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INITIAL VALUE PROBLEM OF THIRD ORDER RANDOM DIFFERENTIAL INCLUSION

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ABSTRACT

In this paper, we prove the existence of solution for the initial value problem of third order random differential inclusion through random fixed point theory. We claim that our result is new to the theory of random differential inclusions.

Keywords: *Random differential inclusion, random solution, caratheodory condition.*

AMS Mathematics Subject Classifications: *60H25, 47H10.*

1. STATEMENT OF THE PROBLEM

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let R be the real and and let $J = [0, T]$ be a closed and bounded interval in R. Consider the initial value problem of third order ordinary random differential inclusion (in short RDI),

$$\left.\begin{aligned} x'''(t,\omega) \in F(t, x(t,\omega), \omega) + G(t, x(t,\omega), \omega) + H(t, x(t,\omega), \omega) \quad \text{a.e.} \quad t \in J \\ x(0,\omega) = q_0(\omega), x'(0,\omega) = q_1(\omega), x''(0,\omega) = q_2(\omega) \end{aligned}\right\} \tag{1.1}$$

for all $\omega \in \Omega$, where $q_0, q_1, q_2 : \Omega \to R$ is measurable, $F, G : J \times R \times \Omega \to \mathcal{P}_p(R)$.

By a random solution for the RDI (1.1) we mean a measurable function $x : \Omega \to AC(J, R)$ satisfying for each $\omega \in \Omega$, $x'''(t,\omega) = v_1(t,\omega) + v_2(t,\omega) + v_3(t,\omega)$ $\forall t \in J$ and $x(0,\omega) = q_0(\omega), x'(0,\omega) = q_1(\omega), x''(0,\omega) = q_2(\omega)$ for some measurable functions $v_1, v_2, v_3 : \Omega \to L^1(J, R)$ with $v_1(t,\omega) \in F(t, x(t,\omega), \omega)$ a.e. $t \in J$ and $v_2(t,\omega) \in G(t, x(t,\omega), \omega)$, $v_3(t,\omega) \in H(t, x(t,\omega), \omega)$ a.e. $t \in J$, where $AC(J, R)$ is the space of absolutely continuous real-valued functions on *J*. The RDI (1.1) has not been discussed earlier in the literature. In this paper, we prove the existence of solution for the initial value problem of third order random differential inclusion (1.1) through random fixed point theory.

2. AUXIALARY RESULTS

we need the following lemma to prove the result.

Lemma 2.1(Dhage [4]): Let $(\Omega, \mathcal{A}, \mu)$ be a complete $\sigma-finite$ measure space and let X be separable Banach space. If $F, G, H : \Omega \to \mathcal{P}_{cl}(X)$ are three multi-valued random operators, then the sum F+G+H defined by $(F + G + H)(\omega) = F(\omega) + G(\omega) + H(\omega)$ is again a multi-valued random operator on Ω.

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Corollary 2.2: Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let $[a,b]$ be a random interval in a separable Banach space X. Let $A, B, C : \Omega \times [a,b] \to \mathcal{P}_{cl}(X)$ be three right monotone increasing multi-valued random operators satisfying for each $\omega \in \Omega$,

(a) $A(\omega)$ is a multi-valued contraction,

(b) $B(\omega)$, $C(\omega)$ are completely continuous, and

(c) $A(\omega)x + B(\omega)x + C(\omega)x \in [a,b]$ for all $x \in [a,b]$.

Furthermore, if the cone K in X is normal, then the random operator inclusion $x \in A(\omega)x + B(\omega)x + C(\omega)x$ has a random solution in $[a,b]$.

