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GROWTH OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLE BASED ON RELATIVE ORDER

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ABSTRACT

In this paper we introduce the idea of comparative growth properties of entire functions of two complex variable on the basis relative order, relative L-order and relative L^*-order.

Keywords: *Entire function, relative order, relative L-order, relative L^* order, growth, poly disc.*

INTRODUCTION

In the poly disc, f be an entire function of two complex variables holomorphic

$$U = \{(z_1, z_2) : |z_i| \le r_i, i = 1,2 \; \forall \; r_1 \ge 0, r_2 \ge 0\}$$

and

$$M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \le r_i, i = 1,2\}$$

From maximum principle and Hartog's theorem in [2], $M_f(r_1, r_2)$ is increasing function of r_1, r_2 in [2].

Definition 1: In [1, 2] the order ${}_{v_2}\rho_f$ and the lower order ${}_{v_2}\lambda_f$ of an entire function f of two complex variables are defined as

$${}_{v_2}\rho_f = \lim_{r_1, r_2 \to \infty} \sup \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)}$$

and

$${}_{v_2}\lambda_f = \lim_{r_1, r_2 \to \infty} \inf \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)}$$

If f is regular then order and lower order are the same .An entire function of two complex variables which is not equal is called irregular .For single variable, above definition concides with the classical definition of order in [18] L.Bernal in [3, 4] introduced the definition of relative order between two entire functions of single variable. During the past decades, several authors in see [13, 14, 15, 16] made close investigations on the properties of relative order of entire functions of single variable. In facts, some works relating to the growth estimates of composite entire functions of single variable on the basis of relative order of entire functions have been explored in [6, 7, 8, 9, 10]. In case of relative order Banerjee and Dutta [5] to define the relative order of entire functions of two complex variables as follows.

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Definition 2: The relative order between two entire functions of two complex variables denoted by

$$ {}_{v_2}\rho_g(f) = \lim_{r_1,r_2\to\infty}\sup \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log(r_1r_2)} $$

where g is an entire function holomorphic in the closed polydisc $U=\{(z_1,z_2):|z_i|\le r_i, i=1,2\ \forall, r_1\ge 0$, $r_2\ge 0\}$ and if $g(z)=e^{z_1z_2}$ the definition coincides with definition 1 in see [5] the relative lower order of f with respect to g

$$ {}_{v_2}\lambda_g(f) = \lim_{r_1,r_2\to\infty}\inf \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log(r_1r_2)} $$

for two complex variable relative order and relative lower order of entire function with respect to another entire functions are the same is said to be regular relative growth not regular is called irregular relative growth.

Definition 3: The relative hyper order ${}_{v_2}\overline{\rho}_g(f)$ and relative hyper lower order ${}_{v_2}\overline{\lambda}_g(f)$ of an entire function f of two complex variable with respect to another entire g of two complex variable are defined as follows:

$$ {}_{v_2}\overline{\rho}_g(f) = \lim_{r_1,r_2\to\infty}\inf \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log(r_1r_2)} $$

and

$$ {}_{v_2}\overline{\lambda}_g(f) = \lim_{r_1,r_2\to\infty}\inf \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log(r_1r_2)} $$

Somasundaram and Thamizharasi [18] introduced the notion of L-order (L-lower order) for entire functions where $L=L(r_1,r_2)$ is a positive continuos functions increasing slowely i.e., $L(ar_1,ar_2)\approx L(r_1,r_2)$ as $r_1,r_2\to\infty$ for every positive constant 'a' their definition are as follows:

Definition 4: [18] The L-order ${}_{v_2}\rho_f^L$ and the L-lower order ${}_{v_2}\lambda_f^L$ of an entire function f of two variable defined as follows:

$$ {}_{v_2}\rho_f^L = \lim_{r_1,r_2\to\infty}\sup \frac{\log^{[2]} M_f(r_1,r_2)}{\log[(r_1r_2)L(r_1,r_2)]} $$

and

$$ {}_{v_2}\lambda_f^L = \lim_{r_1,r_2\to\infty}\inf \frac{\log^{[2]} M_f(r_1,r_2)}{\log[(r_1r_2)L(r_1,r_2)]} $$

Similarly one can define the L-hyper order and ${}_{v_2}\overline{\rho}_f^L$ the L-hyper lower order of ${}_{v_2}\overline{\lambda}_f^L$ an entire function f.

