly convex for $1 < p < \infty$. For properties of uniformly convex Banach space, every uniformly convex Banach space is reflexive .

To have an extension of the Banach theorem to nonexpansive maps, we need to impose some geometric conditions on the domain of the nonexpansive map. See the following

Let X be a Banach space, $C \subset X$ nonempty, closed, bounded and convex, and let $T : C \to C$ be a ***nonexpansive*** map. The problem is whether T admits a fixed point in C. The answer, in general, is false .

Example 2.2[9]: Let $X = c_0$ (the space of all sequences of scalars converging to zero) with the supremum norm. Setting $C = \{y \in X : \|y\| \leq 1\}$, the map $T : C \to C$ defined by $f(x) = (1, x_0, x_1, \ldots)$, for $x = (x_0, x_1, x_2, \ldots) \in C$ is nonexpansive but clearly admits no fixed point in C .

Things are quite different in uniformly convex Banah spaces.

Theorem 2.3[8](Kirk Fixed Point Theorem): Let X be a uniformly convex Banach space and $C \subseteq X$ be a nonempty, closed, bounded and convex. If T:$C \to C$ is a nonexpansive mapping, then T has a fixed point .

Definition 2.4[10]: A family $T(t)$, $0 \le t < \infty$ of bounded linear operators on a Banach space X is called a (one – parameter) ***semigroup*** on X if it is satisfies the following conditions:

$T(t+s) = T(t)T(s)$, $t,s \ge 0$ *and* $T(0) = I$. (I is the identity operator on X)

Definition 2.5[10]: The infinitesimal generator A of the ***semigroup*** $T(t)$ on Banach space X is defined by $Ax = lim_{t\to 0^+} \frac{T(t)x-x}{t}$, where the limit exists and the domain of A is $D(A)=\{x\in X: lim_{t\to 0^+} \frac{T(t)x-x}{t}$, exists$\}$.

Definition 2.6[10]: A *semigroup* $T(t)$, $0 \le t < \infty$ of boubded linear operators on Banach space X is said to be ***strongly continuous semigroup* (or *C*o-semigroup)** if:

$$\| T(t)x-x \|_X \to 0 \quad as\ t \to 0^+ \text{ for all } x \in X.$$

3. CONTROLLABILITY OF NONLINEAR CONTROL PROBLEMS

In this section we will study the controllability of mild solution to the problem **(1.1)** in uniformly convex Banach space by using ***semigroup theorem*** (C_0– semigroup), and ***"Kirk fixed point theorem"* .**

Let $A_1 : S \to S$, be the linear operator, defined by,
$A_1z = Az$, $z \in D(A_1)$, where $D(A_1) = \{z \in D(A) : \tau z = 0\}$

Now, We assume the following basic hypothesis:

(H₁) $D(A) \subset D(\tau)$ and the restriction of τ to $D(A)$ is continuous relative to graph norm of $D(A_1)$.

(H₂) The operator A_1 is the infinitesimal generator of a C_0-semigroup $T(.)$ such that $\max_{t\ge 0} \| T(t) \|_p \le M$.

(H₃) There exists a linear continuous operator $B_2 : U \longrightarrow S$, such that $AB_2 \in L(U,S)$, $\tau(B_2u) = B_1u$, for all $u \in U$. Also $B_2u(t)$ is continuously differentiable, and $\| B_2u \|_p \le L \| B_1u \|$ $\forall u \in U$, where L is a constant.

(H₄) For all $t \in (0, b]$ and $u \in U$, $T(t)B_2u \in D(A_1)$. Moreover, there exists a positive function $v_0 \in L^1(0,b)$, such that $\| A_1T(t)B_2 \|_p \le v_0(t)$ a.e., $t \in (0, b)$.

(H₅) The nonlinear operators, f & K where $f : J\times S \to S$, $K: J\times S \to S$ satisfies lipshitz condition on the seconed argument $\| K(t,z_1) - K(t,z_2) \|_P \le M_4 \| z_1 - z_2 \|_P$, where M_4 is constant and $z_1, z_2 \in S_0$,
and $\| f(t,z_1) - f(t,z_2) \|_P \le M_1 \| z_1 - z_2 \|_P$, where M_1 is constant and $z_1, z_2 \in S_0$,
and $M_3 = \max_{t\to J} \| f(t,0) \|_P$, and $k_1 = \max_{t\in J} \| z(t) \|_P$.