3. EXISTENCE RESULTS

We seek the solutions of RDI (1.1) in the function space $C(J,R)$ of continuous real-valued functions on *J*. Define a norm $\|\cdot\|$ in $C(J,R)$ by $\|x\| = \sup_{t \in J} |x(t)|$ and the order relation $\le$ in $C(J,R)$ by $x \le y \Leftrightarrow y - x \in K$, where the cone *K* in $C(J,R)$ is defined by

$$K = \{x \in C(J,R) \mid x(t) \ge 0 \quad \text{for all} \quad t \in J\}.$$

For any measurable function $x : \Omega \to C(J,R)$, let

$$S^1_{F(\omega)}(x) = \left\{ v \in \mathcal{M}(\Omega, L^1(J,R)) \middle| v(t,\omega) \in F(t, x(t,\omega), \omega) \quad \text{a.e.} \quad t \in J \right\}. \tag{3.1}$$

This is our set of selection functions. The integral of the random multi-valued function *F* is defined as

$$\int_0^t F(s, x(s,\omega), \omega)\, ds = \left\{ \int_0^t v(s,\omega)\, ds : v \in S^1_{F(\omega)}(x) \right\}.$$

We need the following definitions in the sequel.

Definition 3.1: A multi-valued function $F : J \times R \times \Omega \to \mathcal{P}_{cp}(R)$ is called Caratheodory if for each $\omega \in \Omega$,

(i) $t \mapsto F(t, x, \omega)$ is measurable for each $x \in \mathbb{R}$, and

(ii) $x \mapsto F(t, x, \omega)$ is an upper semi-continuous almost everywhere for $t \in J$.

Again, a Caratheodory multi-valued function *F* is called L^1-Caratheodory if for each real number r > 0 there exists a measurable function $h_r : \Omega \to L^1(J,R)$ such that for each $\omega \in \Omega$

$$\|F(t,x,w)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t,x,\omega)\} \le h_r(t,\omega) \quad \text{a.e.} \quad t \in J \text{ for all } x \in R \text{ with } |x| \le r$$

Furthermore, a Caratheodory multi-valued function *F* is called L^1_X-Caratheodory if

(iii) there exists a measurable function $h : \Omega \to L^1(J,R)$ such that

$\|F(t,x,\omega)\|_{\mathcal{P}} \le h(t,\omega)$ a.e. $t \in J$ for all $x \in R$, and the function *h* is called a growth function of *F* on $J \times R \times \Omega$.

Definition 3.2: A multi-valued function $F : J \times R \times \Omega \to P_{cp}(R)$ is called s-Caratheodory if for each $\omega \in \Omega$,

(i) $(t,\omega) \mapsto F(t,x,\omega)$ is measurable for each $x \in R$, and

(ii) $x \mapsto F(t,x,\omega)$ is an Hausdorff continuous almost everywhere for $t \in J$.

Furthermore, a s-Caratheodory multi-valued function *F* is called s-L^1- Caratheodory if

(iii) for each real number r > 0 there exists a measurable function $h_r : \Omega \to L^1(J,R)$ such that for each $w \in \Omega$

(iv) $\|F(t,x,\omega)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t,x,\omega)\} \le h_r(t,\omega)$ a.e. $t \in J$ for all $x \in R$ with $|x| \le r$.

Then we have the following lemmas which are well-known in the literature.

Lemma 3.1 (Lasota and Opial [8]): Let E be a Banach space. If dim (E) < ∞ and $F : J \times E \times \Omega \to \mathcal{P}_{c\mathcal{P}}(E)$ is L^1- Caratheodory, then $S^1_{F(\omega)}(x) \neq \emptyset$ for each $x \in E$.

Lemma 3.2 (Lasota and Opial [8]): Let E be a Banach space, F a Caratheodory multi-valued operator with $S^1_{f(\omega)} \neq \emptyset$, and $\ell : L^1(J, E) \to C(J, E)$ be a continuous linear mapping. Then the composite operator

$$\mathcal{L} \circ S^1_{F(\omega)} : C(J, E) \to \mathcal{P}_{bd,cl}(C(J, E))$$

is a closed graph operator on $C(J, E) \times C(J, E)$.

Lemma 3.3 (Caratheodory theorem [5]): Let E be a Banach space. If $F : J \times E \to \mathcal{P}_p(E)$ is s-Caratheodory, then the multi-valued mapping $(t, x) \mapsto F(t, x(t))$ is jointly measurable for each measurable function $x : J \to E$.

We consider the following set of hypotheses in the sequal.

(A_1) The multi-valued mapping $(t, \omega) \mapsto F(t, x, \omega)$ is jointly measurable for all $x \in R$.

