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A NOTE ON BOOLEAN TERNARY SEMIRINGS

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ABSTRACT

The main purpose of this note is to prove that a commutative Boolean ternary semiring of characteristic two is isomorphic to a Boolean ring. Further, we construct a ternary semiring (not necessarily commutative), provided a semiring with absorbing zero and some of their properties are obtained. Finally, the existence of a non-commutative Boolean ternary semiring which is not of characteristic two is illustrated.

***Keywords**: Ternary semiring, Boolean ternary semiring, multiplicatively idempotent, Ternary Boolean algebra.*

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§0.INTRODUCTION

The notion of Ternary semiring was introduced by T. K. Dutta and S.Kar and studied their properties extensively (see [3], [19], [20] and [21]). More Precisely, A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by [1] is said to be a *ternary semiring* (in short TSR) if T is an additive commutative semi group satisfying the following conditions :

i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,
ii) $[(a + b)cd] = [acd] + [bcd]$,
iii) $[a(b + c)d] = [abd] + [acd]$,
iv) $[ab(c + d)] = [abc] + [abd]$ for all $a; b; c; d; e \in$ T.

For the convenience we write $x_1x_2x_3$ instead of $\left[x_1x_2x_3\right]$. For the definition of semiring and undefined terms in this paper we refer [7], [13] [19], [20], [21] and [22] and Z_0^- will denote the set of all non positive integers. It is clear that a binary operation can be considered as a ternary operation on the underlying nonempty set; therefore, every semiring can be regarded as a natural example for a ternary semiring, whereas Z_0^- forms a ternary semiring with respect to usual addition (+) and multiplication (·) as a ternary operation which is not a semiring. In this paper, we investigate few interesting properties of a ternary semiring in which every element is multiplicative idempotent (see Definition 1.1 (iii)), called Boolean Ternary semiring (BTSR).

§1. PRELIMINARIES

Definition 1.1: Let T be a TSR, $a \in T$. Then a is said to be

(i) **Additive zero** if $a + x = x + a = x$ for all $x \in$ T,
(ii) The additive zero 0 in T is called an **absorbing zero** if ab0 = a0b = 0ab = 0 for all a, b in T,
(iii) ***Multiplicatively idempotent*** element if $aaa = a$ *(simply we write* $a^3 = a$),
(iv) ***Additive idempotent*** element if $a + a = a$,
(v) An element e of T is called **unital element** if aee = eea = eae = a for all a in T.

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Remark 1.2:
(i) In rings, every zero is absorbing, but in ternary semiring not every zero is absorbing, which is evident from Example 1.4(2), that the element 1 is not an absorbing zero.
(ii) If e is a unital element of TSR T, then abe = aeb = eab for all a, b in T.

Definition 1.3: A ternary semiring T is called:
(i) ***"Commutative"*** if and only if $abc = bca = cab = bac = cba = acb$ for all $a, b, c \in$ T.
(ii) **"Regular"** if and only if, to each $a \in T$ corresponds an element $a' \in T$ such that $aa'a = a$.
(iii) **"Boolean"** if and only if every element in T is multiplicatively idempotent.

Examples 1.4: Some interesting examples for ternary semiring are
1. Let T be the set of all $n \times n$ real skew-symmetric matrices over ring of integers that commutes with each other. Then T is a TSR with addition of matrices and matrix multiplication as the ternary operation, whereas the set S of all commuting $n \times n$ real symmetric matrices over the set of non negative integers forms a semiring with respect to matrix addition and multiplication.
2. Let $O = \{1,3,5,7,\ldots\}$ be the set of all odd positive integers. If we define $\oplus$ and $\circ$ on O as $a \oplus b = \max\{a,b\}$ And $a \circ b \circ c = a + b + c$ for all a,b,c in O, where + indicates the usual addition of integers. Then (O, $\oplus$, $\circ$) is a commutative TSR in which every element is additive idempotent but not multiplicatively idempotent.
3. Let $T = \{5,10,15\}$. If we define on T as $a \oplus b = LCM\{a,b\}$ and $a \circ b \circ c = GCD\{a,b,c\}$, where LCM and GCD stand for the least common multiple and greatest common divisor of positive integers, *T* is a commutative TSR with additive zero element 5. Further, every element of T is both additive and multiplicatively idempotent.

Definition 1.5: A ternary semiring (T, +, ·) is additive cancellative if for a, b, c in T
(i) a + b = a + c implies b = c,
(ii) a + b = c + b implies a = c.

