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CONVERGENCE THEOREM FOR JUNGCK-AGARWAL et. al ITERATIVE SCHEME IN CONVEX METRIC SPACES

MEENAKSHI GUGNANI[1], PREETY*[2]

[1]Department of Mathematics,
Sh. L. N. Hindu College, Rohtak-124001(Haryana), India.

[2]Department of Mathematics,
Maharshi Dayanand University, Rohtak-124001(Haryana), India.

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ABSTRACT

In this paper we prove the strong convergence result for a pair of nonself mappings using Jungck S- iterative scheme in Convex metric spaces satisfying certain contractive condition. The results are the generalization of some existing results in the literature.

Keywords: *Fixed Point, Iterative schemes, contractive condition, convex metric spaces.*

1. INTRODUCTION AND PRELIMINARIES

In 1970, Takahashi [16] introduced the notion of convex metric space and studied the fixed point theorems for nonexpansive mappings. He defined that a map $W : X^2 \times [0,1] \to X$ is a convex structure in X if

$$d(u,W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda)d(u,y)$$

for all $x, y, u \in X$ and $\lambda \in [0,1]$. A metric space (*X*, *d*) together with a convex structure *W* is known as convex metric space and is denoted by (X, d, W). A nonempty subset *C* of a convex metric space is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$.

Remark 1.1: Every normed space is a convex metric space, where a convex structure $W(x, y, z;\alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z,$ for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1.$ In fact,

$$\begin{aligned} d(u,W(x, y, z;\alpha, \beta, \gamma)) &= \|u - (\alpha x + \beta y + \gamma z)\| \\ &\le \alpha\|u - x\| + \beta\|u - y\| + \gamma\|u - z\| \\ &= \alpha d(u,x) + \beta d(u, y) + \gamma d(u,z), \end{aligned}$$

for all $u \in X$. But there exists some convex metric spaces which cannot be embedded into normed spaces.

Let X be a Banach space, Y an arbitrary set, and $S,T : Y \to X$ such that $T(\mathrm{Y}) \subseteq \mathrm{S}(\mathrm{Y})$. For $x_0 \in Y$, consider the following iterative scheme:

$$Sx_{n+1} = Tx_n,\ n = 0,1,2.... \tag{1.1}$$

is called Jungck iterative scheme and was essentially introduced introduced by Jungck [1] in 1976 and it becomes the Picard iterative scheme when $S = I_d$ (identity mapping) and *Y* = *X*.

For $\alpha_n \in [0,1]$, Singh et al. [2] defined the Jungck-Mann iterative scheme as

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n,\ n = 0,1,2.... \tag{1.2}$$

Corresponding Author: Preety[2], [2]Department of Mathematics,*
Maharshi Dayanand University, Rohtak-124001(Haryana), India.

For $\alpha_n, \beta_n, \gamma_n \in [0,1]$, Olatinwo defined the Jungck Ishikawa [3] (see also [4, 5]) and Jungck-Noor [6] iterative schemes as

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n,$$
$$Sy_n = (1-\beta_n)Sx_n + \beta_n Tx_n,\ n = 0,1,2.... \tag{1.3}$$
$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n,$$
$$Sy_n = (1-\beta_n)Sx_n + \beta_n Tz_n,$$
$$Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n,\ n = 0,1,2.... \tag{1.4}$$

respectively.

Jungck Agarwal *et al.* [18] iteration is given as:

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ Sy_n &= (1-\beta_n)Sx_n + \beta_n Tx_n \end{aligned} \tag{1.5}$$

And Agarwal et al. [12] iterative scheme is given as:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n \end{aligned} \tag{1.6}$$

Remark 1.2: If $X = Y$ *and* $S = I_d$ (identity mapping), then the Jungck-Noor (1.4), Jungck-Ishikawa (1.3), Jungck-Mann (1.2) and Jungck Agarwal *et al.* (1.5) iterative schemes become the Noor [9], Ishikawa [10], Mann [11] and the Agarwal et al.iterative [12] schemes respectively.

Jungck [1] used the iterative scheme (1.1) to approximate the common fixed points of the mappings S and T satisfying the following Jungck contraction:

$$d(Tx, Ty) \le d(Sx, Sy),\ 0 \le \alpha < 1. \tag{1.7}$$

Olatinwo [3] used the following more general contractive definition than (1.7) to prove the stability and strong convergence results for the Jungck-Ishikawa iteration process:

(a) There exists a real number $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and for all $x, y \in Y$, we have

$$d(Tx,Ty) \le \varphi d(Sx,Tx) + ad(Sx,Sy). \tag{1.8}$$

(b) There exists a real number $M \ge 0$, $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and for all $x, y \in Y$, we have

$$d(Tx,Ty) \le \frac{\varphi d(Sx,Tx) + ad(Sx,Sy)}{1 + Md(Sx,Tx)} \tag{1.9}$$

Now we give the above iterative schemes in the setting of convex metric spaces:

Let (X, d, W) be a convex metric spaces. For $x_0 \in X$, we have

(1.1.1) Jungck Picard iterative scheme:

$$Sx_{n+1} = Tx_n,\ n = 0,1,2....$$

(1.1.2) Jungck Mann iterative scheme:

$$Sx_{n+1} = W(Sx_n, Tx_n, \alpha_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1].