(A_2) $F(t, x, w)$ is closed and bounded for each $(t, \omega) \in J \times \Omega$ and $x \in R$.

(A_3) F is integrably bounded on $J \times \Omega \times R$.

(A_4) There is a function $\ell \in \mathcal{M}(\Omega, L^1(J, R))$ such that for each $\omega \in \Omega$,

$$d_H(F(t, x, \omega), F(t, y, \omega)) \leq \ell(t, \omega)|x - y| \text{ a.e. } t \in J \text{ for all } x, y \in R.$$

(A_5) The multi-valued mapping $x \mapsto S^1_{F(\omega)}(x)$ is right monotone increasing in $x \in C(J, R)$ almost everywhere for $t \in J$. and

(B_1) The multi-valued mapping $(t, \omega) \mapsto G(t, x(t, \omega), \omega)$, $(t, \omega) \mapsto H(t, x(t, \omega), \omega)$ are jointly measurable for all measurable $x : \Omega \to C(J, R)$.

(B_2) $G(t, x, \omega)$, $H(t, x, \omega)$ are closed and bounded for each $(t, \omega) \in J \times \Omega$ and $x \in R$.

(B_3) G , H are L^1-Caratheodory.

(B_4) The multi-valued mapping $x \mapsto S^1_{F(\omega)}(x)$ is right monotone increasing in $x \in C(J, R)$ almost everywhere for $t \in J$.

(B_5) RDI (1.1) has a strict lower random solution a and a strict upper random solution b with $a \leq b$ on $J \times \Omega$.

MAIN RESULT

Theorem 3.1: Assume that the hypotheses $(A_1) - (A_5)$ and $(B_1) - (B_5)$ hold. If $\|\ell(\omega)\|_{L^1} < 1,$ then the RDI (1.1) has a random solution in [a, b] defined on $J \times \Omega$

Proof: Let $X = C(J, R)$. Define a random order interval $[a, b]$ in *X* which is well defined in view of hypothesis (B_5). Now the RDI (1.1) is equivalent to the random integral inclusion,

$$x'''(t, \omega) \in q_0(\omega) + q_1(\omega)\omega + q_2(\omega)\frac{\omega^2}{2} + \int_0^t \frac{(t-s)^2}{2} F(s, x(s, \omega), \omega)\, ds$$
$$+ \int_0^t \frac{(t-s)^2}{2} G(s, x(s, \omega), \omega)\, ds, \quad t \in J. \tag{3.2}$$

for all $\omega \in \Omega$. Define three multi-valued operators $A, B, C : \Omega \times [a, b] \to \mathcal{P}_{\mathcal{P}}(X)$ by

$$A(\omega)x = \left\{ u \in \mathcal{M}(\Omega, X) \mid u(t, \omega) = q_0(\omega) + \int_0^t v_1(s, \omega)\, ds, v_1 \in S^1_{F(\omega)}(x) \right\}$$
$$= \left(\mathcal{K}_1 \circ S^1_{F(\omega)}\right)(x) \tag{3.3}$$

And

$$B(\omega)x=\left\{u\in\mathcal{M}(\Omega,X)\,|\,u(t,\omega)=q_0(\omega)+q_1(\omega)\omega+\int_0^t v_2(s,\omega)ds, v_2\in S^1_{F(\omega)}(x)\right\}$$
$$=\left(\mathcal{K}_2\circ S^1_{F(\omega)}\right)(x)$$

And

$$C(\omega)x=\left\{u\in\mathcal{M}(\Omega,X)\,|\,u(t,\omega)=q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t v_3(s,\omega)ds, v_3\in S^1_{F(\omega)}(x)\right\}$$
$$=\left(\mathcal{K}_3\circ S^1_{F(\omega)}\right)(x)$$

Where, $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3:\mathcal{M}(\Omega,L^1(J,R))\to\mathcal{M}(\Omega,C(J,R))$ are continuous operators defined by

$$\mathcal{K}_1 v_1(t,\omega)=q_0(\omega)+\int_0^t v_1(s,\omega)ds,$$
$$\mathcal{K}_2 v_2(t,\omega)=q_0(\omega)+q_1(\omega)\omega+\int_0^t v_2(s,\omega)ds, \tag{3.5}$$

And
$$\mathcal{K}_3 v_3(t,\omega)=q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t v_3(s,\omega)ds. \tag{3.6}$$

Clearly, the operators $A(\omega)$ and $B(\omega)$ are well defined in view of hypotheses (A_3) and (B_3). We will show that $A(\omega)$, $B(\omega)$ and $C(\omega)$ satisfy all the conditions of Corollary 2.2.

