International Journal of Mathematical Archive-6(11), 2015, 185-194
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ECCENTRICITY PROPERTIES OF $BG_4(G)$

[1]M. BHANUMATHI, [2]M. KAVITHA*

[1]Associate Professor, Govt. Arts College for Women, Pudukkottai-622 001, (T.N.), India.

[2]Research Scholar, Govt. Arts College for Women, Pudukkottai-622 001, (T.N.), India.

(Received On: 13-11-15; Revised & Accepted On: 30-11-15)

ABSTRACT

Let G be a simple (p, q) graph with vertex se V(G) and edge set E(G). $B_{\overline{G},NINC,\overline{K}q}(G)$ *is a graph with vertex set* $V(G)\cup E(G)$ *and two vertices are adjacent if and only if they correspond to two non adjacent vertices of G, a vertex and an edge not incident to it in G. For simplicity, denote this graph by* $BG_4(G)$*, Boolean graph fourth kind of G. In this paper, eccentricity properties of* $BG_4(G)$ *and its complement* $\overline{BG_4}(G)$ *are studied.*

Keywords: *Eccentricity, Boolean graph* $BG_4(G)$*.*

2010 Mathematics Subject Classification: *05C69, 05C12.*

1. INTRODUCTION

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For a graph theoretic terminology refer to Harary [4], Buckley and Harary [3].

Let G be a connected graph and u be a vertex of G. The eccentricity e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max \{d(u, v) : u \in V\}$. The radius r(G) is the minimum eccentricity of the vertices, whereas the diameter diam(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central vertex if e(v) = r(G). The center C(G) is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a peripheral vertex if e(v) = diam(G). The periphery P(G) is the set of all such vertices. For a vertex v, each vertex at distance e(v) from v is an eccentric node of v.

A subgraph of G is a graph having all of its vertices and edges in G. It is a spanning subgraph if it contains all the vertices of G. If H is a subgraph of G, then G is a super graph of H. For any set S of vertices in G, the induced subgraph < S > is the maximal subgraph with vertex set S.

A graph G is complete if every pair of its vertices is adjacent. K_n denotes the complete graph on n vertices.

The complement $\overline{G}$ of a graph G is the graph with vertex set V(G) such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A self-complementary graph is isomorphic to its complement.

A graph G is connected if there is a path joining each pair of vertices. A component of a graph is a maximal connected subgraph. If a graph has only one component, then it is connected. Otherwise it is disconnected. The diameter diam(G) of a connected graph G is the length of any, longest geodesic (diametral path).

The Line graph L(G) of a graph G is the graph whose vertices correspond to the edges of G and two vertices of L(G) are adjacent if and only if the corresponding edges of G are adjacent.

The Total graph T(G) of a graph G is the graph whose vertices correspond to the set of vertices and edges of G and two vertices of T(G) are adjacent if and only if the corresponding elements of G are adjacent or incident.

*Corresponding Author: M. Kavitha**
[2]Research Scholar, Govt. Arts College for Women, Pudukkottai-622 001, (T.N.), India.

Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $B_{\overline{G},NINC,\overline{K}_q}(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two non adjacent vertices of G, a vertex and an edge not incident to it in G. For simplicity, denote this graph by $BG_4(G)$, Boolean graph fourth kind of G. In [1] properties of $BG_4(G)$ is studied. The vertices of $BG_4(G)$, which are in V(G) are called point vertices and vertices in E(G) are called line vertices. $V(BG_4(G)) = V(G) \cup E(G)$, $E(BG_4(G)) = E(\overline{T}(G) - E(\overline{L}(G))$, where T(G) is the total graph of G and L(G) is the line graph of G.

In [2, 5, 6, 7] Janakiraman, Bhanumathi and Muthammai have defined and studied the properties of Boolean graphs. Motivated by this, here we study the eccentric properties of Boolean graph $BG_4(G)$.

2.1 Eccentricity Properties of $BG_4(G)$

In this section, radius and diameter of $BG_4(G)$ are found out. Throughout this section, if $v \in V(G)$ and $e \in E(G)$, the corresponding vertices of $BG_4(G)$ are denoted by v' and e'.

Observation 2.1.1: If G is totally disconnected then the eccentricity of every vertex in $BG_4(G)$ is one and $BG_4(G)$ is K_p.

Proposition 2.1.1: If $BG_4(G)$ is connected, eccentricity of a point vertex is 1, 2 or 3.