(H₆) The nonlinear operator Q: $J\times S \to S$ is continuous and there exist a constants M_2, M_5, such that for all $z_1, z_2 \in S_0$, we have :
$\| Q(t, K(t,z_1)) - Q(t,K(t,z_2)) \|_p \le M_2 \| K(t,z_1) - K(t,z_2) \|_p \le M_2 M_4 \| z_1 - z_2 \|_p$ and $M_5 = \max_{t\in J} \| Q(t,K(t,0)) \|_P$.

(H₇) $B : U \longrightarrow S$ is a bounded linear operator, $\| B \|_p \le c$, where c is a positive constant.

(H₈) The linear operator W from $L^2(J,U)$ into S, defined by
$Wu = \int_0^b T(t-s)Bu(s)ds$. induces a bounded inverse operator $\widetilde{W}^{-1}$ defined on $L^2(J,U)/\ker(W)$, and there exist positive constant $k_2 > 0$ such that $\| \widetilde{W}^{-1} \|_P \le k_2$ **.**

● The construction of $\widetilde{W}^{-1}$ is outlined as follows [11,12] .

Let $Y = L^2[J,U]/\ker(W)$. Since $\ker(W)$ is closed, Y is Banach space under the norm

$$\| [u] \|_Y = \inf_{u\in[u]} \| u \|_{L^2[J,U]} = \inf_{W\hat{u}=0} \| u + \hat{u} \|_{L^2[J,U]},$$

where $[u]$ are the equivalence classes of u.

Define $\widetilde{W}: Y \to X$ by $\widetilde{W}[u] = Wu$, $u \in [u]$.

Then, $\widetilde{W}$ is one-to-one and $\| \widetilde{W}[u] \|_x \le \| W \| \, \| [u] \|_Y$.

Also, $V = \text{Range}(W)$ is a Banach space with the norm $\| v \|_v = \| \widetilde{W}^{-1} v \|_Y$.

To see this, note that this norm is equivalent to the graph norm on $D(\widetilde{W}^{-1})=\text{Range}(\widetilde{W})$. $\widetilde{W}$ is bounded, and since $D(\widetilde{W}) = Y$ is closed, $\widetilde{W}^{-1}$ is closed. So, the above norm makes Range $(W) = V$, a Banach space. Moreover,

$$\| Wu \|_v = \| \widetilde{W}^{-1}Wu \|_Y = \| \widetilde{W}^{-1}W[u] \| = \| [u] \| = \inf_{u\in[u]} \| u \| \le \| u \|; \text{ So, } W \in \mathcal{L}(L^2[J,U], V).$$

Since $L^2[J,U]$ is reflexive and $\ker(W)$ is weakly closed, the infimum is actually attained. Therefore, for any $v \in V$, a control $u \in L^2[J,U]$ can be chosen such that $u = \widetilde{W}^{-1}v$.

3.1 Controllability Result of the Problem (1.1):

It should be noted that to define and find the mild solution of the boundary value control problem (1.1).

Now, let $z(t)$ be a solution of **(1.1)**, then we can define a function :

$$x(t) = z(t) - B_2u(t) . \tag{3.1}$$

From our assumptions, we see that $x(t)\in D(A_1)$. Hence equation **(1.1)** can be written in term of A_1 and B_2, as follows :

$$\dot{z}(t) = A[x(t) + B_2u(t)] + Bu(t) + f(t,z(t)) + Q(t,K(t,z(t))), \text{ a.e in } J = [0,b]$$
$$= Ax(t) + AB_2u(t) + Bu(t) + f(t,z(t)) + Q(t,K(t,z(t)))$$

Since $x(t) \in D(A_1)$ then $A_1x(t) = Ax(t)$, and ;

$$\left.\begin{aligned} &\dot{z}(t) = A_1x(t) + AB_2u(t) + Bu(t) + f(t,z(t)) + Q(t,K(t,z(t))), \text{ a.e in } J \\ &z(t) = x(t) + B_2u(t), \\ &z(0) = z_0 \end{aligned}\right\} \tag{3.2}$$

from condition **(H_3)**, we have $B_2u(t)$ is continuously differentiable, if z is continuously differentiable on J, then by definition of the mild solution $x(t) = z(t) - B_2u(t)$, can be defined as a mild solution to Cauchy problem [10]:

$$\frac{d}{dt}x(t) = \frac{d}{dt}z(t) - B_2\frac{d}{dt}u(t).$$

By equation (3.2), one gets that,

$$\left.\begin{aligned} &\dot{x}(t) = A_1x(t) + AB_2u(t) + (Bu)(t) + f(t,z(t)) + Q(t,K(t,z(t))) - B_2\frac{d}{dt}u(t), \text{ a.e in } J \\ &x(0) = z_0 - B_2u(0) . \end{aligned}\right\} \tag{3.3}$$