Step-I: First, we show that *A* is closed valued multi-valued random operator on $\Omega\times[a,b]$. Observe that the operator $A(\omega)$ is equivalent to the composition $\mathcal{K}_1\circ S^1_{F(w)}$ of two operators on $L^1(J,R)$, where $\mathcal{K}_1:\mathcal{M}(\Omega,L^1(J,R))\to X$ is the continuous operator. To show $A(\omega)$ has closed values, it then suffices to prove that the composition operator $\mathcal{K}_1\circ S^1_{F(\omega)}$ has closed values on [*a, b*]. Let $x\in[a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_{F(\omega)}(x)$ converging to *v* in measure. Then, by the definition of $S^1_{F(\omega)}$, $v_n(t,\omega)\in F(t,x(t,\omega),\omega)$ a.e. for $t\in J$. Since $F(t,x(t,\omega),\omega)$ is closed, $v(t,\omega)\in F(t,x(t,\omega),\omega)$ a.e. for $t\in J$. Hence, $v\in S^1_{F(\omega)}$ (*x*). As a result, $S^1_{G(\omega)}(x)$ is a closed set in $L^1(J,R)$ for each $\omega\in\Omega$. From the continuity of $\mathcal{K}_1$, it follows that $\left(\mathcal{K}_1\circ S^1_{F(\omega)}\right)(x)$ is a closed set in *X*. Therefore, $A(\omega)$ is a closed-valued multi-valued operator on [*a, b*] for each $\omega\in\Omega$. Next, we show that $A(\omega)$ is a multi-valued random operator on [*a, b*]. First, we show that the multi-valued mapping $(\omega,x)\mapsto S^1_{F(\omega)}(x)$ is measurable. Let $f\in\mathcal{M}(\Omega,L^1(J,R))$ be arbitrary. Then we have

$$d\left(f,S^1_{F(\omega)}(x)\right)=\text{in}\ \left\{\|f(\omega)-h(\omega)\|_{L^1}:h\in S_{F(\omega)}(x)\right\}$$
$$=\inf\left\{\int_0^T\left|f(t,\omega)-h(t,\omega)\right|dt:h\in S_{F(\omega)}(x)\right\}$$
$$=\int_0^T\inf\left\{\left|f(t,\omega)-z\right|:z\in F(t,x(t,\omega),\omega)\right\}dt$$
$$=\int_0^T d\left(f(t,\omega)\right),F(t,x(t,\omega),\omega)dt.$$

But by hypothesis (A_1), the mapping $(t,\omega)\mapsto F(t,x,\omega)$ is measurable, and by (A_4), the mapping $x\mapsto F(t,x,\omega)$ is Hausdorff continuous. Hence by Caratheodory theorem, the map $(t,\omega)\mapsto F(t,x(t,\omega),\omega)$ is measurable for all measurable function $x:\Omega\to C(J,R)$. It is known that the multi-valued mapping $z\mapsto d(z,F(t,x,\omega))$ is continuous, Hence the mapping multi-valued mapping $(t,x,\omega,z)\mapsto d(z,F(t,x,\omega))$ measurable. Hence we deduce that the mapping $(t,x,\omega,f)\mapsto d(f(t,\omega),F(t,x(t,\omega),\omega)$ is measurable from $J\times X\times\Omega\times L^1(J,R)$ into R^+. Now the integral is the limit of the finite sum of measurable functions, and so, $d\left(f,S^1_{F(\omega)}(x)\right)$ is measurable. As a result, the multi-valued mapping (.,.) $\to S^1_{F(\cdot)}(\cdot)$ is jointly measurable.