Proof: Consider a point v in V(G). If G has an isolated vertex then the eccentricity of that vertex in $BG_4(G)$ is one.
Assume G has no isolated vertex.

To find $d(u', v')$ in $BG_4(G)$ where $u, v \in V(G)$. G has no isolated vertex. Take $u, v \in V(G)$. If u and v are non adjacent in G then $d(u', v') = 1$ in $BG_4(G)$. Suppose u and v are adjacent in G. If u and v have a common non incident edge or non adjacent vertex then we have a shortest path $u'w'v'$ or $u'e'v'$. Hence $d(u', v') = 2$.

Suppose u and v does not have a common non incident edge or vertex in G, but u has a non adjacent vertex w and v has a non incident edge e and vice versa. Then we have a shortest path $u'w'e'v'$. since w is non incident with e in G. Hence, $d(u', v') = 3$.

Suppose a vertex is adjacent to other vertices and incident with all the edges of G, then that vertex is isolated in $BG_4(G)$.

To find $d(v', e')$ in $BG_4(G)$ where $e \in E(G)$ and $v \in V(G)$.

If e is not incident with v in G then $d(v', e') = 1$ in $BG_4(G)$.

Suppose e is incident with v in G. Let $e = vv_1 \in E(G)$. In $BG_4(G)$, e' is not incident to v'. If there exists another vertex v_2, which is not adjacent to v in G, then $v'v_2'e'$ is a shortest path in $BG_4(G)$ and hence $d(v', e') = 2$ in $BG_4(G)$.

If there exists no such vertex, then $\deg_G v = p-1$ and if there exist non incident edge e_1 in G, then $v'e_1'v_3'e'$ (where v_3 is not incident to e_1 and e in G) is a shortest path and hence $d(v', e') = 3$ in $BG_4(G)$.

Suppose this is also not possible. That is, if $\deg_G v = p-1$ and if there does not exist non incident edge $e_1$in G, then that vertex is isolated vertex in $BG_4(G)$ that means $BG_4(G)$ is disconnected.

Hence, if $BG_4(G)$ is connected, then eccentricity of point vertices is 1, 2 or 3.

Proposition 2.1.2: If $BG_4(G)$ is connected, then the eccentricity of a line vertex is 2, 3 or 4.

Proof: From the previous theorem $d(e', v') = 1, 2$ or 3 in $BG_4(G)$.

Now to find $d(e_1', e_2')$ for $e_1, e_2 \in E(G)$

Case-(i): e_1 and e_2 are non adjacent

If there exist a vertex v which is not incident with both e_1 and e_2 then $e_1'v'e_2'$ is a shortest path and hence $d(e_1', e_2') = 2$ in $BG_4(G)$. If there exists no vertex v, not incident with both e_1 and e_2 and there is no edge adjacent to both e_1 and e_2. Consider the edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in G, $e_1'v_2'u_1'e_2'$ is a shortest path in $BG_4(G)$ then $d(e_1', e_2') = 3$ in $BG_4(G)$.

Suppose u_1 and v_1 are adjacent to all other vertices in G and there are only four vertices in G, then $e_1'w_1'e'w_2'e_2'$ is a shortest path in $BG_4(G)$. Hence, $d(e_1', e_2') = 4$ in $BG_4(G)$.

Case-(ii): e_1 and e_2 are adjacent

Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in G. e_1' is adjacent to v_2' and e_2' is adjacent to u_1' and also u_1' and v_2' also adjacent. Hence, $e_1'v_2'u_1'e_2'$ is a shortest path in $BG_4(G)$. Hence, $d(e_1', e_2') = 3$.

Hence, distance from e′ to any other vertex is 1, 2, 3 or 4 in $BG_4(G)$. This implies that the eccentricity of a line vertex is 2, 3 or 4.

Theorem 2.1.1: Let G be a (p, q) graph. Eccentricity of a point vertex is one if and only if G has an isolated vertex.

Proof: Let u be an isolated vertex in G. Then the vertex u′ is adjacent to all other line vertices and point vertices of $BG_4(G)$. Hence, deg u′ = p+q-1 and eccentricity of a vertex u′ is one in $BG_4(G)$.

On the other hand, assume that eccentricity of a point vertex u is one in $BG_4(G)$. This implies, deg u = p+q–1. Therefore, u is not adjacent to any vertex in G. Hence, u is an isolated vertex in G.

Note 2.1: If G has an isolated vertex then $BG_4(G)$ is connected and radius of $BG_4(G)$ is one.