Definition 1.6: Let T be TSR with additive zero 0. If there exists the least positive integer n such that a + a +… + a = 0 (n arguments on the left hand side, in this case we write na = 0) for each a in T, it is called the characteristic of T; we denote it by Char (T).

The following lemma is useful in the sequel.

Lemma 1.7: Let (T, +, ·) be a ternary semi ring of characteristic two. Then
1) T is additive cancellative.
2) For a, b in T, a + b = 0 implies a = b.
3) ab0 = a0b = 0ab = 0 for all a, b in T.

Proof: Routine.

Remark 1.8: The converse of Lemma 1.7 is not necessarily true as it is evident from the fact that (Z_0^-, +, ·) is a TSR, in which (1), (2) and (3) of Lemma 1.7 hold but Char (Z_0^-) ≠ 2.

Definition 1.9: (see [18]) A system (R, +, ·) is a Boolean semiring if and only if the following properties hold:
1. (R, +) is an additive (abelian) group (whose "zero" will be denoted by "0")
2. (R, ·) is a semigroup of idempotents in the sense, a·a = a, for all a in R
3. a(b+c) = ab + ac and
4. abc = bac, for all a, b, c in R (weak commutative).

Example 1.10: (see [18]) Let (G, +) be any abelian group and define a·b = a, for all a, b in G. Then (G, +, ·) is a Boolean semiring.

§2. ASSOCIATED TERNARY SEMIRING

We now provide a method to construct a ternary semiring (not necessarily commutative), provided a semiring with absorbing zero 0.

Let (S, +, ·) be a semiring with absorbing zero 0 and $M_2^-(S)$ the set of all matrices of the form $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$, where $a,b \in S$, forms a ternary semiring with respect to addition $\oplus$ and matrix multiplication $\circ$ defined as $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & a+c \\ b+d & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & e \\ f & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \cdot d \cdot e \\ b \cdot c \cdot f & 0 \end{bmatrix}$ for all $a,b,c,d,e,f \in \mathrm{S}$. (Indeed, the ternary operator $\circ$ is the usual matrix multiplication as ternary operator over S). We shall call this $M_2^-(S)$, the ternary semiring associated with S. Further, the set $M_2^+(S) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a,b \in S \right\}$ forms a semiring with respect to matrix addition and matrix multiplication and the matrix ternary semiring $M_2(S)$ over a semiring *S* is a direct sum of $M_2^-(S)$ and $M_2^+(S)$ as a ternary semirings.

We now discuss certain properties of semirings in connection with their associated ternary semirings.

Theorem 2.1: Let $(M_2^-(S), \oplus, \circ)$ be the ternary semiring associated with a semiring (*S*, +, ·). Then the following statements hold:

(i) *S* can be regarded as ternary subsemiring of $M_2^-(S)$.

(ii) If $M_2^-(S)$ is commutative, then *S* is commutative.

(iii) If $M_2^-(S)$ is Boolean then *S* is Boolean.

(iv) If $M_2^-(S)$ is regular, then *S* is regular.

(v) If e is multiplicative identity in S, then $\begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}$ is bi-unital element in $M_2^-(S)$

Proof: Proof of (v) is clear, and one can prove (ii), (iii), (iv) as simple consequences of (i). We prove (i): If we define ψ: (*S*, +, ·) $\rightarrow (M_2^-(S), \oplus, \circ)$ as ψ (a) = $\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ for all a in *S*, then ψ is a ternary homomorphism and one-one, therefore, S can be considered as ternary subsemiring of $M_2^-(S)$.

Theorem 2.2: (page 3 of [7]) If (*S*, +, ·) is a semiring without identity element (hemi ring), then we can canonically embed it in a semiring with identity element in the following manner: Let *R* = *S* × Z_0^+ and define operations of addition and multiplication on S by setting $(r,n)+(r',n')=(r+r',n+n')$ and $(r,n)\cdot(r',n')=(nr'+n'r+rr',nn')$ for all $(r,n),(r',n')$ in *R*, where Z_0^+ is the set of non negative integers. Then (*R*, +, ·) is a semiring with multiplicative identity (0, 1), called **the Dorroh extension** of *S* by Z_0^-.

Theorem 2.3: Every Associated ternary semiring without unital element can be embedded into a ternary semiring with unital element.

Proof: Let $M_2^-(S)$ be the ternary semiring associated with a semiring *S*.

Assume that $M_2^-(S)$ is a ternary semiring without the unital element.