Definition 5: The definition of relative L-order of f with respect to g denoted by ${}_{v_2}\rho_g^L(f)$ as follows ${}_{v_2}\rho_g^L(f)=\inf\{\mu>0\ :\ M_f(r_1,r_2)<M_g[(r_1r_2)L(r_1,r_2)]^{\mu}$ for all $r_1,r_2>r_0(\mu)>0\}$ or

$$ {}_{v_2}\rho_g^L(f) = \lim_{r_1,r_2\to\infty}\sup \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)L(r_1,r_2)]} $$

Similarly we may define the relative order L-lower order of f of two complex variable with respect to g denoted by

$$ {}_{v_2}\lambda_g^L(f) = \lim_{r_1,r_2\to\infty}\inf \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)L(r_1,r_2)]} $$

Definition 6: The relative L-hyper order ${}_{v_2}\overline{\rho}_g^L(f)$ and the L-hyper lower order ${}_{v_2}\overline{\lambda}_g^L(f)$ of an entire function f of two complex variable with respect to another entire function g of two complex variable are defined as follows:

$$ {}_{v_2}\overline{\rho}_g^L(f) = \lim_{r_1,r_2\to\infty}\sup \frac{\log^{[2]} M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)L(r_1,r_2)]} $$

$$_{v_2}\overline{\lambda}_g^{L}(f)=\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[2]}M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)L(r_1,r_2)]}$$

the more generalized concept of L^*-order(L^*-lower order) of an entire function is L^*-order(L^*-lower order) their definition as follows:

Definition 7: The L^*-order and L^*-lower order of an entire function f denoted by

$$_{v_2}\rho_f^{L^*}=\lim_{r_1,r_2\to\infty}\sup\frac{\log^{[2]}M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)e^{L(r_1,r_2)}]}$$

$$_{v_2}\lambda_f^{L^*}=\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[2]}M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)e^{L(r_1,r_2)}]}$$

the more generalized concept of relative L-order and the relative L-hyper oder are the relative L^*-order and relative L^*-hyper order respectively their definitions as follows:

Definition 8: The relative L^*-order of an entire function f of two complex variable with respect to another entire function g, denoted by

$$_{v_2}\rho_g^{L}(f)=\inf\{\mu>0\ :\ M_f(r_1,r_2)<M_g[(r_1r_2)e^{L(r_1,r_2)}]^{\mu}\ \text{for all}\ r_1,r_2>r_0(\mu)>0\}$$

$$_{v_2}\rho_g^{L^*}(f)=\lim_{r_1,r_2\to\infty}\sup\frac{\log M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)e^{L(r_1,r_2)}]}$$

and

$$_{v_2}\lambda_g^{L^*}(f)=\lim_{r_1,r_2\to\infty}\inf\frac{\log M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)e^{L(r_1,r_2)}]}.$$

Definition 9: The relative L^*-hyper order $_{v_2}\overline{\rho}_g^{L^*}(f)$ and relative L^*-hyper lower order $_{v_2}\overline{\lambda}_g^{L^*}(f)$ of entire function f with respect to another entire function g are defined as follows:

$$_{v_2}\overline{\rho}_g^{L^*}(f)=\lim_{r_1,r_2\to\infty}\sup\frac{\log^{[2]}M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)e^{L(r_1,r_2)}]}$$

and

$$_{v_2}\overline{\lambda}_g^{L^*}(f)=\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[2]}M_g^{-1}M_f(r_1,r_2)}{\log[(r_1r_2)e^{L(r_1,r_2)}]}.$$

Some results on the comparative growth properties of entire of two complex variable functions on the basis of relative order(relative lower order),relative L-order(relative L-lower order) and relative L^*-order(relative L^*-lower order)have been proved earlier.

In this section we present the main results of the paper.

Theorem 1: Let f,g and h be three entire functions such that $0<_{v_2}\lambda_g(f)\le_{v_2}\rho_g(f)<\infty$ and $0<_{v_2}\lambda_g(h)\le_{v_2}\rho_g(h)<\infty$ then

$$\frac{_{v_2}\lambda_g(f)}{_{v_2}\rho_g(h)}\le\lim_{r_1,r_2\to\infty}\inf\frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)}\le\frac{_{v_2}\lambda_g(f)}{_{v_2}\lambda_g(h)}$$

$$\le\lim_{r_1,r_2\to\infty}\sup\frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)}\le\frac{_{v_2}\rho_g(f)}{_{v_2}\lambda_g(h)}.$$