Theorem 2.1.2: Let G and $BG_4(G)$ be connected. Eccentricity of a vertex is one in G and has at least one non incident edge in G if and only if eccentricity of that vertex in $BG_4(G)$ is three.

Proof: Let $v \in V(G)$ and $e \in E(G)$ which is non incident with v in G.

Assume e(v) = 1 in G. Then the vertex v is adjacent to all vertices of G. Hence v′ is not adjacent to any point vertex in $BG_4(G)$ and v′ is adjacent to e′ in $BG_4(G)$.

By proposition 2.1.1 we have d(v′, u′) = 3 where u is adjacent to v and incident with e in G. Hence, e(v′) = 3.

On the other hand, assume that eccentricity of a point vertex in $BG_4(G)$ is three. By proposition 2.1.1, we have $deg_G v = p-1$, and eccentricity of v is one in G. Suppose all the edges of G are incident with v, then v will be isolated in $BG_4(G)$.

So G has at least one edge that is not incident with v. Hence, the theorem is proved.

Theorem 2.1.3: Let G be a graph. If radius of G is greater than one then eccentricity of a point vertex in $BG_4(G)$ is two.

Proof: Let G be a connected graph. Assume $u \in V(G)$, v is not adjacent to u in G. This implies that, in $BG_4(G)$, u′ and v′ are adjacent.

Suppose u is adjacent to v in G and $w \in G$ is not adjacent to both u and v then v′w′u′ is a shortest path in $BG_4(G)$.

Hence, $d(u', v') \leq 2$ and $d(v', e') \leq 2$ since $e(v) \neq 1$ in G. Hence, eccentricity of a point vertex is two in $BG_4(G)$.

Theorem 2.1.4: Eccentricity of a line vertex is 2 in $BG_4(G)$ if and only if G has more than four vertices and $r(G) \neq 1$.

Proof: Let G be a graph with at least five vertices and $r(G) \neq 1$. In G, any two edges e_1 and e_2 have a common non incident vertex. Hence $e_1'v'e_2'$ is a shortest path from e_1' to e_2'.

Hence $d(e_1', e_2') = 2$ in $BG_4(G)$.

Every line vertex e′ corresponding to e = uv is adjacent to p−2 point vertices in $BG_4(G)$ and since $r(G) \neq 1$, every vertex u or v in G has at least one non adjacent vertex w in G. Then there exist shortest path e′w′u′ or e′w′v′ in $BG_4(G)$.

Therefore, d(e′, v′) = 2. Hence, eccentricity of a line vertex is 2.

On the other hand, assume that in $BG_4(G)$ the eccentricity of a line vertex is 2. Let $e = uv \in E(G)$, In $BG_4(G)$, e(e′) =2. This implies that distance between e′ and other line vertices are exactly 2. Hence, for any two edges e_1 and e_2 in G, there is a common non incident vertex. This implies $p \geq 5$.

Also, $e(e')=2$ implies $d(u', e') = 2$ and $d(v', e') = 2$ in $BG_4(G)$.which implies there exist $w \in V(G)$ such that w is not adjacent to both u and v in G. (That is, $u'w'e'$, $v'w'e'$ are paths in BG4(G)). Thus $r(G) > 1$. Hence, the theorem is proved.

Theorem 2.1.5: $G = K_4$ if and only if eccentricity of each line vertex is four in $BG_4(G)$.

Proof: Assume that $G = K_4$. Let $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$. A line vertex e_1' is adjacent to v_3' and v_4' and we have a shortest path $e_1'v_4'e_2'$ in $BG_4(G)$. In $BG_4(G)$, $d(e_1', e_2') = 2$ for adjacent edges in G.

For non adjacent edges we have a shortest path of distance 4. Hence, eccentricity of a line vertex is 4.

Conversely, assume eccentricity of every line vertex is four. Hence $d(e_1',u') \le 4$ in $BG_4(G)$ and $d(e_1', e_2') \le 4$ in $BG_4(G)$ for $u \in V(G)$ and $e_1, e_2 \in E(G)$, $|V(G)| \le 4$ since eccentricity of a line vertex is four (by theorem 2.1.4). $\overline{G}$ has anyone edge, then eccentricity of a line vertex is three, which is a contradiction to the assumption. Hence, $G = K_2$, K_3 or K_4.

If $G = K_2$ or K_3 then $BG_4(G)$ is disconnected. Hence, $G = K_4$.

