International Journal of Mathematical Archive-6(11), 2015, 81-86
IJMA Available online through www.ijma.info ISSN 2229 – 5046

HARARY EQUIENERGETIC GRAPHS

HARISHCHANDRA S. RAMANE[1], VINAYAK V. MANJALAPUR*[2]

[1,2]Department of Mathematics, Karnatak University - Dharwad – 580000, India.

(Received On: 15-10-15; Revised & Accepted On: 05-11-15)

ABSTRACT

The Harary matrix or reciprocal distance matrix of a graph G is defined as $RD(G) = [r_{ij}]$, in which $r_{ij} = \frac{1}{d_{ij}}$ if $i \neq j$ and $r_{ij} = 0$ if $i = j$, where d_{ij} is the distance between the i^{th} and j^{th} vertex of G. The Harary energy $HE(G)$ of G is defined as the sum of the absolute values of the eigenvalue of the Harary matrix of graph G. Two graphs G_1 and G_2 are said to be Harary equienergetic if $HE(G_1) = HE(G_2)$. In this paper we obtain the Harary eigenvalues and Harary energy of the join of regular graphs of diameter less than or equal to two and thus construct the Harary equienergetic graphs on n vertices, for all $n \geq 6$ having different Harary eigenvalues.

Mathematics Subject Classification: *05C50, 05C12.*

Keywords: *Harary eigenvalues, Harary energy, Harary equienergetic graphs, join of graphs.*

1. INTRODUCTION

Let G be a simple, connected graph with n vertices $v_1, v_2, \ldots, v_n$. The *distance* between the vertices v_i and v_j, denoted by $d_{ij} = d(v_i, v_j)$ is the length of shortest path joining them. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G [1].

The *Harary matrix* (also called as *reciprocal distance matrix* [10]) of a graph G is an $n \times n$ matrix $(G) = [r_{ij}]$, in which

$$r_{ij} = \begin{cases} \frac{1}{d_{ij}}, & if\ i \neq j \\ 0, & otherwise \end{cases}$$

where d_{ij} is the distance between v_i and v_j. The Harary matrix of a graph was introduced by Ivanciuc *et al.* [9] which has importance in the study of molecules in QSPR (Quantitative Structure Property Relationship) models [9]. The *characteristic polynomial* of $RD(G)$is defined as $\psi(G : \mu) = \det(\mu I - RD(G))$, where I is the identity matrix of order n. The eigenvalues of the Harary matrix $RD(G)$, denoted by $\mu_1, \mu_2, \ldots, \mu_n$ are said to be the *Harary eigenvalues* or *H-eigenvalues* of G and their collection is called the *Harary spectrum* or *H-spectrum* of G.

Since the Harary matrix of G is symmetric, its eigenvalues are real and can be ordered as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. Two non-isomorphic graphs are said to be *Harary cospectral* or *H-cospectral* if they have same H-eigenvalues. The results on H-eigenvalues of a graph are obtained in [2, 4, 5, 8, 13].

The *Harary energy* or *H-energy* of a graph G, denoted by $HE(G)$, is defined as [6]

$$HE(G) = \sum_{i=1}^{n} |\mu_i|. \tag{1}$$

The Eq. (1) is defined in full analogy to the *ordinary graph energy* [7] defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of G. Details about the graph energy can be found in [11].

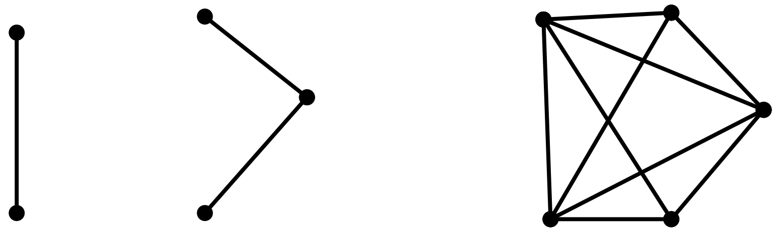
The graphs G_1 and $G_2$are said to be *Harary equienergetic* or *H-equienergetic* if $HE(G_1) = HE(G_2)$. For obvious reason, H-cospectral graphs are H-equienergetic. Therefore it is interesting to obtain non H-cospectral graphs on same number of vertices having equal H-energy. In [12] the H-energy of line graphs of certain regular graphs was obtained and thus obtained the pairs of H-equienergetic graphs.