Proof: From the definition of relative order and relative lower order we have for arbitrary positive ε and for all large values of r_1, r_2

$$\log M_g^{-1} M_f(r_1,r_2) \geq ({}_{v_2}\lambda_g(f) - \varepsilon)\log r_1 r_2 \tag{1}$$

and

$$\log M_g^{-1} M_h(r_1,r_2) \geq ({}_{v_2}\rho_g(f) + \varepsilon)\log r_1 r_2 \tag{2}$$

Now from (1) and (2) it follows for all large values of r_1, r_2

$$\frac{\log M_g^{-1} M_f(r_1,r_2)}{\log M_g^{-1} M_h(r_1,r_2)} \geq \frac{{}_{v_2}\lambda_g(f) - \varepsilon}{{}_{v_2}\rho_g(h) + \varepsilon}$$

As $\varepsilon > 0$ is arbitrary, we obtain that

$$\lim_{r_1,r_2\to\infty}\inf \frac{\log M_g^{-1} M_f(r_1,r_2)}{\log M_g^{-1} M_h(r_1,r_2)} \geq \frac{{}_{v_2}\lambda_g(f)}{{}_{v_2}\rho_g(h)} \tag{3}$$

Again for a sequence of values of r_1, r_2 tending to infinity

$$\log M_g^{-1} M_f(r_1,r_2) \leq ({}_{v_2}\lambda_g(f) + \varepsilon)\log r_1 r_2 \tag{4}$$

and for all large values of r_1, r_2

$$\log M_g^{-1} M_h(r_1,r_2) \geq ({}_{v_2}\lambda_g(f) - \varepsilon)\log r_1 r_2 \tag{5}$$

So combining (4) and (5) we get

$$\frac{\log M_g^{-1} M_f(r_1,r_2)}{\log M_g^{-1} M_h(r_1,r_2)} \leq \frac{{}_{v_2}\lambda_g(f) + \varepsilon}{{}_{v_2}\lambda_g(h) - \varepsilon}$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$\lim_{r_1,r_2\to\infty}\inf \frac{\log M_g^{-1} M_f(r_1,r_2)}{\log M_g^{-1} M_h(r_1,r_2)} \leq \frac{{}_{v_2}\lambda_g(f)}{{}_{v_2}\lambda_g(h)} \tag{6}$$

Also for sequence of values of r_1, r_2 tending to infinity,

$$\log M_g^{-1} M_h(r_1,r_2) \leq ({}_{v_2}\lambda_g(h) + \varepsilon)\log r_1 r_2 \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of r_1, r_2 tending to infinity

$$\frac{\log M_g^{-1} M_f(r_1,r_2)}{\log M_g^{-1} M_h(r_1,r_2)} \geq \frac{{}_{v_2}\lambda_g(f) - \varepsilon}{{}_{v_2}\lambda_g(h) + \varepsilon}$$

Choosing $\varepsilon \to 0$ we get from above that

$$\frac{\log M_g^{-1} M_f(r_1,r_2)}{\log M_g^{-1} M_h(r_1,r_2)} \geq \frac{{}_{v_2}\lambda_g(f)}{{}_{v_2}\lambda_g(h)} \tag{8}$$

Also for all large values of r_1, r_2

$$\log M_g^{-1} M_f(r_1,r_2) \leq ({}_{v_2}\rho_g(f) + \varepsilon)\log r_1 r_2 \tag{9}$$

so from (5) and (9) it follows for all large values of r_1, r_2,

$$\frac{\log M_g^{-1} M_f(r_1,r_2)}{\log M_g^{-1} M_h(r_1,r_2)} \leq \frac{{}_{v_2}\rho_g(f)}{{}_{v_2}\lambda_g(h)} \tag{10}$$

Thus the theorem follows from (3) (6), (8) and (10).The following theorem can be proved in the line Theorem 1 and so the proof is omitted.

Theorem 2: Let f, g and h be three entire functions such that $0 < {}_{v_2}\lambda_g^L(f) \leq {}_{v_2}\rho_g^L(f) < \infty$ and $0 < {}_{v_2}\lambda_g^L(h) \leq {}_{v_2}\rho_g^L(h) < \infty$ then

$$\frac{{}_{v_2}\lambda_g^L(f)}{{}_{v_2}\rho_g^L(h)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\lambda_g^L(f)}{{}_{v_2}\lambda_g^L(h)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\rho_g^L(f)}{{}_{v_2}\lambda_g^L(h)}$$

Using the more generalized concept of relative L-order (relative L-lower order) and relative L^*-order (relative L^*-lower order) of an entire functions of two complex variable with respect to another entire function we may state the following theorem without proof.