Note 2.2: If $G=K_4$, eccentricity of every point vertex is 3 and eccentricity of every line vertex is 4 in $BG_4(G)$. Therefore, $BG_4(K_4)$ is bi-eccentric with diameter 4.

Theorem 2.1.6: $G \ne K_4$, $p = 4$ and G has at least two edges if and only if eccentricity of each line vertex is three in $BG_4(G)$.

Proof: Assume that G has four vertices and $G \ne K_4$. Therefore $\overline{G}$ has at least one edge. Any edge in G is non incident with other two points in G. Then the line vertex e' is adjacent to non incident vertex u' in $BG_4(G)$.

If e_1 and e_2 are non adjacent, by proposition 2.2.2, $d(e_1', e_2') = 3$ in $BG_4(G)$ since $G \ne K_4$ and $p = 4$.

If e_1 and e_2 are adjacent then $d(e_1', e_2') = 3$ in $BG_4(G)$.

If e_1 is non incident with v_1 then $d(e_1', v_1') = 1$ and e_1 is incident with v_2 then $d(e_1', v_2') = 3$. Hence, eccentricity of a line vertex is 3.

On the other hand, assume that eccentricity of each line vertex is three in $BG_4(G)$. Hence $d(e_1', u_1') \le 3$ and $d(e_1', e_2') \le 3$ in $BG_4(G)$, for $u \in V(G)$ and $e_1, e_2 \in E(G)$. Hence, by theorem 2.1.4, $p \le 4$ and $G \ne K_4$ by theorem 2.1.5. Suppose $p \le 3$, $BG_4(G)$ is disconnected. Hence, $G \ne K_4$ and $p = 4$.

Theorem 2.1.7: If $G = K_p$, $p \ge 5$, then $BG_4(G)$ is a bi eccentric graph with radius 2.

Proof: Let $G = K_p$, $p \ge 5$.

Case-(i): To find $d(v_1', v_2')$ in $BG_4(G)$ where $v_1, v_2 \in V(G)$.

Any two vertices of G are adjacent in G. So the point vertices of $BG_4(G)$ are non adjacent. Every pair of vertices have the common non incident edge e in G. Then there exist a shortest path $v_1'e'v_2'$ in $BG_4(G)$. Hence $d(v_1', v_2') = 2$ in $BG_4(G)$.

Case-(ii): To find $d(v', e')$ in $BG_4(G)$ where $v \in V(G)$ and $e \in E(G)$.

Suppose a vertex v is incident to an edge e in G. Let $e = uv$, $e_1 = uv_1$, $e_2 = v_2v_3$ be the edges of G. There exist a shortest path $v'e_2'v_1'e_1'$ in $BG_4(G)$. Hence $d(v', e') = 3$ in $BG_4(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $BG_4(G)$ where $e_1, e_2 \in E(G)$.

$\overline{K_q}$ is an induced subgraph of $BG_4(G)$. Thus it follows that $d(e_1', e_2') \ne 1$. Every pair of edges has a common non incident vertex v in G since G has more than four vertices. Therefore, there exist a shortest path $e_1'v'e_2'$ in $BG_4(G)$. Hence, $d(e_1', e_2') = 2$.

Hence, radius of $BG_4(G)$ is two and diameter is three.

Note 2.3:

(i) If $G = K_3$, $BG_4(G)$ is a disconnected graph.

(ii) If $G = K_4$, the diameter of $BG_4(G)$ is four by theorem 2.1.5.

Theorem 2.1.8: If G is a graph with radius 1 and diameter 2, then $BG_4(G)$ has isolated vertex or $diam(BG_4(G)) = 3$.

Proof: Let G be a graph with radius 1 and diameter 2. Assume $e(v) = 1$ in G. Suppose v is incident with all the edges of G, then v' is isolated in $BG_4(G)$. If not, by proposition 2.1.1, $e(v') = 3$ in $BG_4(G)$ and $e(e') \leq 3$ for $G \neq K_n$. Hence, $diam(BG_4(G)) = 3$.

Theorem 2.1.9: If G is a 2-self centered graph with $p \geq 5$, then $BG_4(G)$ is 2-self centered.

Proof: Let G be a 2-self centered graph with $p \geq 5$. By theorem 2.1.4, in $BG_4(G)$ eccentricity of a point vertex is 2 since $r(G) \neq 1$. By theorem 2.1.4, in $BG_4(G)$ eccentricity of a line vertex is 2 since G has more than four vertices and $r(G) \neq 1$. Therefore, $BG_4(G)$ is a 2-self centered graph.