Corresponding Author: Vinayak V. Manjalapur[2]*
[1,2]Department of Mathematics, Karnatak University, Dharwad – 580000, India.

In this paper we obtain the characteristic polynomial of the Harary matrix of the join of two regular graphs whose diameter is less than or equal to two and thereby construct pairs of non H-cospectral, H-equienergetic graphs on n vertices for all $n \geq 6$.

2. *H*-SPECTRA AND *H*-ENERGY OF JOIN OF GRAPHS

Definition: The join of two graphs G_1 and G_2, denoted by $G_1 \nabla G_2$, is a graph obtained from G_1 and G_2 by joining each vertex of G_1 to all vertices of G_2.



G_1 G_2 $G_1 \nabla G_2$

Theorem 2.1: *Let G_i be an r_i- regular graph on n_i vertices and $diam(G_i) \leq 2$, $i = 1, 2$. Then the characteristic polynomial of the Harary matrix of $G_1 \nabla G_2$ is*

$$\psi(G_1 \nabla G_2 : \mu) = \frac{[(\mu - X)(\mu - Y) - n_1 n_2]}{(\mu - X)(\mu - Y)} \psi(G_1 : \mu)\psi(G_2 : \mu), \tag{2}$$

where $X = \frac{n_1 + r_1 - 1}{2}$ *and* $Y = \frac{n_2 + r_2 - 1}{2}$.

Proof: $\psi(G_1 \nabla G_2 : \mu) = \det\big(\mu I - RD(G_1 \nabla G_2)\big)$

$$= \begin{vmatrix} \mu I_{n_1} - RD(G_1) & -J_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & \mu I_{n_2} - RD(G_2) \end{vmatrix} \tag{3}$$

where J is the matrix whose all entries are equal to unity. The determinant (3) can be written as

$$\begin{vmatrix} \mu & -r_{12} & \dots & -r_{1n_1} & -1 & 1 & \dots & -1 \\ -r_{21} & \mu & \dots & -r_{2n_1} & -1 & 1 & \dots & -1 \\ \vdots & & \vdots & & & & \vdots & \\ -r_{n_1 1} & -r_{n_1 2} & \dots & \mu & -1 & 1 & \dots & -1 \\ -1 & 1 & \dots & -1 & \mu & -r'_{12} & \dots & -r'_{1n_2} \\ -1 & 1 & \dots & -1 & -r'_{21} & \mu & \dots & -r'_{2n_2} \\ \vdots & & \vdots & & & & \vdots & \\ -1 & 1 & \dots & -1 & -r'_{n_2 1} & -r'_{n_2 2} & \dots & \mu \end{vmatrix} \tag{4}$$

in which $r_{ij} = \frac{1}{d_{ij}}$, where d_{ij} is the distance between the vertices v_i and v_j in G_1 and $r'_{ij} = \frac{1}{d'_{ij}}$, where d'_{ij} is the distance between the vertices u_i and u_j in G_2. Since G_i is an r_i -regular graph and $diam(G_i) \leq 2$, every vertex of G_i is at distance one from r_i vertices and at distance 2 from remaining $(n_i - 1 - r_i)$ vertices, $i = 1, 2$. Therefore

$$\sum_{j=1}^{n_1} r_{ij} = \frac{n_1 + r_1 - 1}{2} \qquad for\ i = 1, 2, \dots, n_1 \tag{5}$$

and

$$\sum_{j=1}^{n_2} r'_{ij} = \frac{n_2 + r_2 - 1}{2} \qquad for\ i = 1, 2, \dots, n_2 \tag{6}$$

Performing following operations in the sequence on the determinant (4) we get (7).

(i) Subtract the row $(n_1 + 1)$ from the rows $(n_1 + 2), (n_1 + 3), \dots, (n_1 + n_2)$.