Theorem 3: Let f, g and h be three entire functions such that $0 < {}_{v_2}\lambda_g^{L^*}(f) \leq {}_{v_2}\rho_g^{L^*}(f) < \infty$ and $0 < {}_{v_2}\lambda_g^{L^*}(h) \leq {}_{v_2}\rho_g^{L^*}(h) < \infty$ then

$$\frac{{}_{v_2}\lambda_g^{L^*}(f)}{{}_{v_2}\rho_g^{L^*}(h)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\lambda_g^{L^*}(f)}{{}_{v_2}\lambda_g^{L^*}(h)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\rho_g^{L^*}(f)}{{}_{v_2}\lambda_g^{L^*}(h)}.$$

Theorem 4: Let f, g and h be three entire functions such that $0 < {}_{v_2}\lambda_g(f) \leq {}_{v_2}\rho_g(f) < \infty$ and $0 < {}_{v_2}\rho_g(h) < \infty$ then

$$\liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\rho_g(f)}{{}_{v_2}\rho_g(h)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)}.$$

Proof: From definition of relative order we get for sequence of values of r_1, r_2 tending to infinity

$$\log M_g^{-1}M_h(r_1,r_2) \geq ({}_{v_2}\rho_g(h) - \varepsilon)\log r_1 r_2 \tag{11}$$

Now from (9) and (11) it follows for a sequence of values of r_1, r_2 tending to infinity,

$$\frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\rho_g(f) + \varepsilon}{{}_{v_2}\rho_g(h) - \varepsilon}$$

as $\varepsilon(>0)$ is arbitrary we obtain that

$$\liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\rho_g(f)}{{}_{v_2}\rho_g(h)} \tag{12}$$

Again, for a sequence of values of r_1, r_2 tending to infinity

$$\log M_g^{-1}M_f(r_1,r_2) \geq ({}_{v_2}\rho_g(f) - \varepsilon)\log r_1 r_2 \tag{13}$$

So combining (2) and (13) we get for sequence of values of r_1, r_2 tending to infinity,

$$\frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \geq \frac{{}_{v_2}\rho_g(f) - \varepsilon}{{}_{v_2}\rho_g(h) + \varepsilon}$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \geq \frac{{}_{v_2}\rho_g(f)}{{}_{v_2}\rho_g(h)} \tag{14}$$

Thus the theorem follows from (12) and (14). The following two theorems can be carried out in the line of theorem 4 and therefore we omit their proofs.

Theorem 5: Let f, g and h be three entire functions such that $0 <_{v_2} \lambda_g^L(f) \leq_{v_2} \rho_g^L(f) < \infty$ and $0 <_{v_2} \rho_g^L(h) < \infty$ then

$$\liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\rho_g^L(f)}{{}_{v_2}\rho_g^L(h)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)}.$$

Theorem 6: Let f, g and h be three entire functions such that $0 < {}_{v_2}\lambda_g^{L^*}(f) \leq {}_{v_2}\rho_g^{L^*}(f) < \infty$ and $0 < {}_{v_2}\rho_g^{L^*}(h) < \infty$ then

$$\liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \frac{{}_{v_2}\rho_g^{L^*}(f)}{{}_{v_2}\rho_g^{L^*}(h)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)}.$$

the following theorem is a natural consequence of theorem 1 and theorem 4.

Theorem 7: Let f, g and h be three entire functions such that $0 <_{v_2} \lambda_g(f) \leq {}_{v_2}\rho_g(f) < \infty$ and $0 < {}_{v_2}\lambda_g(h) \leq {}_{v_2}\rho_g(h) < \infty$ then

$$\liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \min\left\{\frac{{}_{v_2}\lambda_g(f)}{{}_{v_2}\lambda_g(h)}, \frac{{}_{v_2}\rho_g(f)}{{}_{v_2}\rho_g(h)}\right\}$$

$$\leq \max\left\{\frac{{}_{v_2}\lambda_g(f)}{{}_{v_2}\lambda_g(h)}, \frac{{}_{v_2}\rho_g(f)}{{}_{v_2}\rho_g(h)}\right\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)}$$

the proof is omitted. Analogously the following two theorems without proof.