Cor 2.1.1: If G is a 2-self centered graph with four vertices, then $BG_4(G)$ is a bi-eccentric graph with radius 2.

Proof: Let G be a 2-self centered graph with four vertices. By theorem 2.1.2, eccentricity of a point vertex is 2 in $BG_4(G)$ and by theorem 2.1.6, eccentricity of a line vertex is 3 in $BG_4(G)$. Hence, $BG_4(G)$ is a bi-eccentric graph with radius 2.

Theorem 2.1.10: If G is a graph with radius 2 and diameter 3 then $BG_4(G)$ is a graph with radius 2 and diameter 3.

Proof: Let G be a graph with radius 2 and diameter 3. Consider the vertices $u, v \in V(G)$.

Case-(i): To find $d(u', v')$ in $BG_4(G)$ where $u, v \in V(G)$.

If u and v are non adjacent in G, then they are adjacent in $BG_4(G)$.

If u and v are adjacent in G and $e(u) = 3$ or $e(v) = 3$, then u and v have the common non incident vertex in G. Therefore, $d(u', v') = 2$ in $BG_4(G)$. If $e(u) = e(v) = 3$, then u and v have the common non incident vertex in G. Hence $d(u', v') = 2$ in $BG_4(G)$. If $e(u) = e(v) = 2$ and u and v does not have a common non incident edge and not have a common non adjacent vertex, then $d(u', v') = 3$ in $BG_4(G)$. If there exist a common non incident edge or common non adjacent vertex then $d(u', v') = 2$ in $BG_4(G)$.

Case-(ii): To find $d(u', e')$ in $BG_4(G)$ where $u \in V(G)$ and $e \in E(G)$.

If a vertex u is incident with an edge $e = uv$ in G, then $d(u', e') = 2$ in $BG_4(G)$ since $e(u) = 2$ or $e(v) = 3$ (that is the eccentric vertex of u in G is adjacent to u and e in $BG_4(G)$).

If u is not incident with e in G, then $d(u', e') = 1$ in $BG_4(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $BG_4(G)$ where $e_1, e_2 \in E(G)$.

By proposition 2.1.2, $d(e_1', e_2') = 2$ or 3.

Hence, $BG_4(G)$ is a graph with radius 2 and diameter 3.

Theorem 2.1.11: If G is a graph with radius 2 and diameter 4, then $BG_4(G)$ is a 2-self centered graph.

Proof: Let G be a graph with radius 2 and diameter 4.

Case-(i): To find $d(u', v')$ in $BG_4(G)$ where $u, v \in V(G)$.

Let $uv = e \in E(G)$. The non adjacent vertices of G are adjacent in $BG_4(G)$. If the eccentricity of a vertex $v \in V(G)$ is 3 or 4, then u and v have the common non adjacent vertex w in G. Therefore, there exists a shortest path $u'w'v'$ in $BG_4(G)$. Hence $d(u', v') = 2$ in $BG_4(G)$.

If u and v have eccentricity 2, then $d(u', v') = 2$ or 3 in $BG_4(G)$. (The proof is similar as in theorem 2.1.10 case (i)). By theorem 2.1.9, $d(u', v') = 2$ in $BG_4(G)$.

Case-(ii): To find d(u′, e′) in $BG_4(G)$, where u ∈ V(G), e∈E(G).

If u is not incident with e in G, then d(u′, e′) = 1 in $BG_4(G)$. If u is incident with e in G then d(u′, e′) = 2. (the proof is similar to theorem 2.1.10 case(ii)).

Case-(iii): To find $d(e_1', e_2')$ in $BG_4(G)$, where $e_1, e_2 \in E(G)$.

The graph G has more than four vertices since diameter of G is four. By the theorem 2.1.4, eccentricity of a line vertex is 2.

Thus, the eccentricity of every point vertex and line vertex is two in $BG_4(G)$. Hence, $BG_4(G)$ is 2-self centered.

Theorem 2.1.12: If G is a graph with r(G) ≥ 3, then $BG_4(G)$ is a 2-self centered graph.

Proof: Let G be a graph with r(G) ≥ 3.

Case-(i): To find d(u′, v′) in $BG_4(G)$, where u, v∈V(G).

If u and v are non adjacent in G, then d(u′, v′) = 1 in $BG_4(G)$.

If u and v are adjacent in G, then u and v have a common non adjacent vertex in G.

By the definition, d(u′, v′) = 2 in $BG_4(G)$.