By condition (H_2), we have $T(t)$, $t \geq 0$ is the C_0-semigroup generated by the linear operator A_1, and x(t) is a solution of (3.3), then the function $H(s) = T(t-s)x(s)$ is differentiable for $0 < s < t$ [10], and

$$\frac{d}{ds}H(s) = T(t-s)\frac{d}{ds}x(s) + x(s)\frac{d}{ds}T(t-s) .$$

Thus by equation (3.3) we get that

$$\frac{d}{ds}H(s) = T(t-s)[A_1x(s) + AB_2u(s) + (Bu)(s) + f(s,z(s)) + Q(s,K(s,z(s))) - B_2\frac{d}{ds}u(s)] + x(s)[-AT(t-s)]$$

$$\frac{d}{ds}H(s) = T(t-s)A_1x(s) + T(t-s)AB_2u(s) + T(t-s)(Bu)(s) + T(t-s)f(s,z(s))$$
$$+ T(t-s)Q(s,K(s,z(s))) - T(t-s)B_2\frac{d}{ds}u(s) - T(t-s)A\,x(s),$$

and since $A_1x(t) = Ax(t)$, we see that,

$$\frac{d}{ds}H(s) = T(t-s)AB_2u(t) + T(t-s)(Bu)(t) + T(t-s)f(s,z(s)) + T(t-s)Q(s,K(s,z(s))) - T(t-s)B_2\frac{d}{ds}u(s).$$

On integrating both sides from 0 to t, yields :

$$\begin{aligned} H(t) - H(0) = &\int_0^t T(t-s)AB_2u(s)ds + \int_0^t T(t-s)Bu(s)ds + \int_0^t T(t-s)f(s,z(s))ds \\ &+ \int_0^t T(t-s)Q(s,K(s,z(s)))ds - \int_0^t T(t-s)B_2\frac{d}{ds}u(s)ds \end{aligned} \tag{3.4}$$

From the definition of $H(s) = T(t-s)x(s)$, we have :

$$H(t) = T(t-t)x(t) = T(0)[z(t) - B_2u(t)] = z(t) - B_2u(t),\ T(0) = I, \tag{3.5}$$

and

$$H(0) = T(t-0)x(0) = T(t)[z(0) - B_2u(0)] = T(t)z_0 - T(t)B_2u(0). \tag{3.6}$$

Now, by integrating the term $\int_0^t T(t-s)B_2\frac{d}{ds}u(s)ds$ in (3.4); by parts, we get that :

$$\begin{aligned} \int_0^t T(t-s)B_2\frac{d}{ds}u(s)ds &= T(t-s)B_2u(s)\Big|_0^t + \int_0^t u(s)A_1T(t-s)B_2ds \\ &= B_2u(t) - T(t)B_2u(0) + \int_0^t u(s)A_1T(t-s)B_2ds \end{aligned} \tag{3.7}$$

Substitutting (3.5), (3.6) and (3.7) in equation (3.4), one gets:

$$\begin{aligned} z(t) - B_2u(t) - T(t)z_0 + T(t)B_2u(0) = &\int_0^t T(t-s)AB_2u(s)ds + \int_0^t T(t-s)Bu(s)ds + \\ &+ \int_0^t T(t-s)f(s,z(s))ds + \int_0^t T(t-s)Q(s,K(s,z(s)))ds - B_2u(t) \\ &+ T(t)B_2u(0) - \int_0^t u(s)A_1T(t-s)B_2ds, \end{aligned}$$

therefore

$$z(t) = T(t)z_0 + \int_0^t T(t-s)AB_2u(s)ds + \int_0^t T(t-s)Bu(s)ds + \int_0^t T(t-s)\, f(s,z(s))ds + \int_0^t T(t-s)\, Q(s,K(s,z(s)))ds - \int_0^t A_1T(t-s)B_2u(s)ds \,. \qquad (3.8)$$

So according to the result above, the following definition has been presented.

Definition 3.1: A function $z : [0,b] \to S$ given by (3.8) is called a mild solution to the problem (1.1) if z is continuous on $[0,b]$, continuously differentiable on $(0,b)$ and $z(s) \in S$, for $0 < s < t$.