(ii) Add the columns $(n_1 + 2), (n_1 + 3), \dots, (n_1 + n_2)$ to the column $(n_1 + 1)$ and $|B|$. apply Eq. (6), where $Y = \frac{n_2 + r_2 - 1}{2}$: (7)

$$\begin{vmatrix} \mu & -r_{12} & \dots & -r_{1n_1} & -n_2 \\ -r_{21} & \mu & \dots & -r_{2n_1} & -n_2 \\ \vdots & & \vdots & & \\ -r_{n_1 1} & -r_{n_1 2} & \dots & \mu & -n_2 \\ -1 & -1 & \dots & -1 & \mu - Y \end{vmatrix}$$

where

$$|B| = \begin{vmatrix} \mu + r'_{12} & -r'_{23} + r'_{13} & \dots & -r'_{2n_2} + r'_{1n_2} \\ -r'_{32} + r'_{12} & \mu + r'_{13} & & -r'_{3n_2} + r'_{1n_2} \\ \vdots & & \vdots & \\ -r'_{n_2 2} + r'_{12} & -r'_{n_2 3} + r'_{13} & \dots & \mu + r'_{1n_2} \end{vmatrix} \quad (8)$$

The first determinant in (7) is of order $(n_1 + 1)$.

Performing following operations in the sequence on the first determinant of (7) we get (9).

(i) Subtract the first row from the rows $2, 3, \dots, n_1$.

(ii) Add columns $2, 3, \dots, n_1$ to the first column of and apply Eq. (5), where $X = \frac{n_1 + r_1 - 1}{2}$:

$$\begin{vmatrix} \mu - X & -r_{12} & \dots & -r_{1n_1} & -n_2 \\ 0 & \mu + r_{12} & \dots & -r_{2n_1} + r_{1n_1} & 0 \\ \vdots & & \vdots & & \\ 0 & -r_{n_1 2} + r_{12} & \dots & \mu + r_{1n_1} & 0 \\ -n_1 & -1 & \dots & -1 & \mu - Y \end{vmatrix} |B|. \quad (9)$$

Expand it along the first column to obtain(10):

$$\{(\mu - X)\Delta_1 - (-1)^{n_1} n_1 \Delta_2\} \ |B| \quad (10)$$

where

$$\Delta_1 = \begin{vmatrix} \mu + r_{12} & -r_{23} + r_{13} & \dots & -r_{2n_1} + r_{1n_1} & 0 \\ -r_{32} + r_{12} & \mu + r_{13} & & -r_{3n_1} + r_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -r_{n_1 2} + r_{12} & -r_{n_1 3} + r_{13} & \dots & \mu + r_{1n_1} & 0 \\ -1 & -1 & & -1 & \mu - Y \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} -r_{12} & -r_{13} & \dots & -r_{1n_1} & -n_2 \\ \mu + r_{12} & -r_{23} + r_{13} & \dots & -r_{2n_1} + r_{1n_1} & 0 \\ -r_{32} + r_{12} & \mu + r_{13} & & -r_{3n_1} + r_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -r_{n_1 2} + r_{12} & -r_{n_1 3} + r_{13} & \dots & \mu + r_{1n_1} & 0 \end{vmatrix}$$

The expression (10) can be written as

$$\{(\mu - X)(\mu - Y)|A| - (-1)^{n_1} n_1 (-1)^{n_1 + 1}(-n_2)|A|\} \ |B| = \{(\mu - X)(\mu - Y) - n_1 n_2\}|A||B| \quad (11)$$

Where,

$$|A| = \begin{vmatrix} \mu + r_{12} & -r_{23} + r_{13} & \cdots & -r_{2n_1} + r_{1n_1} \\ -r_{32} + r_{12} & \mu + r_{13} & & -r_{3n_1} + r_{1n_1} \\ \vdots & & \vdots & \\ -r_{n_1 2} + r_{12} & -r_{n_1 3} + r_{13} & \cdots & \mu + r_{1n_1} \end{vmatrix} \qquad (12)$$