Case-(ii): To find d(u′, e′) in $BG_4(G)$, where u ∈ V(G), e∈E(G).

If u is incident with an edge e in G, then u and e have the common non incident (adjacent) vertex in G. By proposition 2.1.2, we have d(u′, e′) = 2 in $BG_4(G)$.

If u is non incident with an edge e in G, then e′ and v′ are adjacent in $BG_4(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $BG_4(G)$, where $e_1 = u_1v_1$, $e_2 = u_2v_2 \in E(G)$.

G has more than five vertices since r(G) = 3. By theorem 2.1.4, $d(e_1', e_2') = 2$.

Therefore, e(u′) = 2 and e(e′) = 2 in $BG_4(G)$. Hence, $BG_4(G)$ is a 2-self centered graph.

2.2 Eccentricity properties of $\overline{BG_4}(G)$

In this section, radius and diameter of $\overline{BG_4}(G)$ are found out. Throughout this section, if v∈V(G) and e∈E(G), the corresponding vertices of $\overline{BG_4}(G)$ are denoted by v′ and e′.

Proposition 2.2.1: If $\overline{BG_4}(G)$ is connected, eccentricity of a point vertex is 1, 2 or 3.

Proof: Consider a point vertex v in V(G).

To find d(u′, v′) in $\overline{BG_4}(G)$ **where u, v ∈ V(G).**

Case-(i): If a vertex u has degree p-1 and incident with all the edges of G, then in $\overline{BG_4}(G)$, u′ is adjacent to all the line vertices and point vertices. Hence, eccentricity of point vertex v is one.

Case-(ii): If a vertex u is adjacent to v in G then d(u′, v′)=1 in $\overline{BG_4}(G)$. If a vertex u is non adjacent to v in G, then $u'e_1'e_2'v'$ is a shortest path and d(u′, v′) = 3 in $\overline{BG_4}(G)$, where edge e_1 is incident with u and edge e_2 is incident with v in G.

To find d(u′, e′) in $\overline{BG_4}(G)$ where e∈E(G).

Case-(i): If a vertex u is incident with an edge e in G, then d(u′, e′) = 1 in $\overline{BG_4}(G)$.

Case-(ii): If a vertex u is not incident with an edge e in G, then there exist a shortest path $u'e_1'e'$ and d(u′, e′)= 2 in $\overline{BG_4}(G)$, where e_1 is incident with u in G.

Hence, distance from u′ to any other vertex is 1, 2 or 3 in $\overline{BG_4}(G)$. Hence, the eccentricity of a point vertex is 1, 2 or 3.

Proposition 2.2.2: If $\overline{BG_4}(G)$ is connected, then the eccentricity of a line vertex is 1 or 2.

Proof: From the proposition 2.2.1 $d(e_1', v') = 1$ or 2.

Now to find $d(e_1', e_2')$ for $e_1, e_2 \in E(G)$.

K_q is an induced sub graph of $\overline{BG_4}(G)$. Hence, $d(e_1', e_2') = 1$. Hence the eccentricity of a line vertex is 1 or 2.

Observation 2.2.1: If a graph G has a vertex of degree p−1 and incident with all the edges of G, that is $G = K_{1, p-1}$, then radius of $\overline{BG_4}(G)$ is one.

Theorem 2.2.1: If $G = K_p$, then $\overline{BG_4}(G)$ is a two self centered graph.

Proof: Let $G = K_p$.

Case-(i): To find $d(v_1', v_2')$ in $\overline{BG_4}(G)$, where $v_1, v_2 \in V(G)$.

All the vertices of K_p are adjacent to each other and G is an induced sub graph of $\overline{BG_4}(G)$. Hence, $d(v_1', v_2') = 1$ in $\overline{BG_4}(G)$.

Case-(ii): To find d(v′, e′) in $\overline{BG_4}(G)$, where v ∈ V(G) and e∈E(G).

If a vertex v is incident with an edge e, then d(v′, e′) = 1 in $\overline{BG_4}(G)$.

Suppose a vertex v is not incident with an edge e, then there exists a shortest path $v'e_1'e'$ in $\overline{BG_4}(G)$, where e_1 is incident with v in G. Therefore, d(v′, e′) = 2 in $\overline{BG_4}(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $\overline{BG_4}(G)$ where $e_1, e_2 \in E(G)$.

Since K_q is an induced sub graph of $\overline{BG_4}(G)$. $d(e_1', e_2') = 1$ in $\overline{BG_4}(G)$.