By (v) of Theorem 2.1, S is a semiring without identity. Let *R* be the Dorroh extension of *S* by Z_0^- as in the Theorem 2.2. Then $M_2^-(R)$ is a ternary semiring with unital element $\begin{bmatrix} 0 & (0,1) \\ (0,1) & 0 \end{bmatrix}$. If we define ψ: $(M_2^-(S), \oplus, \circ) \rightarrow (M_2^-(R), \oplus, \circ)$ by $\psi\left(\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & (a,0) \\ (b,0) & 0 \end{bmatrix}$ for all $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in M_2^-(S)$, then ψ is an injective ternary homomorphism. This completes the proof.

§3. MAIN RESULTS

Throughout this section, T will always denote a ternary semiring with absorbing zero 0 and unless otherwise stated a ternary semiring means a ternary semiring with absorbing zero. The notion of Boolean ternary semiring (BTSR) was originally introduced by D. M Rao *et al.* and established the following result (see Definition IV.1 and Theorem IV.4 of [13]).

Theorem 3.1: (see [13]) If T is a BTSR, then (i) $a + a = 0$. (ii) $a + b = 0$ implies $a = b$. (iii) $aba = bab$.

The following example shows that (i) and (iii) of theorem 3.1 is false.

Example 3.2: Let $S = \{0,1,2,3,4\}$. Define + and · on S as in the following Cayley tables.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	2
2	2	3	4	2	3
3	3	4	2	3	4
4	4	2	3	4	2

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	3	2
3	0	3	3	3	3
4	0	4	2	3	4

Then (S, +, ·) is a commutative semiring

with additive zero 0 (see page 19 of [17]). It is clear that $0^3 = 0$, $1^3 = 1$, $2^3 = 2$, $3^3=3$, $4^3 = 4$. Since every semiring is TSR, S is a commutative Boolean Ternary semiring in which (i) and (iii) of theorem 3.1 fails to hold in S. For this, $a + a \neq 0$ for all $0 \neq a \in$ S and if a = 2, b=4 then aba ≠ bab.

Theorem 3.3: Let T be a Boolean Ternary semiring. Then the following statements hold:

1) For all a in T, 2a = 8a.
2) If $a \in T$ is an additively invertible element of T, then 6a = 0.
3) If T has a bi-unital element e, then e is the only multiplicatively invertible element of T.
4) In addition, if T is additive cancellative TSR, then $T = \bigcup_{a \in T} T_a$, where $T_a = \{a, 2a, 3a, 4a, 5a, 6a\}$.

Proof:

(i) Let $a \in T$. Then $(a+ a)^3 = (a+ a)$
$\Rightarrow (a+ a) (a+ a) (a+ a) = (a+ a)$
$\Rightarrow (a + a) a (a + a)+ (a + a)a(a + a) = (a + a)$
$\Rightarrow a a(a + a) + aa (a + a) + (a + a)aa +(a + a)aa = (a + a)$
$\Rightarrow 8a = 2a$. (Since $a^3 = a$ for all a in T)

(ii) Let a, b in T be such that $a+ b = 0$. Then $2a + 2b = 0$. Since $8a = 2a$, we have
$6a = 6a + 0 = 6a + 2a +2b = 8a + 2b = 2a + 2b = 0$.

(iii) Let a, b in T be such that $abt = atb = tab = e$ for all t in T, where e is the biunital element in T.
Then a $= aee = a(bae)(abe) = (aba)e(abe) = eee=e$.

(iv) Let $a \in T$. Then $(a+ a)^3 = (a+ a)$

$\Rightarrow 8a = 2a$
$\Rightarrow 7a+a = a + a$
$\Rightarrow 7a = a$ (by additive cancellativity)
$\Rightarrow 8a = 2a, 9a = 3a, 10a = 4a, 11a = 5a, 12a =6a$. This completes the proof.

Theorem 3.4: Let (T, +, ·) is a commutative Boolean ternary semiring and char (T) = 2. If we define ∘ on T as a ∘ b = a·a·b for all a, b in T, then (T, +, ∘) is a Boolean ring. Further, if we define ψ: (T, +, ·) → (T, +, ∘) as ψ (a) = a for all a in T, then ψ is isomorphism, considering (T, +, ∘) as a ternary semiring.

First we establish the following Lemma under the hypothesis of Theorem 3.4.

Lemma 3.5: For any a, b in T, aba = bab.