Definition 3.2: The system (1.1) is said to be controllable on the interval $J = [0,b]$ if, for every $z_0, z_1 \in S$, there exists a control $u \in L^2(J,U)$ such that the mild solution $z(.)$ of (1.1) satisfying $z(b) = z_1$.

3.2 Main Result

In this section we will prove the theorem that deals with the controllability of the problem **(1.1).**

Theorem 3.1: Let the hypothesis **(H_1) – (H_8)** are satisfied for the nonlinear boundary value control problem **(1.1).**

$\dot{z}(t) = Az(t) + (Bu)(t) + f(t,z(t)) + Q(t,K(t,z(t)))$, a.e., in $J=[0,b]$,

$\tau z(t) = B_1u(t)$, $z(0) = z_0$ **.**

Assume further that

(H_9) There exists a constant $k_3 > 0$, such that : $\int_0^b v_0(t) \le k_3$ **.**

(H_{10}) $M\|z_0\|_p + h_1+h_2+h_3+h_4+[bk_2M\|AB_2\|_p + k_2k_3 + bk_2Mc][\|z_1\|_p + M\|z_0\|_p + h_1+h_2+h_3+h_4] \le r$, where r is a positive constant **.**

where $h_1=bMM_1k_1$, $h_2=bMM_3$, $h_3=bMM_2M_4k_1$, $h_4=bMM_5$.

(H_{11}) $\lambda = [[bMM_1 + bMM_2M_4] + [bMk_2\|AB_2\|_p + k_2k_3 + bk_2Mc]\,[bMM_1 + bMM_2M_4]]$ be such that $0 \le \lambda \le 1$**.**

Then the problem **(1.1)** is controllable on J **.**

***Proof*:** By using definition **(3.2)** and equation **(3.8)** we get that

$$z_1 = z(b) = T(b)z_0 + \int_0^b T(b-s)AB_2u(s)ds + \int_0^b T(b-s)Bu(s)ds + \int_0^b T(b-s)\, f(s,z(s))ds + \int_0^b T(b-s)\, Q(s,K(s,z(s)))ds - \int_0^t A_1T(t-s)B_2u(s)ds$$

For an arbitrary function $z(.)$, condition **(H_8)** leads to

$z_1 = T(b)z_0 + Wu + \int_0^b T(b-s)\, f(s,z(s))ds + \int_0^b T(b-s)\, Q(s,K(s,z(s)))ds$**.** Therefore,

$Wu = z_1 - T(b)z_0 - \int_0^b T(b-s)f(s,z(s))ds - \int_0^b T(b-s)Q(s,K(s,z(s)))ds$**.**

From constraction of $\widetilde{W}^{-1}$ in **(H_8)**, we have that $u(t) = \widetilde{W}^{-1}(Wu(t))$, then

$$u(t) = \widetilde{W}^{-1}\Big(z_1 - T(b)z_0 - \int_0^b T(b-s)\, f(s,z(s))ds - \int_0^b T(b-s)Q(s,K(s,z(s)))ds\Big)(t)) \qquad (3.9)$$

Now one will show that, when using this control the operator defined by

$$(\Phi z)(t) = T(t)z_0 + \int_0^t T(t-s)AB_2u(s)ds + \int_0^t T(t-s)Bu(s)ds + \int_0^t T(t-s)\, f(s,z(s))ds + \int_0^t T(t-s)\, Q(s,K(s,z(s)))ds - \int_0^t A_1T(t-s)B_2u(s)ds,$$

has a fixed point. This fixed point is then a solution of equation **(3.8).**

Clearly, $(\Phi z)(b) = z_1$, which means that the control u steers the nonlinear control system from the initial z_0 to z_1 in time b, provided we can obtain a fixed point of the nonlinear operator Φ**.**