The determinant (12) can be written as

$$|A| = \frac{1}{(\mu - X)} \begin{vmatrix} \mu - X & -r_{12} & -r_{13} & \cdots & -r_{1n_1} \\ 0 & \mu + r_{12} & -r_{23} + r_{13} & \cdots & -r_{2n_1} + r_{1n_1} \\ 0 & -r_{32} + r_{12} & \mu + r_{13} & & -r_{3n_1} + r_{1n_1} \\ \vdots & \vdots & & \vdots & \\ 0 & -r_{n_1 2} + r_{12} & -r_{n_1 3} + r_{13} & \cdots & \mu + r_{1n_1} \end{vmatrix} \qquad (13)$$

From Eq. (5) the sum of the i-th row in (13) is $\mu + r_{i1}$ for $i = 2,3,\ldots,n_1$.

Therefore, by subtracting the columns $2,3,\ldots,n_1$ of (13) from the first column, we obtain (14):

$$|A| = \frac{1}{(\mu - X)} \begin{vmatrix} \mu & -r_{12} & -r_{13} & \cdots & -r_{1n_1} \\ -\mu - r_{21} & \mu + r_{12} & -r_{23} + r_{13} & \cdots & -r_{2n_1} + r_{1n_1} \\ -\mu - r_{31} & -r_{32} + r_{12} & \mu + r_{13} & & -r_{3n_1} + r_{1n_1} \\ \vdots & \vdots & & \vdots & \\ -\mu - r_{n_1 1} & -r_{n_1 2} + r_{12} & -r_{n_1 3} + r_{13} & \cdots & \mu + r_{1n_1} \end{vmatrix} \qquad (14)$$

Add the first row of (14) to the rows $2, 3, \ldots . n_1$ to obtain (15):

$$|A| = \frac{1}{(\mu - X)} \begin{vmatrix} \mu & -r_{12} & -r_{13} & \cdots & -r_{1n_1} \\ -r_{21} & \mu & -r_{23} & \cdots & -r_{2n_1} \\ -r_{31} & -r_{32} & \mu & & -r_{3n_1} \\ \vdots & \vdots & & \vdots & \\ -r_{n_1 1} & -r_{n_1 2} & -r_{n_1 3} & \cdots & \mu \end{vmatrix}$$

$$|A| = \frac{1}{(\mu - X)} \; \psi(G_1 : \mu) . \qquad (15)$$

In a similar manner we can show that from (8),

$$|B| = \frac{1}{(\mu - Y)} \; \psi(G_2 : \mu) \qquad (16)$$

Substituting (15) and (16) back into (11) gives Eq. (2).

Theorem 2.2: *Let G_i be an r_i- regular graph on n_i vertices and $diam(G_i) \le 2$, $i = 1, 2$. Then* $HE(G_1 \nabla G_2) = HE(G_1) + HE(G_2) - (X + Y) + \sqrt{(X - Y)^2 + 4n_1 n_2}$,

where $X = \frac{n_1 + r_1 - 1}{2}$ *and* $Y = \frac{n_2 + r_2 - 1}{2}$.

Proof: From Theorem 2.1,

$$\psi(G_1 \nabla G_2 : \mu) = \frac{[(\mu - X)(\mu - Y) - n_1 n_2]}{(\mu - X)(\mu - Y)} \; \psi(G_1 : \mu)\psi(G_2 : \mu),$$

which gives that

$$(\mu - X)(\mu - Y)\psi(G_1 \nabla G_2 : \mu) = [(\mu - X)(\mu - Y) - n_1 n_2] \; \psi(G_1 : \mu)\psi(G_2 : \mu).$$

Let $P_1(\mu) = (\mu - X)(\mu - Y)\psi(G_1 \nabla G_2 : \mu)$

and $P_2(\mu) = [(\mu - X)(\mu - Y) - n_1 n_2]\ \psi(G_1 : \mu)\psi(G_2 : \mu)$.

The roots of $P_1(\mu) = 0$ are X, Y and the H-eigenvalues of $G_1 \nabla G_2$.