Define a function ϕ on $J \times X \times \Omega$ by

$$\phi(t,x,\omega) = \left(\mathcal{K}_1 S^1_{F(\omega)}\right)(x)(t) = \int_0^t F\left(s, x(s,\omega,\omega)\right) ds.$$

We shall show that $\phi(t,x,\omega)$ is continuous in t in the Hausdorff metric on R. Let $\{t_n\}$ be a sequence in converging to $t \in J$. Then we have

$$\begin{aligned}
& d_H\left(\phi(t_n,x,\omega), \phi(t,x,\omega)\right) \\
&= d_H\left(\int_0^{t_n} F\left(s,x(s,\omega),\omega\right)ds, \int_0^{t} F\left(s,x(s,\omega),\omega\right)ds\right) \\
&= d_H\left(\int_J \mathcal{X}_{[0,t_n]}(s) F\left(s,x(s,\omega),\omega\right)ds, \int_J \mathcal{X}_{[0,t]}(s) F\left(s,x(s,\omega),\omega\right)ds\right) \\
&= d_H\left(\int_J \mathcal{X}_{[0,t_n]}(s) F\left(s,x(s,\omega),\omega\right)ds, \int_J \mathcal{X}_{[0,t]}(s) F\left(s,x(s,\omega),\omega\right)ds\right) \\
&= \int_J \left|\mathcal{X}_{[0,t_n]}(s) - \mathcal{X}_{[0,t]}(s)\right| \left\|F\left(s,x(s,\omega),\omega\right)\right\|_{\mathcal{P}} ds \\
&= \int_J \left|\mathcal{X}_{[0,t_n]}(s) - \mathcal{X}_{[0,t]}(s)\right| h_r(s,\omega)\,ds \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus the multi-valued mapping $t \mapsto \phi(t,x,\omega)$ is continuous, and hence, and by Lemma 3.3, the mapping

$$(t,x,\omega) \mapsto \int_0^t F\left(s,x(s,\omega),\omega\right)ds$$

is measurable. Consequently, $A(\omega)$ is a random multi-valued operator on [a, b]. Similarly, it can be shown that $B(\omega)$, $C(\omega)$ is a closed-valued multi-valued operator on [*a*, *b*] and the mapping $(t,x,\omega) \mapsto \int_0^t G\left(s,x(s,\omega),\omega\right)ds$ is measurable. Again, since the sum of two measurable multi-valued functions is measurable, the mapping $(t,x,\omega) \mapsto q_0(\omega) + q_1(\omega)\omega + q_2(\omega)\frac{\omega^2}{2} + \int_0^t G\left(s,x(s,\omega),\omega\right)ds$ is measurable.

Step-II: Next we show that $A(\omega)$ is a multi-valued contraction on *X*. Let $x, y \in X$ be any two element and let $u_1 \in A(\omega)(x)$. Then $u_1 \in X$ and

$$u_1(t,\omega) = \int_0^t v_1(s,\omega)ds \quad \text{for some } v_1 \in S^1_{F(\omega)}(x).$$

Since

$$d_H\left(F\left(t,x(t,\omega),\omega\right), F\left(t,y(t,\omega),\omega\right)\right) \le \ell(t,\omega)\left|x(t,\omega) - y(t,\omega)\right|,$$

We obtain that there exists a $w \in F\left(t, y(t,\omega), \omega\right)$ such that

$$\left|v_1(t,\omega) - w\right| \le \ell(t,\omega)\left|x(t,\omega) - y(t,\omega)\right|.$$

Thus, the multi-valued operator *U* defined by

$$U(t,\omega) = S^1_{F(\omega)}(y)(t) \cap K(\omega)(t),$$

where $K(\omega)(t) = \left\{w \middle| \left|v_1(t,\omega) - w\right| \le \ell(t,\omega)\left|x(t,\omega) - y(t,\omega)\right|\right\}$

has nonempty values and is measurable. Let v_2 be a measurable selection function for *U*. Then there exists $v_2 \in F\left(t, y(t,\omega), \omega\right)$ with $\left|v_1(t,\omega) - v_2(t,\omega)\right| \le \ell(t,\omega)\left|x(t,\omega) - y(t,\omega)\right|$, a.e. on *J*.