Therefore, radius of $\overline{BG_4}(G)$ is a 2-self centered graph.

Theorem 2.2.2: If $G \neq K_{1, n}$ is a graph with radius 1 and diameter 2, then $\overline{BG_4}(G)$ is a 2-self centered graph.

Proof: Let G be a graph with radius 1 and diameter 2. Consider $v_1, v_2 \in V(G)$, $d(v_1', v_2') = 1$ in $\overline{BG_4}(G)$ if v_1 and v_2 are adjacent in G. $d(v_1', v_2') = 2$ in $\overline{BG_4}(G)$ if v_1 and v_2 are non adjacent in G since diameter of G is 2. $d(v_1', e') =1$ in $\overline{BG_4}(G)$ if v_1 is non incident with an edge e in G since there exist a shortest path $v_1'e_1'e'$ in $\overline{BG_4}(G)$, where $e_1 = v_1v_2$ in G.

$d(e_1', e_2') = 1$ in $\overline{BG_4}(G)$ since K_q is an induced sub graph of G.

Therefore, $\overline{BG_4}(G)$ is a 2-self centered graph.

Theorem 2.2.3: If G is a 2-self centered graph then $\overline{BG_4}(G)$ is a 2-self centered graph.

Proof: Assume G is a 2-self centered graph.

Case-(i): To find $d(v_1', v_2')$ in $\overline{BG_4}(G)$ where $v_1, v_2 \in V(G)$.

Suppose v_1 and v_2 are adjacent in G, then $d(v_1', v_2') = 1$ in $\overline{BG_4}(G)$. Suppose v_1 and v_2 are non adjacent in G, $d(v_1', v_2') = 2$ in $\overline{BG_4}(G)$, since G is an induced sub graph of $\overline{BG_4}(G)$.

Case-(ii): To find $d(v', e')$ in $\overline{BG_4}(G)$, where $v \in V(G)$ and $e \in E(G)$.

If a vertex v is incident with e in G, then $d(v', e') = 1$ in $\overline{BG_4}(G)$.

If a vertex v is not incident with e in G, then there exists a shortest path $v'e_1'e'$ in $\overline{BG_4}(G)$ where e_1 is incident with v in G. Therefore, $d(v', e') = 2$ in $\overline{BG_4}(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $\overline{BG_4}(G)$, where $e_1, e_2 \in E(G)$.

In $\overline{BG_4}(G)$, K_q is an induced sub graph. Hence, $d(e_1', e_2') = 1$ in $\overline{BG_4}(G)$.

Therefore, $\overline{BG_4}(G)$ is a 2-self centered graph.

Theorem 2.2.4: If G is a graph with radius 2 and diameter 3, then $\overline{BG_4}(G)$ is a graph with radius 2 and diameter 3.

Proof: Assume a graph G with radius 2 and diameter 3.

Case-(i): To find $d(v_1', v_2')$ in $\overline{BG_4}(G)$ where $v_1, v_2 \in V(G)$.

The adjacent vertices of G are adjacent in $\overline{BG_4}(G)$. Consider a vertex v such that $e(v) = 2$ and w is an eccentric vertex of v. Then, $d(v', w') = 2$ in $\overline{BG_4}(G)$, otherwise $d(u', v') = 3$ in $\overline{BG_4}(G)$ since u and v are at distance 3 in G.

Case-(ii): To find $d(v', e')$ in $\overline{BG_4}(G)$ where $v \in V(G)$ and $e \in E(G)$.

By proposition 2.2.1, $d(v', e') = 1$ or 2 in $\overline{BG_4}(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $\overline{BG_4}(G)$ where $e_1, e_2 \in E(G)$.

In $\overline{BG_4}(G)$ $d(e_1', e_2') = 1$ since K_q is an induced sub graph of $\overline{BG_4}(G)$.

Hence, $\overline{BG_4}(G)$ is a graph with radius 2 and diameter 3.

Theorem 2.2.5: If G is a graph with radius 2 and diameter 4, then $\overline{BG_4}(G)$ is a bi-eccentric graph with radius two.

Proof: Let G be a graph with radius 2 and diameter 4.

Case-(i): To find $d(v_1', v_2')$ in $\overline{BG_4}(G)$ where $v_1, v_2 \in V(G)$.

If v_1 and v_2 are adjacent in G, then $d(v_1', v_2') = 1$ in $\overline{BG_4}(G)$.