Let $S=L_p(J,X)$, for $1 < p < \infty$ with X a real Banach space and $S_0=\{z : z \in C(J,X) \subset S,\ z(0)= z_0, \|z(t)\|_p \le r, \text{ for } t \in J\}$, where r is a positive constant**.** Then S_0 is clearly a bounded, closed, convex subset of S [9]. Now we define a mapping, $\Phi : S \to S_0$ by,

$$(\Phi z)(t) = T(t)z_0 + \int_0^t T(t-s)\, f(s,z(s))ds + \int_0^t T(t-s)\, Q(s,K(s,z(s)))ds + \int_0^t [T(t-s)AB_2 - A_1T(t-s)B_2 + T(t-s)B]u(s)ds$$

$$(\Phi z)(t) = T(t)z_0 + \int_0^t T(t-s)\, f(s,z(s))ds + \int_0^t T(t-s)\, Q(s,K(s,z(s)))ds + \int_0^t [T(t-\eta)AB_2 - A_1T(t-\eta)B_2 + T(t-\eta)B]\, \widetilde{W}^{-1}[z_1 - T(b)z_0 - \int_0^b T(b-s)\, f(s,z(s))ds - \int_0^b T(b-s)Q(s,K(s,z(s)))ds)(s))ds](\eta)d\eta \qquad (3.10)$$

Taking the norm of both sides of **(3.10)**

$$\|(\Phi z)(t)\|_p = \| T(t)z_0 + \int_0^t T(t-s)f(s,z(s))ds + \int_0^t T(t-s)Q(s,K(s,z(s)))ds + \int_0^t [T(t-\eta)AB_2 - A_1T(t-\eta)B_2 + T(t-\eta)B]\,\widetilde{W}^{-1}[z_1 - T(b)z_0 - \int_0^b T(b-s)\,f(s,z(s))ds - \int_0^b T(b-s)Q(s,K(s,z(s)))ds)(s))ds](\eta)d\eta \|_p$$

$$\|(\Phi z)(t)\|_p \le \|T(t)z_0\|_p + \|\int_0^t T(t-s)f(s,z(s))ds\|_p + \|\int_0^t T(t-s)Q(s,K(s,z(s)))ds\|_p + \| \int_0^t [T(t-\eta)AB_2 - A_1T(t-\eta)B_2 + T(t-\eta)B]\,\widetilde{W}^{-1}[z_1 - T(b)z_0 - \int_0^b T(b-s) f(s,z(s))ds - \int_0^b T(b-s)Q(s,K(s,z(s)))ds)(s))ds](\eta)d\eta \|_p$$

$$\|(\Phi z)(t)\|_p \le \|T(t)z_0\|_p + \int_0^t \|T(t-s)[f(s,z(s)) - f(s,0) + f(s,0)]ds\|_p + \int_0^t \|T(t-s)\,[Q(s,K(s,z(s))) - Q(s,K(s,0)) + Q(s,K(s,0))]ds\|_p + \| \int_0^t [T(t-\eta)AB_2 - A_1T(t-\eta)B_2 + T(t-\eta)B]\,\widetilde{W}^{-1}[\|z_1\|_p + \|T(b)z_0\|_p + \|\int_o^b T(b-s)\,f(s,z(s)) - f(s,0) + f(s,0)]ds\|_p + \|\int_0^b T(b-s)[Q(s,K(s,z(s))) - Q(s,K(s,0)) + Q(s,K(s,0))]ds\,\|_p](\eta)d\eta\,\|_p$$

$$\|(\Phi z)(t)\|_p \le \|T(t)\|_p\|z_0\|_p + \int_0^t \|T(t-s)\|_p[\|f(s,z(s)) - f(s,0)\|_p + \|f(s,0)\|_p]ds + \int_0^t \|T(\mathrm{t-s})\|_p[\|Q(s,K(s,z(s))) - Q(s,K(s,0))\|_p + \|Q(s,K(s,0))\|_p]ds + \int_0^t [\|T(t-\eta)\|_p\|AB_2\|_p + \|A_1T(t-\eta)B_2\|_p + \|T(t-\eta)\|_p\|B\|_p]\,\|\widetilde{W}^{-1}\|_p\mathbf{[}\|z_1\|_p + \|T(b)\|_p\|z_0\|_p + \int_0^b \|T(b-s)\|_p[\|f(s,z(s)) - f(s,0)\|_p + \|f(s,0)\|_p]ds + \int_0^b \|T(b-s)\|_p[\|Q(s,K(s,z(s))) - Q(s,K(s,0))\|_p + \|Q(s,K(s,0))\|_p\,ds\mathbf{]]}(\eta)d\eta$$