Define $u_2(t,\omega)=\int_0^t v_2(s,\omega)ds.$ It follows that $u_2 \in A(\omega)(y)$ and

$$\begin{aligned}
\left|u_1(t,\omega)-u_2(t,\omega)\right| &\le \left|\int_0^t v_1(s,\omega)ds-\int_0^t v_2(s,\omega)ds\right| \\
&\le \int_0^t \left|v_1(s,\omega)-v_2(s,\omega)\right|ds \\
&\le \int_0^t \ell(t,\omega)\left|x(s,\omega)-y(s,\omega)\right|ds \\
&\le \left\|\ell(\omega)\right\|_{L^1}\left\|x(\omega)-y(\omega)\right\|.
\end{aligned}$$

Taking the supremum over *t*, we obtain

$$\left\|u_1(\omega)-u_2(\omega)\right\| \le \left\|\ell(\omega)\right\|_{L^1}\left\|x(\omega)-y(\omega)\right\|.$$

From this and the analogous inequality obtained by interchanging the role of x and y we obtain

$$d_H\left(A(\omega)(x),A(\omega)(y)\right) \le \left\|\ell(\omega)\right\|_{L^1}\left\|x(\omega)-y(\omega)\right\|,$$

for all $x,y\in X$. This shows that $A(\omega)$ is multi-valued random contraction on *X*, since $\left\|\ell(\omega)\right\|_{L^1}<1$ for each $\omega\in\Omega$.

Step-III: Next, we show that $B(\omega)$ is completely continuous for each $\omega\in\Omega$. First, we show that $B(\omega)([a,b])$ is compact for each $\omega\in\Omega$. Let $\{y_n(\omega)\}$ be a sequence in $B(\omega)([a,b])$ for some $\omega\in\Omega$. We will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in *X*.

Case-I: First, we show that $\{y_n(\omega)\}$ is uniformly bounded sequence. By the definition of $\{y_n(\omega)\}$, we have a $v_n(\omega)\in S^1_{G(\omega)}(x)$ for some $x\in[a,b]$ such that

$$y_n(t,\omega)=q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t v_n(s,\omega)ds,\quad t\in J.$$

Therefore,

$$\begin{aligned}
\left|y_n(t,\omega)\right| &\le \left|q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right|+\int_0^t\left|v_n(s,\omega)\right|ds \\
&\le \left|q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right|+\int_0^t\left\|F(s,x_n(s,\omega),\omega)\right\|_{\mathcal{P}}ds \\
&\le \left|q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right|+\left\|h_r(\omega)\right\|_{L^1}
\end{aligned}$$

for all $t\in J$, where $r=\left\|a(\omega)\right\|+\left\|b(\omega)\right\|$. Taking the supremum over *t* in the above inequality yields,

$$\left\|y_n(\omega)\right\| \le \left|q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right|+\left\|h_r(\omega)\right\|_{L^1}$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)([a,b])$.

Next we show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)([a,b])$. Let $t,\tau\in J$. Then we have

$$\begin{aligned}\left|y_n(t,\omega)-y_n(\tau,\omega)\right| &\le \left|\int_0^t v_n(s,\omega)\,ds-\int_0^\tau v_n(s,\omega)\,ds\right| \\ &\le \left|\int_\tau^t v_n(s,\omega)\,ds\right| \\ &\le \left|\int_\tau^t h_r(s,\omega)\,ds\right| \\ &\le \left|p(t,\omega)-p(\tau,\omega)\right|,\end{aligned}$$

where $p(t,\omega)=\int_0^t h_r(s,\omega)\,ds.$

From the above inequality, it follows that

$$\left|y_n(t,\omega)-y_n(\tau,\omega)\right|\to 0 \quad \text{as} \quad t\to\tau.$$

This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $B(\omega)([a,b]).$ Now $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous for each $\omega\in\Omega$, so it has a cluster point in view of Arzela-Ascoli theorem. As a result, $B(\omega)$ is a compact multi-valued random operator on [*a*, *b*] and similarly for $C(\omega)$.