(1.1.3) Jungck Ishikawa iterative scheme:

$$Sx_{n+1} = W(Sx_n, Ty_n, \alpha_n)$$
$$Sy_n = W(Sx_n, Tx_n, \beta_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

(1.1.4) Jungck Noor iterative scheme:

$$Sx_{n+1} = W(Sx_n, Ty_n, \alpha_n)$$
$$Sy_n = W(Sx_n, Tz_n, \beta_n)$$
$$Sz_n = W(Sx_n, Tx_n, \gamma_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

(1.1.5) Jungck Agarwal iterative scheme:

$$Sx_{n+1} = W(Tx_n, Ty_n, \alpha_n)$$
$$Sy_n = W(Sx_n, Tx_n, \beta_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in [0,1].

(1.1.6) Jungck Agarwal iterative scheme:

$$x_{n+1} = W(Tx_n, Ty_n, \alpha_n)$$
$$y_n = W(x_n, Tx_n, \beta_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in [0,1].

We will need the following definition to prove our main result:

Definition 1.3 (see [14, 15]): Let f and g be two self-maps on X. A point x in X is called (1) a fixed point of f if $(x) = x$; (2) coincidence point of a pair (f, g) if $fx = gx$; (3) common fixed point of a pair (f, g) if $x = fx = gx$. If $w = fx = gx$ for some x in X, then w is called a point of coincidence of f and g. A pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points.

Now we will give our main results:

2. CONVERGENCE RESULTS

Theorem 2.1: Let (X, d, W) be an arbitrary Convex metric space and let $S, T : Y \to X$ be nonself –operators on an arbitrary set Y satisfying contractive condition (1.8) , (1.9). Assume that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of X and $Sz = Tz = p$ (say). Let $\varphi : R^+ \to R^+$ be monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck –Agarwal *et. al* iteration process defined by (1.1.5), where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the Jungck –Agarwal *et. al* iterative process $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Also, p will be the unique common fixed point of S,T provided that Y=X, and S and T are weakly compatible.

Proof: First, we prove that z is the unique coincidence point of S and T by using condition (1.8). Let $C(S, T)$ be the set of the coincidence points of S and T Suppose that there exists $z_1, z_2 \in C(S, T)$ such that $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$. If $p_1 = p_2$, then $Sz_1 = Sz_2$ and since S is injective, it follows that $z_1 = z_2$. If $p_1 \neq p_2$, then from condition (1.8), for mappings S and T, we have

$$\begin{aligned} 0 \le d(p_1, p_2) &= d(Tz_1, Tz_2) \\ &\le \varphi d(Sz_1, Tz_2) + ad(Sz_1, Sz_2) \\ &= ad(p_1, p_2) \end{aligned}$$

which implies that $(1-a)d(p_1, p_2) \le 0.$ So we have $(1-a) > 0,$

Since $a \in [0,1)$, but $d(p_1, p_2) \le 0$, which is as contradiction since norm is negative. So we have $d(p_1, p_2) = 0,$ that is $p_1 = p_2 = p.$ Since $p_1 = p_2$, then we have that

$p_1 = Sz_1 = Tz_1 = Sz_2 = Tz_2 = p_2$, leading to $Sz_1 = Sz_2 \Rightarrow z_1 = z_2 = z.$

Hence z is unique coincidence point of S and T.

Now we prove that iterative process $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Using condition (1.7) and (1.8), we have

$$\begin{aligned} d(Sx_{n+1}, p) &= d(W(Tx_n, Ty_n, \alpha_n), p) \\ &\le (1-\alpha_n) d(Tx_n, p) + \alpha_n d(Ty_n, p) \\ &= (1-\alpha_n) d(Tz, Tx_n) + \alpha_n d(Tz, Ty_n) \\ &\le (1-\alpha_n)\left[\frac{\varphi d(Sz,Tz) + ad(Sz,Sx_n)}{1+Md(Sz,Tz)}\right] + \alpha_n\left[\frac{\varphi d(Sz,Tz) + ad(Sz,Sy_n)}{1+Md(Sz,Tz)}\right] \\ &\le (1-\alpha_n) d(Sx_n, p) + \alpha_n a d(Sy_n, p) \end{aligned} \tag{2.1.1}$$

For $d(Sy_n, p)$, we have

$$\begin{aligned} d(Sy_n, p) &= d(W(Sx_n, Tx_n, \beta_n), p) \\ &\le (1-\beta_n) d(Sx_n, p) + \beta_n d(Tx_n, p) \\ &\le (1-\beta_n) d(Sx_n, p) + \beta_n\left[\frac{\varphi d(Sz,Tz) + ad(Sz,Sx_n)}{1+Md(Sz,Tz)}\right] \\ &\le (1-\beta_n + a\beta_n) d(Sx_n, p) \end{aligned} \tag{2.1.2}$$