Theorem 8: Let f, g and h be three entire functions such that $0 < {}_{v_2}\lambda_g^L(f) \leq {}_{v_2}\rho_g^L(f) < \infty$ and $0 < {}_{v_2}\lambda_g^L(h) \leq {}_{v_2}\rho_g^L(h) < \infty$ then

$$\liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \min\left\{\frac{{}_{v_2}\lambda_g^L(f)}{{}_{v_2}\lambda_g^L(h)}, \frac{{}_{v_2}\rho_g^L(f)}{{}_{v_2}\rho_g^L(h)}\right\}$$

$$\leq \max\left\{\frac{{}_{v_2}\lambda_g^L(f)}{{}_{v_2}\lambda_g^L(h)}, \frac{{}_{v_2}\rho_g^L(f)}{{}_{v_2}\rho_g^L(h)}\right\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)}.$$

Theorem 9: Let f, g and h be three entire functions such that $0 < {}_{v_2}\lambda_g^{L^*}(f) \leq {}_{v_2}\rho_g^{L^*}(f) < \infty$ and $0 < {}_{v_2}\lambda_g^{L^*}(h) \leq {}_{v_2}\rho_g^{L^*}(h) < \infty$ then

$$\liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)} \leq \min\left\{\frac{{}_{v_2}\lambda_g^{L^*}(f)}{{}_{v_2}\lambda_g^{L^*}(h)}, \frac{{}_{v_2}\rho_g^{L^*}(f)}{{}_{v_2}\rho_g^{L^*}(h)}\right\}$$

$$\leq \max\left\{\frac{{}_{v_2}\lambda_g^{L^*}(f)}{{}_{v_2}\lambda_g^{L^*}(h)}, \frac{{}_{v_2}\rho_g^{L^*}(f)}{{}_{v_2}\rho_g^{L^*}(h)}\right\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log M_g^{-1}M_h(r_1,r_2)}$$

We state the following theorems without proof based on relative hyper order, relative L-hyper order and relative L^*-hyper order of an entire function with respect to another entire function.

Theorem 10: Let f, g and h be three entire functions such that $0 < {}_{v_2}\overline{\lambda}_g(f) \le {}_{v_2}\overline{\rho}_g(f) < \infty$ and $0 < {}_{v_2}\overline{\lambda}_g(h) \le {}_{v_2}\overline{\rho}_g(h) < \infty$ then

$$\frac{{}_{v_2}\overline{\lambda}_g(f)}{{}_{v_2}\overline{\rho}_g(h)} \le \liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\lambda}_g(f)}{{}_{v_2}\overline{\lambda}_g(h)}$$

$$\le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\rho}_g(f)}{{}_{v_2}\overline{\lambda}_g(h)}.$$

Theorem 11: Let f, g and h be three entire functions such that $0 <_{v_2} \overline{\lambda}_g(f) \le_{v_2} \overline{\rho}_g(f) < \infty$ and $0 <_{v_2} \overline{\rho}_g(h) < \infty$ then

$$\liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\rho}_g(f)}{{}_{v_2}\overline{\rho}_g(h)} \le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)}.$$

Theorem 12: Let f, g and h be three entire functions such that $0 < {}_{v_2}\overline{\lambda}_g(f) \le {}_{v_2}\overline{\rho}_g(f) < \infty$ and $0 < {}_{v_2}\overline{\lambda}_g(h) \le {}_{v_2}\overline{\rho}_g(h) < \infty$ then

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \min\left\{\frac{{}_{v_2}\overline{\lambda}_g(f)}{{}_{v_2}\overline{\lambda}_g(h)}, \frac{{}_{v_2}\overline{\rho}_g(f)}{{}_{v_2}\overline{\rho}_g(h)}\right\}$$

$$\le \max\left\{\frac{{}_{v_2}\overline{\lambda}_g(f)}{{}_{v_2}\overline{\lambda}_g(h)}, \frac{{}_{v_2}\overline{\rho}_g(f)}{{}_{v_2}\overline{\rho}_g(h)}\right\} \le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)}.$$