By conditions from **(H_1) − (H_9),** and since $\|B\|_p \le c$, then we get that

$$\|(\Phi z)(t)\|_p \le M\|z_o\|_p + \int_0^t M[M_1\|z(s)\|_p + M_3]ds + \int_0^t M[M_2M_4\|z(s)\|_p + M_5]ds + \int_0^t [M\|AB_2\|_p + v_0(t) + Mc]k_2\mathbf{[}\|z_1\|_p + M\|z_0\|_p + bM[M_1\|z(s)\|_p + M_3] + bM[M_2M_4\|z(s)\|_p + M_5]\mathbf{]}(\eta)\,d\eta$$

Since $z \in S$, then $\|z(s)\|_p \le k_1$, and then :

$$\|(\Phi z)(t)\|_p \le M\|z_0\|_p + bMM_1k_1 + bMM_3 + bMM_2M_4k_1 + bMM_5 + [bk_2M\|AB_2\|_p + k_2k_3 + bk_2Mc]\mathbf{[}\|z_1\|_p + M\|z_0\|_p + bMM_1k_1 + bMM_3 + bMM_2M_4k_1 + bMM_5]$$

By condition **(H_{10})**, we have

$$\|(\Phi z)(t)\|_p \le M\|z_0\|_p + h_1 + h_2 + h_3 + h_4 + [bk_2M\|AB_2\|_p + k_2k_3 + bk_2Mc]\mathbf{[}\|z_1\|_p + M\|z_0\|_p + h_1 + h_2 + h_3 + h_4] \le r$$

Since f,K and Q are continuous and $\|(\Phi z)(t)\|_p \le r$, it follows that Φ is also continuous and maps S_0 into itself**.**

Seconed, we have to show that Φ is *nonexpansive mapping* from S_0 into S_0. For $z_1(t), z_2(t) \in S_0$ and from the definition of $(\Phi z)(t)$ in equation **(3.10)**, we have

$$\|\Phi z_1(t) - \Phi z_2(t)\|_p = \|T(t)z_0 + \int_0^t T(t-s)f(s,z_1(s))ds + \int_0^t T(t-s)Q(s,K(s,z_1(s)))ds + \int_0^t [T(t-\eta)AB_2 - A_1T(t-\eta)B_2 + T(t-\eta)B]\widetilde{W}^{-1}[z_1 - T(b)z_0 - \int_0^b T(b-s)\,f(s,z_1(s))ds - \int_0^b \mathrm{T}(b-s)Q(s,K(s,z_1(s)))ds](\eta)d\eta - T(t)z_0 - \int_0^t T(t-s)\,f(s,z_2(s))ds - \int_0^t T(t-s)\,Q(s,K(s,z_2(s)))ds - \int_0^t [T(t-\eta)AB_2 - A_1T(t-\eta)B_2 + T(t-\eta)B]\,\widetilde{W}^{-1}[z_1 - T(b)z_0 - \int_0^b T(b-s)f(s,z_2(s))ds - \int_0^b T(b-s)Q(s,K(s,z_2(s)))ds](\eta)d\eta\|_p$$

$$\|\Phi z_1(t) - \Phi z_2(t)\|_p = \|\int_0^t T(t-s)[f(s,z_1(s)) - f(s,z_2(s))]ds + \int_0^t T(t-s)[Q(s,K(s,z_1(s))) - Q(s,K(s,z_2(s)))]ds + \int_0^t [T(t-\eta)AB_2 - A_1T(t-\eta)B_2 + T(t-\eta)B]\widetilde{W}^{-1}\mathbf{[}\int_0^b T(b-s)\,f(s,z_1(s)) - f(s,z_2(s))]ds + \int_0^b T(b-s)[Q(s,K(s,z_1(s))) - Q(s,K(s,z_2(s))ds)]\mathbf{]}(\eta)d\eta\|_p$$

$$\|\Phi z_1(t) - \Phi z_2(t)\|_p \le \int_0^t \|T(t-s)\|_p\|f(s,z_1(s)) - f(s,z_2(s))\|_p ds + \int_o^t \|T(t\text{-}s)\|_p\|Q(s,K(s,z_1(s))) - Q(s,K(s,z_2(s)))\|_p ds + \|\int_0^t [T(t-\eta)AB_2 - A_1T(t-\eta)B_2 + T(t-\eta)B]\widetilde{W}^{-1}\mathbf{[}\int_0^b T(b-s)[f(s,z_1(s)) - f(s,z_2(s))]\,ds + \int_o^b T(b-s)[Q(s,K(s,z_1(s))) - Q(s,K(s,z_2(s))ds)]\mathbf{]}\,(\eta)d\eta\|_p$$