From (2.1.1) and (2.1.2), we get

$$\begin{aligned} d(Sx_{n+1}, p) &\le (1-\alpha_n) d(Sx_n, p) + \alpha_n a(1-\beta_n + a\beta_n) d(Sx_n, p) \\ &= a[1-\alpha_n\beta_n(1-a)] d(Sx_n, p) \\ &\le [1-\alpha_n(1-a)] d(Sx_n, p) \\ &\le \prod_{k=0}^{n}[1-(1-a)\alpha_k] d(Sx_0, p) \\ &\le e^{-(1-a)\sum_{k=0}^{\infty}\alpha_k} d(Sx_0, p). \end{aligned} \tag{2.1.3}$$

Since $\alpha_k \in [0,1]$, $0 \le a < 1$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, so $\le e^{-(1-a)\sum_{k=0}^{n}\alpha_k} d(Sx_0, p) \to 0$ as $n \to \infty$.

Hence from equation (2.1.3) we get, $d(Sx_{n+1}, p) \to 0$ as $n \to \infty$, that is $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Corollary 2.2: If we take S=I (Identity mapping) then the iterative scheme (1.1.6) becomes Agarwal *et. al* iteration as defined by (1.1.7). Convergence of Agarwal et. al iterative scheme can be proved on the same lines as in Theorem 2.1.

REFERENCES

1. G. Jungck, "Commuting mappings and fixed points," The American Mathematical Monthly, vol. 83, no. 4, pp. 261–263, 1976.
2. S. L. Singh, C. Bhatnagar, and S.N.Mishra, "Stability of Jungck type iterative procedures," International Journal of Mathematics and Mathematical Sciences, no. 19, pp. 3035–3043, 2005.
3. M. O. Olatinwo, "Some stability and strong convergence results for the Jungck-Ishikawa iteration process," Creative Mathematics and Informatics, vol. 17, pp. 33–42, 2008.
4. A. O. Bosede, "Strong convergence results for the Jungck-Ishikawa and Jungck-Mann iteration processes," Bulletin of Mathematical Analysis and Applications, vol. 2, no. 3, pp. 65–73, 2010.
5. J. O. Olaleru and H. Akewe, "On multistep iterative scheme for approximating the common fixed points of contractive-like operators," International Journal of Mathematics and athematical Sciences, vol. 2010, Article ID 530964, 11 pages, 2010.
6. M. O. Olatinwo, "A generalization of some convergence results using a Jungck-Noor three-step iteration process in arbitrary Banach space," Polytechnica Posnaniensis, no. 40, pp. 37–43, 2008.
7. R. Chugh and V. Kumar, "Strong Convergence and Stability results for Jungck-SP iterative scheme," International Journal of Computer Applications, vol. 36, no. 12, 2011.
8. W. Phuengrattana and S. Suantai, "On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval," Journal of Computational and Applied Mathematics, vol. 235, no. 9, pp. 3006–3014, 2011.

9. M. A. Noor, "New approximation schemes for general variational inequalities," Journal ofMathematical Analysis and Applications, vol. 251, no. 1, pp. 217–229, 2000.
10. S. Ishikawa, "Fixed points by a new iteration method," Proceedings of the American Mathematical Society, vol. 44,no. 1, pp. 147–150, 1974.
11. W. R. Mann, "Mean value methods in iteration," Proceedings of the American Mathematical Society, vol. 4, pp. 506–510, 1953.
12. R. P. Agarwal, D. O'Regan, and D. R. Sahu, "Iterative construction of fixed points of nearly asymptotically nonexpansive mappings," Journal of Nonlinear and Convex Analysis, vol. 8, no.1, pp. 61–79, 2007.
13. D. R. Sahu and A. Petrus¸el, "Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces, "Nonlinear Analysis: Theory, Methods & Applications, vol. 74, no. 17, pp. 6012–6023, 2011.
14. N. Hussain, G. Jungck, and M. A. Khamsi, "Nonexpansive retracts and weak compatible pairs in metric spaces," Fixed Point Theory and Applications, vol. 2012, article 100, 2012.
15. G. Jungck and N. Hussain, "Compatible maps and invariant approximations," Journal of Mathematical Analysis and Applications, vol. 325, no. 2, pp. 1003–1012, 2007.
16. W. Takahashi, "A convexity in metric spaces and nonexpansive mapping", Kodai Math. Sem. Rep., vol. 22, 8 pages, 1970.
17. N. Hussain, V. Kumar and M.A. Kutbi, "On rate of Convergence of Jungck type iterative schemes", Abstract and Applied Analysis, vol. 2013, Article ID 132626, 15 pages, 2013.
18. Chugh *et al.*, "On the stability and strong convergence for Jungck-Agarwal *et al.* iteration procedure", International Journal of Computer Applications, vol. 64, no. 7, 6 pages, 2013.

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