

FIXED POINT THEOREMS FOR T_k -CONTRACTIONS IN K-METRIC SPACES

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ABSTRACT

A basic difference between ordinary metric and k -metric is in the triangle inequality. In this paper we have shown that instead of this difference, by restricting the domain of effectively involved constant, some fixed point theorems for T_k -contractions can be obtained as in cone metric spaces.

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1. INTRODUCTION

H. Pajoohesh introduced the concept of k -metrics in 2012 which are valued in lattice ordered groups and that allows to talk about distance in non-abelian lattice ordered groups. He also characterized intrinsic metrics on lattice ordered rings and established that if a lattice ordered ring is representable then every intrinsic metric therein is a k -metric. Being motivated by these facts we study some fixed point theorems for T -contractive mappings in this setting. In this paper we actually translated the results of [1] in the language of k -metric spaces. As in [2] a k -metric, where k is a real number ≥ 1 , on a nonempty set X is a mapping $d : X \times X \rightarrow R$ such that

- (i) $d(x, y) \geq 0 \quad \forall x, y \in X$,
- (ii) $d(x, y) = 0 \Leftrightarrow x = y$,
- (iii) $d(x, y) = d(y, x) \quad \forall x, y \in X$,
- (iv) $d(x, y) \leq k(d(x, z) + d(z, y)) \quad \forall x, y, z \in X$.

The ordered pair (X, d) is called a k -metric space.

Let us consider the mapping $d : R \times R \rightarrow R$ defined by $d(x, y) = (x - y)^2 \quad \forall x, y \in R$. The fact $(a + b)^2 \leq 2(a^2 + b^2) \quad \forall a, b \in R$ ensures that the mapping d enjoys all the properties of being a k -metric for $k = 2$.

From the definition and the example, just given above, it is clear that every metric is a k -metric ($k = 1$), but a k -metric may not be a metric and every k -metric is an l -metric, where $l \geq k$.

Open balls, closed balls, diameter of non empty sets, open sets (A subset O of a k -metric space (X, d) is said to be open in (X, d) if $\forall x \in O \exists \varepsilon > 0$ such that the open ball $B_d(x, \varepsilon) \subset O$), closed sets, closure and interior of a set, convergence of a sequence, Cauchy sequence, completeness of k -metric spaces are defined as in case of metric spaces. It is also seen that every k -metric space is first countable and T_4 .

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2. T_k – CONTRACTIONS AND FIXED POINT THEOREMS

We begin with the following definitions which bear the character of the mappings involved in the theorems of this section.

Definition 1: Let (X, d) be a k -metric space and $T: X \rightarrow X$. T is said to be **sequentially convergent** in X if for every sequence $\{y_n\}$ in X , $\{Ty_n\}$ is convergent implies that $\{y_n\}$ is also convergent.

Definition 2: Let (X, d) be a k -metric space and $T: X \rightarrow X$. T is said to be **sub-sequentially convergent** in X if for every sequence $\{y_n\}$, $\{Ty_n\}$ is convergent implies that $\{y_n\}$ has a convergent subsequence.

Definition 3: Let (X, d) be a k -metric space and $T, S: X \rightarrow X$ are functions. The mapping S is said to be

T_k -**contraction** if there is a constant $\alpha \in \left[0, \frac{1}{k}\right)$ such that

$$d(TSx, TSy) \leq \frac{\alpha}{k} d(Tx, Ty), \quad \forall x, y \in X$$

Theorem 1: Let (X, d) be a complete k -metric space and $T: X \rightarrow X$ be a one to one and continuous function. In addition let $S: X \rightarrow X$ be a T_k -contraction continuous function. Then

- (i) for every $x_0 \in X$, $\lim_{n \rightarrow \infty} d(TS^n x_0, TS^{n+1} x_0) = 0$.
- (ii) there is $y_0 \in X$ such that $\lim_{n \rightarrow \infty} TS^n x_0 = y_0$
- (iii) if T is sub-sequentially convergent, then there is a unique $z_0 \in X$ such that $Sz_0 = z_0$
- (iv) if T is sequentially convergent, then for each $x_0 \in X$, the sequence $\{S^n x_0\}$ converges to z_0 .

Proof: Let $x_1, x_2 \in X$, then

$$\begin{aligned} d(Tx_1, Tx_2) &\leq k[d(Tx_1, TSx_1) + d(TSx_1, Tx_2)] \\ &\leq kd(Tx_1, TSx_1) + k^2[d(TSx_1, TSx_2) + d(TSx_2, Tx_2)] \\ &\leq kd(Tx_1, TSx_1) + k^2 \left[\frac{\alpha}{k} d(Tx_1, Tx_2) + d(TSx_2, Tx_2) \right] \end{aligned}$$

$$d(Tx_1, Tx_2) \leq \frac{k}{1 - k\alpha} [d(Tx_1, TSx_1) + kd(TSx_2, Tx_2)] \text{-----} > (A)$$

(i) Let $x_0 \in X$ and consider the sequence $\{x_n\}$ given by $x_n = Sx_{n-1} = S^2x_{n-2} = \dots = S^n x_0, \forall n \in \mathbf{N}$

Now,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TS^n x_0, TS^{n+1} x_0) \\ &= d(TS(S^{n-1} x_0), TS(S^n x_0)) \\ &\leq \frac{\alpha}{k} d(T(S^{n-1} x_0), T(S^n x_0)) \\ &= \frac{\alpha}{k} d(TS(S^{n-2} x_0), TS(S^{n-1} x_0)) \\ &\leq \frac{\alpha^2}{k^2} d(T(S^{n-2} x_0), T(S^{n-1} x_0)) \end{aligned}$$

continuing this process, we get

$$d(Tx_n, Tx_{n+1}) \leq \frac{\alpha^n}{k^n} d(Tx_0, TSx_0)$$

So $\lim_{n \rightarrow \infty} d(TS^n x_0, TS^{n+1} x_0) = 0$, since $0 \leq \frac{\alpha}{k} < 1$.