Proof: Let $a, b \in T$ then $a + b \in T$
$\Rightarrow (a + b)^3 = (a + b)$
$\Rightarrow (a + b)(a + b)(a + b) = (a + b)$
$\Rightarrow (a + b)a(a + b) + (a + b)b(a + b) = (a + b)$
$\Rightarrow aa(a + b) + ba(a + b) + (a + b)ba + (a + b)bb = (a + b)$
$\Rightarrow a^3 + aab + baa + bab + aba + + bba + abb + b^3 = (a + b)$
$\Rightarrow a + (aab + aab) + (bab + bab) + (aba + bab) + b = a + b$ (By commutativity and multiplicative idempotency of T)
$\Rightarrow a + ab(a + a) + ba(b + b) + (aba + bab) + b = a + b$

Since $a + a = 0$ and additive cancellative laws of T (in view of (1) of Lemma 1.7),

We have $aba + bab = 0$

In view of (2) of Lemma 1.7, we have $aba = bab$. This completes the proof.

Proof of Theorem 3.4: It is clear that $a \circ a = a \cdot a \cdot a = a$, and $a \circ 0 = 0 \circ a = 0$ for all a in T.

Also, $a \circ b = b \circ a$ for all a, b in T

Let $a, b, c \in T$. Then
$$\begin{aligned} a \circ (b \circ c) &= a \circ (bbc) \\ &= aa(bbc) \\ &= (aab)bc \\ &= (aba)bc \\ &= (bab)bc \quad \text{(in view of Lemma 3.5)} \\ &= (abb)bc \\ &= a(bbb)c = abc \quad (\text{since } b^3 = b) \end{aligned}$$

Similarly we can show that $(a \circ b) \circ c = abc$, therefore $(T, \circ)$ is a commutative semigroup.

We now consider $a \circ (b + c) = aa(b+c) = aab + aac = a \circ b + a \circ c$.

Since $(T, \circ)$ is a commutative semigroup, we have
$$\begin{aligned} (a + b) \circ c &= c \circ (a+b) \\ &= c \circ a + c \circ b \\ &= a \circ c + b \circ c \end{aligned}$$

Also, since $a + a = 0$ for all a in T, every element of T has additive inverse.

Hence $(T, +, \circ)$ is a Boolean ring. By routine verification, one can prove ψ is a ternary isomorphism.

It is a well known fact that a Boolean algebra can be turned into Boolean ring and vice versa (see page 5 of [22]). Also, there is a one-to-one correspondence between a ternary Boolean algebra and an abstract Boolean algebra (see [1]). Thus, we have proved that there is a one-to-one correspondence between a commutative BTSR of characteristic two and a ternary Boolean algebra, as a consequence of Theorem 3.4.

Definition 3.6: A commutative semiring $(S, +, \cdot)$ is called Boolean semiring if $a \cdot a = a$ for all a in S.

Definition 3.7: (see [5]) A near ring $(R, +, \cdot)$ is said to be idempotent if $a^2 = a$ for all a in R.

Proofs of the following theorems are routine and hence omitted.

Theorem 3.8: If $(T, +, \cdot)$ is a commutative BTSR satisfying $a \cdot b \cdot a = b \cdot a \cdot b$ for all a, b in T and if we define $\circ$ on T as $a \circ b = a \cdot a \cdot b$ for all a, b in T, then $(T, +, \circ)$ is a Boolean semiring in the sense of definition 3.6.

Theorem 3.9: If $(T, +, \cdot)$ is a commutative BTSR of characteristic two and if we define $\circ$ on T as $a \circ b = a \cdot a \cdot b$ for all a, b in T, then $(T, +, \circ)$ is a Boolean semiring in the sense of definition 1.9.

Theorem 3.10: If (T, +, ·) is a commutative BTSR and if we define ∘ on T as a∘b = a·a·b for all a, b in T, then (T, +, ∘) is an idempotent near ring.

Finally, we provide an example for the existence of a non commutative BTSR in which additive cancellative law fails to hold and not of characteristic two.

Example 3.11: Let (S, +, ·) be a commutative semiring as in Example 3.2. Then the TSR associated with S, $M_2^-(S)$ is a non commutative ternary semiring. Also, $M_2^-(S)$ is not a Boolean ternary semiring as $\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix} \neq \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}$.

However the set $B_2^-(S) = \left\{\begin{bmatrix}0 & a\\ b & 0\end{bmatrix} : \text{either } a = b = 0 \text{ or } a \neq 0, b \neq 0\right\}$ forms a Boolean ternary subsemiring of $M_2^-(S)$ in which commutative and additive cancellative laws fails to hold. For this,

$$\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 3\\ 2 & 0\end{bmatrix} \neq \begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix} + \begin{bmatrix}0 & 4\\ 4 & 0\end{bmatrix} = \begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

does not imply $\begin{bmatrix}0 & 4\\ 4 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and Char($B_2^-(S)$) ≠ 2.

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