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CONVERGENCE THEOREM FOR JUNGCK-AGARWAL et. al ITERATIVE SCHEME IN CONVEX METRIC SPACES

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ABSTRACT

In this paper we prove the strong convergence result for a pair of nonself mappings using Jungck S- iterative scheme in Convex metric spaces satisfying certain contractive condition. The results are the generalization of some existing results in the literature.

Keywords: *Fixed Point, Iterative schemes, contractive condition, convex metric spaces.*

1. INTRODUCTION AND PRELIMINARIES

In 1970, Takahashi [16] introduced the notion of convex metric space and studied the fixed point theorems for nonexpansive mappings. He defined that a map $W : X^2 \times [0,1] \to X$ is a convex structure in X if

$$d(u,W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda)d(u,y)$$

for all $x,y,u \in X$ and $\lambda \in [0,1]$. A metric space (*X*, *d*) together with a convex structure *W* is known as convex metric space and is denoted by (X,d,W). A nonempty subset *C* of a convex metric space is convex if $W(x,y,\lambda) \in C$ for all $x,y \in C$ and $\lambda \in [0,1]$.

Remark 1.1: Every normed space is a convex metric space, where a convex structure $W(x,y,z;\alpha,\beta,\gamma) = \alpha x + \beta y + \gamma z,$ for all $x,y,z \in X$ and $\alpha,\beta,\gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$. In fact,

$$\begin{aligned} d(u,W(x,y,z;\alpha,\beta,\gamma)) &= \|u - (\alpha x + \beta y + \gamma z)\| \\ &\le \alpha\|u-x\| + \beta\|u-y\| + \gamma\|u-z\| \\ &= \alpha d(u,x) + \beta d(u,y) + \gamma d(u,z), \end{aligned}$$

for all $u \in X$. But there exists some convex metric spaces which cannot be embedded into normed spaces.

Let X be a Banach space, Y an arbitrary set, and $S,T : Y \to X$ such that $T(Y) \subseteq S(Y)$. For $x_0 \in Y$, consider the following iterative scheme:

$$Sx_{n+1} = Tx_n,\ n = 0,1,2.... \tag{1.1}$$

is called Jungck iterative scheme and was essentially introduced introduced by Jungck [1] in 1976 and it becomes the Picard iterative scheme when $S = I_d$ (identity mapping) and *Y* = *X*.

For $\alpha_n \in [0,1]$, Singh et al. [2] defined the Jungck-Mann iterative scheme as

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n,\ n = 0,1,2.... \tag{1.2}$$

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For $\alpha_n, \beta_n, \gamma_n \in [0,1]$, Olatinwo defined the Jungck Ishikawa [3] (see also [4, 5]) and Jungck-Noor [6] iterative schemes as

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n,$$
$$Sy_n = (1-\beta_n)Sx_n + \beta_n Tx_n,\ n = 0,1,2.... \tag{1.3}$$

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n,$$
$$Sy_n = (1-\beta_n)Sx_n + \beta_n Tz_n,$$
$$Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n,\ n = 0,1,2.... \tag{1.4}$$

respectively.

Jungck Agarwal *et al.* [18] iteration is given as:

$$Sx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n,$$
$$Sy_n = (1-\beta_n)Sx_n + \beta_n Tx_n \tag{1.5}$$

And Agarwal et al. [12] iterative scheme is given as:

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n Tx_n \tag{1.6}$$

Remark 1.2: If $X = Y$ *and* $S = I_d$ (identity mapping), then the Jungck-Noor (1.4), Jungck-Ishikawa (1.3), Jungck-Mann (1.2) and Jungck Agarwal *et al.* (1.5) iterative schemes become the Noor [9], Ishikawa [10], Mann [11] and the Agarwal et al.iterative [12] schemes respectively.

Jungck [1] used the iterative scheme (1.1) to approximate the common fixed points of the mappings S and T satisfying the following Jungck contraction:

$$d(Tx, Ty) \le d(Sx, Sy),\ 0 \le \alpha < 1. \tag{1.7}$$

Olatinwo [3] used the following more general contractive definition than (1.7) to prove the stability and strong convergence results for the Jungck-Ishikawa iteration process:

(a) There exists a real number $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and for all $x, y \in Y$, we have
$$d(Tx,Ty) \le \varphi d(Sx,Tx) + ad(Sx,Sy). \tag{1.8}$$

(b) There exists a real number $M \ge 0$, $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and for all $x, y \in Y$, we have
$$d(Tx,Ty) \le \frac{\varphi d(Sx,Tx) + ad(Sx,Sy)}{1 + Md(Sx,Tx)} \tag{1.9}$$

Now we give the above iterative schemes in the setting of convex metric spaces:

Let (X, d, W) be a convex metric spaces. For $x_0 \in X$, we have

(1.1.1) Jungck Picard iterative scheme:

$$Sx_{n+1} = Tx_n,\ n = 0,1,2....$$

(1.1.2) Jungck Mann iterative scheme:

$$Sx_{n+1} = W(Sx_n, Tx_n, \alpha_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1].

(1.1.3) Jungck Ishikawa iterative scheme:

$$Sx_{n+1} = W(Sx_n, Ty_n, \alpha_n)$$
$$Sy_n = W(Sx_n, Tx_n, \beta_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

(1.1.4) Jungck Noor iterative scheme:

$$Sx_{n+1} = W(Sx_n, Ty_n, \alpha_n)$$
$$Sy_n = W(Sx_n, Tz_n, \beta_n)$$
$$Sz_n = W(Sx_n, Tx_n, \gamma_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

(1.1.5) Jungck Agarwal iterative scheme:

$$Sx_{n+1} = W(Tx_n, Ty_n, \alpha_n)$$
$$Sy_n = W(Sx_n, Tx_n, \beta_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in [0,1].