$$\begin{aligned}\|\Phi z_1(t)-\Phi z_2(t)\|_p \le & \int_0^t \|T(t-s)\|_p \|f(s,z_1(s))-f(s,z_2(s))\|_p ds \\ & +\int_0^t \|T(t\text{-}s)\|_p \|Q(s,K(s,z_1(s)))-Q(s,K(s,z_2(s)))\|_p s+\int_0^t [\|T(t-\eta)\|_p \|AB_2\|_p \\ & +\| A_1T(t-\eta)B_2\|_p+\|T(t-\eta)\|_p\|B\|_p]\|\widetilde{W}^{-1}\|_p[\int_0^b \|T(b-s)\|_p \|f(s,z_1(s))-f(s,z_2(s))\|_p ds \\ & +\int_0^b \|T(b-s)\|_p\|Q(s,K(s,z_1(s)))-Q(s,K(s,z_2(s))ds)\|_p](\eta)d\eta\end{aligned}$$

By using conditions $(\mathbf{H_1}) - (\mathbf{H_9})$, we have that

$$\begin{aligned}\|\Phi z_1(t)-\Phi z_2(t)\|_p \le & \int_o^t MM_1\|z_1(s)-z_2(s)\|_p ds+\int_0^t MM_2M_4\|z_1(s)-z_2(s)\|_p ds \\ & +\int_0^t [M\|AB_2\|_p+v_0(t)+Mc]k_2[bMM_1\|z_1(s)-z_2(s)\|_p+bMM_2M_4\|z_1(s)-z_2(s)\|_p]\,(\eta)d\eta\end{aligned}$$

$$\begin{aligned}\|\Phi z_1(t)-\Phi z_2(t)\|_p \le & \, bMM_1\|z_1(t)-z_2(t)\|_p+bMM_2M_4\|z_1(t)-z_2(t)\|_p+b[M\|AB_2\|_p \\ & + k_3 + Mc]k_2[bMM_1\|z_1(t)-z_2(t)\|_p+bMM_2M_4\|z_1(t)-z_2(t)\|_p]\end{aligned}$$

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \le [bMM_1+bMM_2M_4]\|z_1(t)-z_2(t)\|_p+[bMk_2\|AB_2\|_p+k_2k_3+bk_2Mc][\,bMM_1+bMM_2M_4]\|z_1(t)-z_2(t)\|_p$$

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \le \mathbf{[}[bMM_1+bMM_2M_4]+[bMk_2\|AB_2\|_p+k_2k_3+bk_2Mc][bMM_1+bMM_2M_4]\mathbf{]}\ \|z_1(t)-z_2(t)\|_p$$

By condition $(\mathbf{H_{11}})$, we get that

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \le \lambda\|z_1(t)-z_2(t)\|_p \le \|z_1(t)-z_2(t)\|_p$$

Therefore Φ is nonexpansive mapping, and hence by Theorem 2.3, there exists a fixed point $z \in S_0$, such that $\Phi z(t) = z(t)$, thus any fixed point of Φ is a mild solution of system (1.1) on J, which satisfies $z(b) = z_1$, and hence the system is controllable on J.

Remark 3.1 : For study the controllability of the nonlinear boundary value control problem **(1.1)** in any Banach space $\mathbb{S} = C(J,X)$, the space of continuous functions $f(t)$ in the interval J=[0,b] with $\|f\| = \max_{0\le t\le b} |f(t)|$.

($\mathbf{T_1}$) Since the set S_0, which is defined in section 1, is closed subset of a Banach space $\mathbb{S}$, then S_0 is a Banach space. Thus, if we assuming that $0 \le \lambda < 1$ in the condition (H_{11}), then we can prove that the operator Φ defined from S_0 into S_0 by equation (3.10) is a contraction mapping from a Banach space into a Banach space. Hence by Theorem 2.1, Φ has a unique fixed point $z(t)$ which is a mild solution to the problem (1.1) on J and satisfies $z(b) = z_1$. Therefore the system (1.1) is controllable on J .

($\mathbf{T_2}$) If we assume that in section 1, the semigroup $T(t)$, $t > 0$ is a compact on a Banach space $\mathbb{S}$, and the nonlinear operators f, K and Q are all uniformly bounded continuous operators, then the operator Φ which is defined from S_0 into S_0 by equation (3.10) satisfies the Schauder Theorem 2.2, and hence Φ has a fixed point $z(t)$ which is a solution to the system (1.1) and satisfies $z(b) = z_1$. Thus the system (1.1) is controllable on J. For more details see [7] .