(ii) For, $m, n \in \mathbf{N}$ with $m > n$,

$$\begin{aligned} d(Tx_n, Tx_m) &= d(TS^n x_0, TS^m x_0) \\ &\leq \frac{k}{1-k\alpha} [d(TS^n x_0, TS^{n+1} x_0) + kd(TS^{m+1} x_0, TS^m x_0)] \text{ (by A)} \\ &\leq \frac{k}{1-k\alpha} \left[\frac{\alpha^n}{k^n} d(Tx_0, TSx_0) + k \frac{\alpha^m}{k^m} d(Tx_0, TSx_0) \right] \\ &\leq \frac{k}{1-k\alpha} \left[\frac{\alpha^n}{k^n} + k \frac{\alpha^m}{k^m} \right] d(Tx_0, TSx_0) \end{aligned}$$

which implies,

$$\begin{aligned} \lim_{n, m \rightarrow \infty} d(Tx_n, Tx_m) &= 0 \\ \lim_{n, m \rightarrow \infty} d(TS^n x_0, TS^m x_0) &= 0 \end{aligned}$$

Therefore, $\{TS^n x_0\}$ is a Cauchy sequence in X . By completeness of X , $\exists y_0 \in X$ such that

$$\lim_{n \rightarrow \infty} TS^n x_0 = y_0$$

(iii) Let T be sub-sequentially convergent, then since $\{TS^n x_0\}$ is convergent the sequence $\{S^n x_0\}$ has a subsequence $\{S^{n_i} x_0\}$ such that

$$\begin{aligned} \lim_{n_i \rightarrow \infty} S^{n_i} x_0 &= z_0 \in X \tag{1} \\ \Rightarrow \lim_{n_i \rightarrow \infty} TS^{n_i} x_0 &= Tz_0, \text{ [Since, } T \text{ is continuous]} \\ \Rightarrow Tz_0 &= y_0 \end{aligned}$$

By (1),

$$\begin{aligned} \lim_{n_i \rightarrow \infty} S^{n_i+1} x_0 &= Sz_0 \text{ [since } S \text{ is discontinuous]} \\ \Rightarrow \lim_{n_i \rightarrow \infty} TS^{n_i+1} x_0 &= TSz_0 \Rightarrow y_0 = TSz_0 \end{aligned}$$

Then, we have

$$\begin{aligned} TSz_0 &= Tz_0 \\ \Rightarrow Sz_0 &= z_0 \text{ [since } T \text{ is oneone]} \end{aligned}$$

Therefore z_0 is a fixed point of S .

We now prove the uniqueness of fixed point of S .

Let $z_0, z_1 \in X$ so that $Sz_0 = z_0$ and $Sz_1 = z_1$. Then

$$\begin{aligned} 0 \leq d(Tz_0, Tz_1) &= d(TSz_0, TSz_1) \leq \frac{\alpha}{k} d(Tz_0, Tz_1) \\ \Rightarrow d(Tz_0, Tz_1) &= 0 \Rightarrow Tz_0 = Tz_1 \Rightarrow z_0 = z_1 \text{ (since } T \text{ is oneone)}. \end{aligned}$$

(iv) This is a special case of (iii).

Note 1: In the above **Theorem 1** if we take T as identity map and $k = 1$ then we obtain the classical Banach fixed point theorem.

Theorem 2: Let (X, d) be a complete k -metric space and $T : X \rightarrow X$ be an injective and continuous function. Suppose that $S : X \rightarrow X$ is a mapping such that S^n is a T_k -contraction for some $n \in \mathbb{N}$. Then S has a unique fixed point in X

Proof: As in general case.

Theorem 3: Let, (X, d) be a complete k -metric space and $T : X \rightarrow X$ be an injective and continuous mapping. For $c > 0$ and $x_0 \in X$, set $B(Tx_0, c) = \{y \in X : d(Tx_0, y) < c\}$. Suppose $S : X \rightarrow X$ be a T_k -contraction continuous mapping for all $x, y \in B(Tx_0, c)$ satisfying $d(TSx_0, Tx_0) < (1 - \alpha) \frac{c}{k}$.

Then S has a unique fixed point in $\overline{B(Tx_0, c)}$.

Proof: Set $x_n = Sx_{n-1} \quad \forall n \in \mathbb{N}$. By the given condition

$$d(Tx_1, Tx_0) = d(TSx_0, Tx_0) \leq (1 - \alpha)c/k < c \Rightarrow Tx_1 \in B(Tx_0, c)$$

Also whenever $Tx \in B(Tx_0, c)$, $TSx \in B(Tx_0, c)$.

For, let $Tx \in B(Tx_0, c)$, then $d(Tx_0, Tx) < c$ and

$$d(Tx_0, TSx) \leq k[d(TSx_0, Tx_0) + d(TSx_0, TSx)]$$

$$\leq k(1 - \alpha) \frac{c}{k} + k \frac{\alpha}{k} d(Tx_0, Tx)$$

$$< (1 - \alpha)c + \alpha c = c$$

$$\Rightarrow TSx \in B(Tx_0, c)$$

So, $TSx_1 = Tx_2 \in B(Tx_0, c)$ and consequently $\{Tx_n\}$ is a sequence in $B(Tx_0, c)$. Now a proof of this theorem follows from the **Theorem 1** and completeness of $\overline{B(Tx_0, c)}$.

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