Therefore the sum of the absolute values of the roots of $P_1(\mu) = 0$ is

$$X + Y + HE(G_1 \nabla G_2). \tag{17}$$

The roots of $P_2(\mu) = 0$ are H-eigenvalues of G_1 and G_2 and

$$\frac{1}{2}\left[(X+Y) \pm \sqrt{(X+Y)^2 - 4(XY - n_1 n_2)}\right].$$

Therefore the sum of the absolute values of the roots of $P_2(\mu) = 0$ is

$$HE(G_1) + HE(G_2) + \left|\frac{1}{2}\left[(X+Y) + \sqrt{(X+Y)^2 - 4(XY - n_1 n_2)}\right]\right| + \left|\frac{1}{2}\left[(X+Y) - \sqrt{(X+Y)^2 - 4(XY - n_1 n_2)}\right]\right|. \tag{18}$$

Since $P_1(\mu) = P_2(\mu)$, equating (17) and (18), we get

$$HE(G_1 \nabla G_2) = HE(G_1) + HE(G_2) - (X+Y) + \left|\frac{1}{2}\left[(X+Y) + \sqrt{(X+Y)^2 - 4(XY - n_1 n_2)}\right]\right| + \left|\frac{1}{2}\left[(X+Y) - \sqrt{(X+Y)^2 - 4(XY - n_1 n_2)}\right]\right|. \tag{19}$$

Since $r_1 \le n_1 - 1$ and $r_2 \le n_2 - 1$,

$$XY = \left(\frac{n_1 + r_1 - 1}{2}\right)\left(\frac{n_2 + r_2 - 1}{2}\right) \le \left(\frac{2n_1 - 2}{2}\right)\left(\frac{2n_2 - 2}{2}\right) < n_1 n_2 .$$

Therefore Eq. (19) reduces to

$$HE(G_1 \nabla G_2) = HE(G_1) + HE(G_2) - (X+Y) + \sqrt{(X+Y)^2 - 4(XY - n_1 n_2)}$$
$$= HE(G_1) + HE(G_2) - (X+Y) + \sqrt{(X-Y)^2 + 4n_1 n_2}.$$

Corollary 2.3: *If F_1 and F_2 are non H-cospectral, H-equienergetic regular graphs on n vertices and of same degree and $diam(F_i) \le 2$, $i = 1, 2$, then for any regular graph G with $diam(G) \le 2$, $HE(F_1 \nabla G) = HE(F_2 \nabla G)$.*

3. CONSTRUCTION OF *H*- EQUIENERGETIC GRAPHS

Theorem 3.1: *There exist a pair of non H-cospectral, H-equienergetic graphs on n vertices for all $n \ge 6$.*

Proof: Consider the graphs F_a and F_b as shown in Fig. 2.

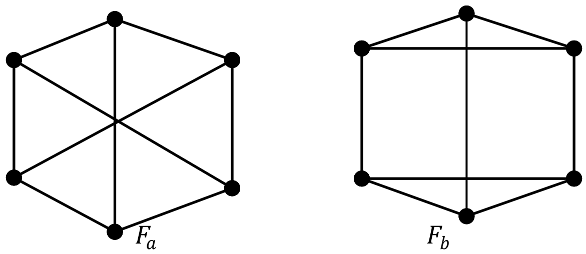


Fig. 2

By direct computation,

$$\psi(F_a : \mu) = \left[(\mu - 4)(\mu + 2)\left(\mu + \tfrac{1}{2}\right)^4\right] \tag{20}$$

And

$$\psi(F_b : \mu) = \left[\mu(\mu - 4)\left(\mu + \tfrac{1}{2}\right)^2\left(\mu + \tfrac{3}{2}\right)^2\right] \tag{21}$$

Both F_a and F_b are regular graphs on 6 vertices and of degree 3. Also $diam(F_i) \le 2$, $i = a, b$, and $HE(F_a) = HE(F_b) = 8$.