($\mathbf{T_3}$) For boundary value problem (1.1), if we assume that the operators f, K and Q are also satisfy Lipchitz condition on the first argument, and since $S=L_P(J,X)$, $1 < p < \infty$ is reflexive Banach space [see example 2.1], then for every $z_0 \in D(A)$ the problem (1.1) has a unique strong solution $z(.)$ on [0,b] given by (3.8) (for more details see [10, Ch.6, Theorem 1.6]) .

4. APPLICATION

Let ϕ be a bounded and open subset of $\mathbb{R}^n$, and let C be boundary control integrodifferential system

$$\begin{aligned}&\frac{\partial y(t,x)}{\partial t} - Ny(t,x) = \sigma_1\left(t, y(t,x), \int_0^t \sigma_2\left(t,s,y(s,x)\right)ds\right) \text{ in } Y=(0,b)\times\phi \\ & y(t,0)= u(t,0)\ , \text{ on } M=(0,b)\times C\ ,\ t\in[0,b] \\ & y(0,x)= y_0(x),\ \text{ for } x\in\phi \qquad (3.11)\end{aligned}$$

where $u\in L^2(M)$, $y_0\in L^2(\phi)$, $\sigma_1 \in L^2(Y)$ and $\sigma_2 \in Y$.

The above problem can be formulated as a boundary control problem of the form (1.1) by suitably taking the spaces, E, X, U and the operators B_1, σ, and x as follows:

Let $E = L^2(\emptyset)$, $X = H^{-\frac{1}{2}}(\Gamma)$, $U=L^2(C)$, $B_1= I$ (the identity operator) and $D(\sigma)=\{y \in L^2(\emptyset); Ny\in L^2(\emptyset)\}$, $\sigma = N$.

The operator x is the "trace" operator such that $xy = y|_C$ is well defined and belongs to $H^{-\frac{1}{2}}(\Gamma)$ for each $y \in D(\sigma)$ (see [14]) and the operator A is given by $A = N, D(A) = H_0^1(\emptyset) \cup H^2(\emptyset)$ (Here $H^k(\emptyset), H^{\propto}(\Gamma)$ and $H_0^1(\emptyset)$ are usual Sobolev Spaces on $\emptyset$, Γ).

Define the linear operator B: $L^2(C) \to L^2(\emptyset)$ by $Bu = W_u$ where W_u is the unique solution to the Dirichlet boundary value problem,

$$DW_u = 0 \text{ in } \emptyset$$
$$W_u = u \text{ in } C$$

In other words (see [15])

$$\int_{\phi} W_u \Delta W dx = \int_{\Gamma} u \frac{\partial W}{\partial n} dx, \text{ for all } W \in H_0^1 \cup H^2(\phi) \quad (3.12)$$

Where $\frac{\partial W}{\partial n}$ denotes the outward normal derivative of W which is well-defined as an element of $H^{\frac{1}{2}}(\Gamma)$. From (3.12) it follows that,

$$\|Wu\|_{L^2(\phi)} \leq C_1 \|u\|_{H^{-1/2}(\Gamma)}, \text{ for all } u \in H^{1/2}(\Gamma)$$

and

$$\|Wu\|_{H^1(\phi)} \leq C_2 \|u\|_{H^{1/2}(\Gamma)}, \text{ for all } u \in H^{1/2}(\Gamma),$$

where C_i , $i=1,2$ are positive constants independent of u.

From the above estimates it follows by an interpolation argument [16] that

$$\|AT(t)B\|_{L(L^2(\Gamma),L^2(\Gamma))} \leq Ct^{-3/4}, \text{ for all } t > 0 \text{ with } v(t) = Ct^{-3/4}$$

Further assume that the bounded invertible operator $\tilde{W}$ exists.

Choose b and other constants, such that satisfying the last condition **(H_{10})**.

Hence, one can see that all the conditions stated in the theorem are satisfied and so the system (3.11) is controllable on (0,b).

5. CONCLUSIONS

1. Generalize nonlinear boundary value control problem by taking *f*, *K*, and *Q* in systems (1.1) any nonlinear operators which are satisfy Lipschitz condition on the seconed argument, and study the controllability of this system by using C_0 – semigroup and Kirk fixed point theorem.
2. The idea of studing the controllability of problems (1.1) by using Banach fixed point theorem and Schauder fixed point theorem are introduced.

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