Theorem 13: Let f, g and h be three entire functions such that $0 < {}_{v_2}\overline{\lambda}_g^L(f) \le {}_{v_2}\overline{\rho}_g^L(f) < \infty$ and $0 < {}_{v_2}\overline{\lambda}_g^L(h) \le {}_{v_2}\overline{\rho}_g^L(h) < \infty$ then

$$\frac{{}_{v_2}\overline{\lambda}_g^L(f)}{{}_{v_2}\overline{\rho}_g^L(h)} \le \liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\lambda}_g^L(f)}{{}_{v_2}\overline{\lambda}_g^L(h)} \le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\rho}_g^L(f)}{{}_{v_2}\overline{\lambda}_g^L(h)}.$$

Theorem 14: Let f, g and h be three entire functions such that $0 < {}_{v_2}\overline{\lambda}_g^L(f) \le {}_{v_2}\overline{\rho}_g^L(f) < \infty$ and $0 < {}_{v_2}\overline{\rho}_g^L(h) < \infty$ then

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\rho}_g^L(f)}{{}_{v_2}\overline{\rho}_g^L(h)} \le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)}.$$

Theorem 15: Let f, g and h be three entire functions such that $0 < {}_{v_2}\overline{\lambda}_g^L(f) \le {}_{v_2}\overline{\rho}_g^L(f) < \infty$ and $0 < {}_{v_2}\overline{\lambda}_g^L(h) \le {}_{v_2}\overline{\rho}_g^L(h) < \infty$ then

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \min\left\{\frac{{}_{v_2}\overline{\lambda}_g^L(f)}{{}_{v_2}\overline{\lambda}_g^L(h)}, \frac{{}_{v_2}\overline{\rho}_g^L(f)}{{}_{v_2}\overline{\rho}_g^L(h)}\right\}$$

$$\le \max\left\{\frac{{}_{v_2}\overline{\lambda}_g^L(f)}{{}_{v_2}\overline{\lambda}_g^L(h)}, \frac{{}_{v_2}\overline{\rho}_g^L(f)}{{}_{v_2}\overline{\rho}_g^L(h)}\right\} \le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)}.$$

Theorem 16: Let f, g and h be three entire functions such that $0 < {}_{v_2}\overline{\lambda}_g^{L^*}(f) \le {}_{v_2}\overline{\rho}_g^{L^*}(f) < \infty$ and $0 < {}_{v_2}\overline{\lambda}_g^{L^*}(h) \le {}_{v_2}\overline{\rho}_g^{L^*}(h) < \infty$ then

$$\frac{{}_{v_2}\overline{\lambda}_g^{L^*}(f)}{{}_{v_2}\overline{\rho}_g^{L^*}(h)} \le \liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\lambda}_g^{L^*}(f)}{{}_{v_2}\overline{\lambda}_g^{L^*}(h)} \le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\rho}_g^{L^*}(f)}{{}_{v_2}\overline{\lambda}_g^{L^*}(h)}.$$

Theorem 17: Let f, g and h be three entire functions such that $0 < {}_{v_2}\overline{\lambda}_g^{L^*}(f) \le {}_{v_2}\overline{\rho}_g^{L^*}(f) < \infty$ and $0 < {}_{v_2}\overline{\rho}_g^{L^*}(h) < \infty$ then

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \frac{{}_{v_2}\overline{\rho}_g^{L^*}(f)}{{}_{v_2}\overline{\rho}_g^{L^*}(h)} \le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)}.$$

Theorem 18: Let f, g and h be three entire functions such that $0 < {}_{v_2}\overline{\lambda}_g^{L^*}(f) \le {}_{v_2}\overline{\rho}_g^{L^*}(f) < \infty$ and $0 < {}_{v_2}\overline{\lambda}_g^{L^*}(h) \le {}_{v_2}\overline{\rho}_g^{L^*}(h) < \infty$ then

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)} \le \min\left\{ \frac{{}_{v_2}\overline{\lambda}_g^{L^*}(f)}{{}_{v_2}\overline{\lambda}_g^{L^*}(h)}, \frac{{}_{v_2}\overline{\rho}_g^{L^*}(f)}{{}_{v_2}\overline{\rho}_g^{L^*}(h)} \right\}$$

$$\le \max\left\{ \frac{{}_{v_2}\overline{\lambda}_g^{L^*}(f)}{{}_{v_2}\overline{\lambda}_g^{L^*}(h)}, \frac{{}_{v_2}\overline{\rho}_g^{L^*}(f)}{{}_{v_2}\overline{\rho}_g^{L^*}(h)} \right\} \le \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_g^{-1} M_f(r_1, r_2)}{\log^{[2]} M_g^{-1} M_h(r_1, r_2)}.$$

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