(1.1.6) Jungck Agarwal iterative scheme:

$$x_{n+1} = W(Tx_n, Ty_n, \alpha_n)$$
$$y_n = W(x_n, Tx_n, \beta_n),\ n = 0,1,2....$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in [0,1].

We will need the following definition to prove our main result:

Definition 1.3 (see [14, 15]): Let f and g be two self-maps on X. A point x in X is called (1) a fixed point of f if $(x) = x$; (2) coincidence point of a pair (f, g) if $fx = gx$; (3) common fixed point of a pair (f, g) if $x = fx = gx$. If $w = fx = gx$ for some x in X, then w is called a point of coincidence of f and g. A pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points.

Now we will give our main results:

2. CONVERGENCE RESULTS

Theorem 2.1: Let (X, d, W) be an arbitrary Convex metric space and let $S, T : Y \to X$ be nonself –operators on an arbitrary set Y satisfying contractive condition (1.8) , (1.9). Assume that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of X and $Sz = Tz = p$ (say). Let $\varphi : R^+ \to R^+$ be monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck –Agarwal *et. al* iteration process defined by (1.1.5), where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the Jungck –Agarwal *et. al* iterative process $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Also, p will be the unique common fixed point of S,T provided that Y=X, and S and T are weakly compatible.

Proof: First, we prove that z is the unique coincidence point of S and T by using condition (1.8). Let $C(S,T)$ be the set of the coincidence points of S and T Suppose that there exists $z_1, z_2 \in C(S,T)$ such that $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$. If $p_1 = p_2$, then $Sz_1 = Sz_2$ and since S is injective, it follows that $z_1 = z_2$. If $p_1 \neq p_2$, then from condition (1.8), for mappings S and T, we have

$$\begin{aligned} 0 \le d(p_1, p_2) &= d(Tz_1, Tz_2) \\ &\le \varphi d(Sz_1, Tz_2) + ad(Sz_1, Sz_2) \\ &= ad(p_1, p_2) \end{aligned}$$

which implies that $(1-a)d(p_1, p_2) \le 0.$ So we have $(1-a) > 0,$

Since $a \in [0,1),$ but $d(p_1, p_2) \le 0,$ which is as contradiction since norm is negative. So we have $d(p_1, p_2) = 0,$ that is $p_1 = p_2 = p.$ Since $p_1 = p_2,$ then we have that

$p_1 = Sz_1 = Tz_1 = Sz_2 = Tz_2 = p_2,$ leading to $Sz_1 = Sz_2 \Rightarrow z_1 = z_2 = z.$

Hence z is unique coincidence point of S and T.

Now we prove that iterative process $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to $p.$

Using condition (1.7) and (1.8), we have

$$\begin{aligned} d(Sx_{n+1},p) &= d(W(Tx_n, Ty_n, \alpha_n), p) \\ &\le (1-\alpha_n)\, d(Tx_n, p) + \alpha_n d(Ty_n, p) \\ &= (1-\alpha_n)\, d(Tz, Tx_n) + \alpha_n d(Tz, Ty_n) \\ &\le (1-\alpha_n)\left[\frac{\varphi d(Sz,Tz) + ad(Sz,Sx_n)}{1+Md(Sz,Tz)}\right] + \alpha_n\left[\frac{\varphi d(Sz,Tz) + ad(Sz,Sy_n)}{1+Md(Sz,Tz)}\right] \\ &\le (1-\alpha_n)\, d(Sx_n, p) + \alpha_n a d(Sy_n, p) \end{aligned} \tag{2.1.1}$$

For $d(Sy_n, p),$ we have

$$\begin{aligned} d(Sy_n, p) &= d(W(Sx_n, Tx_n, \beta_n), p) \\ &\le (1-\beta_n)\, d(Sx_n, p) + \beta_n d(Tx_n, p) \\ &\le (1-\beta_n)\, d(Sx_n, p) + \beta_n\left[\frac{\varphi d(Sz,Tz) + ad(Sz,Sx_n)}{1+Md(Sz,Tz)}\right] \\ &\le (1-\beta_n + a\,\beta_n)\, d(Sx_n, p) \end{aligned} \tag{2.1.2}$$

From (2.1.1) and (2.1.2), we get

$$\begin{aligned} d(Sx_{n+1},p) &\le (1-\alpha_n)\, d(Sx_n, p) + \alpha_n a(1-\beta_n + a\,\beta_n)\, d(Sx_n, p) \\ &= a[1-\alpha_n\beta_n(1-a)]\, d(Sx_n, p) \\ &\le [1-\alpha_n(1-a)]\, d(Sx_n, p) \\ &\le \prod_{k=0}^{n}[1-(1-a)\alpha_k]\, d(Sx_0, p) \\ &\le e^{-(1-a)\sum_{k=0}^{\infty}\alpha_k} d(Sx_0, p). \end{aligned} \tag{2.1.3}$$

Since $\alpha_k \in [0,1],\ 0 \le a < 1$ and $\sum_{n=0}^{\infty}\alpha_n = \infty,$ so $\le e^{-(1-a)\sum_{k=0}^{n}\alpha_k} d(Sx_0, p) \to 0$ as $n \to \infty.$

Hence from equation (2.1.3) we get, $d(Sx_{n+1}, p) \to 0$ as $n \to \infty,$ that is $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to $p.$

Corollary 2.2: If we take S=I (Identity mapping) then the iterative scheme (1.1.6) becomes Agarwal *et. al* iteration as defined by (1.1.7). Convergence of Agarwal et. al iterative scheme can be proved on the same lines as in Theorem 2.1.

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