If v_1 and v_2 are non-adjacent in G. Assume $e(v'_1) = 2$ in G, then $e(v_1') = 2$ in $\overline{BG_4}(G)$. If $e(v'_1) = 3$ in G, then $e(v'_1) = 3$ in $\overline{BG_4}(G)$ by the proposition 2.2.1.

Case-(ii): To find $d(v', e')$ in $\overline{BG_4}(G)$ where $v \in V(G)$ and $e \in E(G)$.

By proposition 2.2.1, $d(v', e') = 1$ or 2 in $\overline{BG_4}(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $\overline{BG_4}(G)$ where $e_1, e_2 \in E(G)$.

K_q is an induced sub graph of $\overline{BG_4}(G)$. Thus $e(e') = 1$ in $\overline{BG_4}(G)$.

Therefore, $\overline{BG_4}(G)$ is a bi-eccentric graph with radius two.

Theorem: 2.2.6 If G is a connected graph $p \geq 3$ with radius greater than 2, then $\overline{BG_4}(G)$ is a bi-eccentric graph with radius 2.

Proof: Let G be a graph with $r(G) \geq 3$.

Case-(i): To find $d(v_1', v_2')$ in $\overline{BG_4}(G)$, where $v_1, v_2 \in V(G)$.

Consider $v_1, v_2 \in V(G)$ and $e_1, e_2 \in E(G)$. If v_1 and v_2 are adjacent, then $d(v_1', v_2') = 1$ in $\overline{BG_4}(G)$.

If v_1 is an eccentric vertex of v_2, then $d(v_1', v_2') = 3$ in $\overline{BG_4}(G)$ since there exist a shortest path $v_1'e_1'e_2'v_2'$ in $\overline{BG_4}(G)$, where $e_1 = v_1w_1$ and $e_2 = v_2w_2$ in G. Hence $d(v_1', v_2') = 3$ in $\overline{BG_4}(G)$.

Case-(ii): To find $d(v', e')$ in $\overline{BG_4}(G)$, where $v \in V(G)$ and $e \in E(G)$.

If a vertex v is incident with an edge e in G, then $d(v', e') = 1$ in $\overline{BG_4}(G)$.

If a vertex v is non-incident with an edge e in G, then there exist a shortest path $v'e_1'e'$ in $\overline{BG_4}(G)$, where $e_1 = vv_1$ in G. Hence, $d(v', e') = 2$ in $\overline{BG_4}(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $\overline{BG_4}(G)$, where $e_1 = uv$, $e_2 = u_1v_1 \in E(G)$.

$d(e_1', e_2') = 1$ in $\overline{BG_4}(G)$ since K_q is an induced sub graph of $\overline{BG_4}(G)$. Hence, eccentricity of a line vertex is 2.

Therefore, $\overline{BG_4}(G)$ is a bi-eccentric graph with radius 2.

CONCLUSION

It is proved that either $BG_4(G)$ is disconnected or it is a graph of diameter at most 4. Also, we have characterized graphs G for which $BG_4(G)$, $\overline{BG_4}(G)$ are 2-self centered, bi-eccentric, etc.

REFERENCES

1. Bhanumathi M., Kavitha M., Boolean Graph Operator $B_{\overline{G},\, NINC,\, \overline{K_q}}$, Elsevier - Procedia Computer Science 47 (2015) 387-393.
2. Bhanumathi. M., (2004), "A study on some structural properties of graphs and some new graph operations of graphs", Thesis, Bharathidasan University, Tamil Nadu, India.
3. Buckley. F, Harary. F, Distance in graphs, Addison–Wesley, Publishing Company (1990).
4. Harary F., Graph theory, Addition - Wesley Publishing Company Reading, Mass (1972).
5. Janakiraman T. N., Muthammai S, Bhanumathi M, On the Boolean function graph of a graph and on its complement, Mathematica Bohemica, 130(2005), 113 – 134.
6. Janakiraman T. N., Bhanumathi M., Muthammai S., Eccentricity properties of the Boolean graphs $BG_2(G)$ and $BG_3(G)$, International Journal of Engineering Science, Advanced Computing and Bio-Technology, Volume 4, Issue 2, pp.32 – 42, 2013.
7. Janakiraman T. N., Bhanumathi M., Muthammai S., Boolean graph $BG_1(G)$ of a graph G, International Journal of Engineering Science, Advanced Computing and Bio-Technology, Volume 6, Issue 1, pp.1– 16, 2015.

Source of support: Nil, Conflict of interest: None Declared