Let F be any r-regular graph on $p \ge 1$ vertices and $diam(F) \le 2$. Then by Theorem 2.2,

$$HE(F_a \nabla F) = HE(F_b \nabla F) = HE(F) + \left(\frac{9 - p - r}{2}\right) + \sqrt{\left(\frac{9 - p - r}{2}\right)^2 + 24p}.$$

Thus, $F_a \nabla F$ and $F_b \nabla F$ are H-equienergetic graphs. By Eqs. (20) and (21), F_a and F_b are non H-cospectral, so from Theorem 2.1, $F_a \nabla F$ and $F_b \nabla F$ are also non H-cospectral. Further $F_a \nabla F$ and $F_b \nabla F$ possesses equal number of vertices $n = 6 + p$, $p = 1, 2, \ldots$.

That the theorem holds also for $n = 6$ is directly verified from Eqs. (20) and (21).

Let K_p be the complete graph on p vertices. It is regular graph of degree $p - 1$. The reciprocal distance matrix of K_p is same as its adjacency matrix. Therefore $HE(K_p) = (p - 1)$ [3]. Using this in Theorem 2.2 we have following result.

Corollary 3.2: If F_a and F_b are the graphs as shown in Fig. 2, then

$$HE\left(F_a \nabla K_p\right) = HE\left(F_b \nabla K_p\right) = p + 3 + \sqrt{p^2 + 14p + 25}.$$

4. CONCLUSION

From Corollary 2.3 it is easy to construct a pair of non H-cospectral, H-equienergetic graphs. In particular from Theorem 3.1 and Corollary 3.2, it is easy to construct a pair of non H-cospectral, H-equienergetic n-vertex graphs for all $n \geq 6$.

ACKNOWLEDGMENT

This work is supported by the University Grants Commission (UGC), Govt. of India through research grant under UPE FAR-II grant No. F 14-3/2012 (NS/PE).

REFERENCES

1. F. Buckley, F. Harary, *Distance in Graphs*, Addison–Wesley, Redwood, 1990.
2. Z. Cui, B. Liu, *On Harary matrix, Harary index and Harary energy*, MATCH Commun. Math. Comput. Chem., 68 (2012), 815 – 823.
3. D. M. Cvetkovi, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application* Academic Press, New York, 1980.
4. K. C. Das, *Maximum eigenvalues of the reciprocal distance matrix,* J. Math. Chem., 47 (2010), 21 – 28.
5. M. V. Diudea, O. Ivanciuc, S. Nikolic, N. Trinajstic, *Matrices of reciprocal distance, polynomials and derived numbers*, MATCH Commun. Math. Comput. Chem., 35 (1997), 41 – 64.
6. A. D. Gungor, A. S. Cevik, *On the Harary energy and Harary Estrada index of a graph,* MATCH Commun. Math. Comput. Chem., 64 (2010), 280 – 296.
7. I. Gutman, *The energy of a graph*, Ber. Math. Stat. Sekt. Forschungsz. Graz, 103 (1978), 1 – 22.
8. F. Huang, X. Li, S. Wang, *On graphs with maximum Harary spectral radius*, arXiv: 1411.6832v1 [math.CO] 25 Nov 2014.
9. O. Ivanciuc, T. S. Balaban, A. T. Balaban, *Design of topological indices. Part 4. Reciprocal distance matrix, related local vertex invariants and topological indices*, J. Math. Chem., 12 (1993), 309 - 318.
10. D. Jenezic, A. Milicevic, S. Nikolic, N. Trinajstic, *Graph Theoretical Matrices in Chemistry,* Uni. Kragujevac, Kragujevac, 2007.
11. X. Li, Y. Shi, I. Gutman, *Graph Energy,* Springer, New York, 2012.
12. H. S. Ramane, R. B. Jummannaver, *Harary spectra and Harary energy of line graphs of regular graphs*, Gulf J. Math. (in press).
13. B. Zhou, N. Trinajstic, *Maximum eigenvalues of the reciprocal distance matrix and the reverse Wiener matrix*, Int. J. Quantum Chem., 108 (2008), 858 – 864.

Source of support: Grants Commission (UGC), Govt. of India, Conflict of interest: None Declared