Next we show that $B(\omega)$,$C(\omega)$ are upper semi-continuous multi-valued random operator on [*a*, *b*]. Let $\{x_n(\omega)\}$ be a sequence in *X* such that $x_n(\omega)\to x_*(\omega).$ Let $\{y_n(\omega)\}$ be a sequence such that $y_n(\omega)\in C(\omega)x_n$ and $y_n(\omega)\to y_*(\omega).$ we will show that $y_*(\omega)\in C(\omega)x_*.$

Since $y_n(\omega)\in C(\omega)x_n$ there exists a $v_n(\omega)\in S^1_{G(\omega)}(x_n)$ such that

$$y_n(t,\omega)=q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t v_n(s,\omega)\,ds,\quad t\in J\cdot$$

We must prove that there is a $v_*(\omega)\in S^1_{G(\omega)}(x_*)$ such that

$$y_*(t,\omega)=q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t v_*(s,\omega)\,ds,\quad t\in J.$$

Consider the continuous linear operator $\mathcal{L}:\mathcal{M}\left(\Omega,L^1(J,R)\right)\to\mathcal{M}\left(\Omega,C(J,R)\right)$ defined by

$$\mathcal{L}v(t,\omega)=\int_0^t v(s,\omega)\,ds,\qquad t\in J.$$

Now $\left\|\left(y_n(\omega)-(q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}q(\omega))\right)-\left(y_*(\omega)-(q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2})\right)\right\|\to 0 \quad a \quad n\to\infty$

From lemma 3.2, it follows that $\mathcal{L}\circ S^1_{G(\omega)}$ is a closed graph operator. Also, from the definition of $\mathcal{L}$, we have

$$y_n(t,\omega)-\left(q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right)\in\left(\mathcal{L}\circ S^1_{G(\omega)}\right)(x_n)$$

Since $y_n(\omega)\to y_*(\omega),$ there is a point $v_*(\omega)\in S^1_{F(\omega)}(x_*)$ such that

$$y_*(t,\omega)=\left(q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right)+\int_0^t v_*(s,\omega)\,ds,\qquad t\in J.$$

This shows that $C(\omega)$ is a upper semi-continuous multi-valued random operator on [a, b]. Thus, $C(\omega)$ is upper semi-continuous and compact and hence a completely continuous multi-valued random operator on [*a*, *b*] and also for $B(\omega)$

.

Step-VI: Next, we show that $A(\omega)$ is a right monotone increasing and multi-valued random operator on [a, b] into itself for each $\omega \in \Omega$. Let $x, y \in [a,b]$ be such that $x \leq y$. Since (A_5) holds, we have that $S^1_{F(\omega)}(x) \leq S^1_{F(\omega)}(y)$. Hence $A(\omega)(x) \overset{i}{\leq} A(\omega)(y)$.Similarly, $B(\omega)(x) \overset{i}{\leq} B(\omega)(y)$, $C(\omega)(x) \overset{i}{\leq} C(\omega)(y)$ From (B_5), it follows that $a \leq A(\omega)a + B(\omega)a + C(\omega)a$ and $A(\omega)b + B(\omega)b + C(\omega)b \leq b$ for all $\omega \in \Omega$. Now $A(\omega)$, $B(\omega)$ and $C(\omega)$ are monotone increasing, so we have for each $\omega \in \Omega$,

$a \leq A(\omega)a + B(\omega)a + C(\omega)a \overset{i}{\leq} A(\omega)x + B(\omega)x \overset{i}{\leq} A(\omega)b + B(\omega)b + C(\omega)b \leq b$ for all $x \in [a,b]$. Hence, $A(\omega)x + B(\omega)x + C(\omega)x \in [a,b]$ for all $x \in [a,b]$.

Thus, the multi-valued random operators $A(\omega)$, $B(\omega)$ and $C(\omega)$ satisfy all the conditions of Corollary 2.2 and hence the random operator inclusion $x \in A(\omega)x + B(\omega)x + C(\omega)x$ has a random solution. This implies that the RDI (1.1) has a random solution on $J \times \Omega$. This completes the proof.

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