International Journal of Mathematical Archive-6(11), 2015, 71-75
Available online through www.ijma.info ISSN 2229 – 5046

GRAPHS FOR DIRECT PRODUCT OF COMMUTATIVE RINGS

G. SHOBHALATHA[1], A. SWETHA*[2]

[1]Research supervisor, [2]Research scholar,
Department of Mathematics, S. K. University, Anantapuramu (A.P.), India.

(Received On: 30-09-15; Revised & Accepted On: 05-11-15)

ABSTRACT

David F.Anderson and Philip S.Livingston studied the properties of the zero divisor graph of a commutative ring. We recall several results of zero divisor graph, total graphs of commutative rings. In this paper we examine the resultant of graphs for the direct product of commutative rings. It is observed that the graphs obtained on the ring of type Z_n, Z_nxZ_n, $Z_nxZ_nxZ_n$……. (where $n \geq 2$) are nothing but either regular graphs or complete bipartite graphs.

Key words: *Commutative, nilpotent, bipartite, complete bipartite, regular.*

I. INTRODUCTION

Graph theory is one of the most flourishing branches of modern Mathematics with wide applications. This paper forms a new bridge between graph theory and the algebraic concept ring R. Here R denote a commutative ring with identity. Istvan Beck first introduce the concept of relating a commutative ring to a graph. It is consider that every element of the ring R was a vertex in the graph and two vertices x, y are connected if and only if xy=0 (x+y=0).

A different method of associating a commutative ring to a graph was proposed by David F.Anderson and Philip S.Livingston. In a such way we obtained some different graphs on commutative rings of type Z_n.

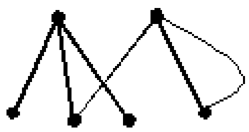
II. BASIC CONCEPT AND PRELIMINARIES

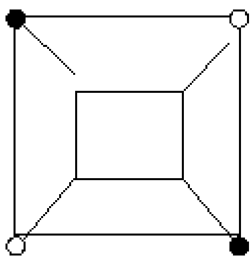
***Nilpotent*element:** An element $a \neq 0$ in a ring R is called nilpotent if $a^n = 0$ for some positives integer n.

Zero-divisor: An element x in a ring R is called a right (or left) zero-divisor if there exists a non-zero element $y \in R$ such that $yx = 0$ (or $xy = 0$).

Bipartite Graph: A bipartite graph is a graph G whose vertex set is partitioned into two disjoint subsets X and Y such that each edge in G has one end in X and the other end in Y. Such a partition (X, Y) is called a bipartition of the graph.

Ex:

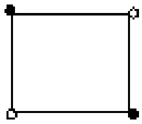




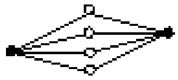
Complete Bipartite graph: A bipartite graph is complete if every vertex of X is joined to all other vertices of Y. We denote the complete bipartite graph by $K_{m,n}$ where m and n represent the number of vertices in the disjoint vertex set X and Y respectively.

Corresponding Author: A. Swetha*[2]
[2]Research scholar, Department of Mathematics, S. K. University, Anantapuramu (A.P.), India.

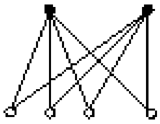
Ex:



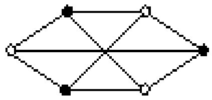
$K_{2,2}$



$K_{2,4}$

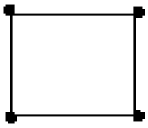


$K_{2,4}$

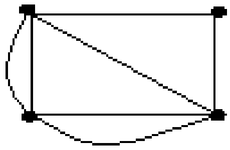


$K_{3,3}$

Regular graph: A graph in which all vertices are of the same degree is called a regular graph. In an r-Regular graph all vertices are of degree r, the number r is called the regularity of a regular graph. Regular graphs are undirected.



2–regular



4–regular

Zero – divisor graph: A zero-divisor graph $\Gamma(R)$ associated to R is the graph whose vertices are the elements of $Z(R)^*$ where. $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero-divisor of R and the vertices x and y are adjacent if and only if $xy = 0$.

III. MAIN RESULTS

Let (R, +, ×) be a commutative ring with identity and let consider R= Z_n where Z_n be a ring of integers modulo n. Let G(R) be the graph of R . Defined by G(R)=(V(R), E(R)) where V(R) be the vertex set of G(R) and E(R) be the edge set of G(R),where the set of all the elements of ring R are consider as the vertices of graph G(R). For any two elements x, y$\in$R be considered as vertices of G(R), if x and y are adjacent in G(R).

Theorem 3.1: Let R=Z_n be a commutative ring of integer modulo n, where n=2^r (r>1). Let S be the set of all nilpotent elements of R such that S ={a$\in$R, a$\neq$0 a^n= 0 for some positive integer n} and the vertex set of G(R) is defined as V(G(R)) = {x,y$\in$R / x and y are adjacent<=> x-y $\in$ s}. Then the graph G(R) is (2^{r-1}-1)**-Regular graph**

Proof: Let R be the commutative ring of integers modulo n. Define S={a$\in$R, a$\neq$0: a^n=0 for some positive integer n} i.e., every element of S is a nilpotent element. Then S$\subseteq$R. Now for any s$\in$S, s-0$\in$ S every element of S is adjacent to 0. Again since n = 2^r, then the set of all nilpotent element are only the even integers in Zn. Therefore S contains all non zero even integers of R. Now if n=2^r, then G(R) = 2^{r-1}-1 (excluding the zero element)

Let x, y$\in$ Z_n where x, y $\neq$ 0. Then x and y are adjacent iff x-y$\in$S.

i.e., x, y$\in$ R <=> x is adjacent to those elements y$\in$ R such that x-y $\in$ S

<=> There are exactly n elements adjacent to each x$\in$R.

<=> V (R) = R and every vertex in G(R) is of degree 2^{r-1}-1.

Then the resultant graph of G(R) = 2^{r-1}-1. That is G(R) is of degree 2^{r-1}-1.

Example 3.1:
(i) Let R be a ring of integer modulo n. When n = 2^r(r $\geq$1). If r=1 then n=2^1 and R = Z_2 = {0, 1}. Let S be the set of all nilpotent elements in Z_2.

i.e., S = {a $\in Z_n$ / a^n =0} S = ϕ. No two elements of Z_2 are adjacent in Γ(R).

Therefore the graph G(R) is a **null graph**.

(ii) Let r=2 => n= 2^2 then R = $Z_{2^2} = Z_4$.

Z_4 = {0, 1, 2, 3} is a ring of integers modulo 4.

Then V (R) = {0, 1, 2, 3} and S ={2}. Here 2-0=2$\in$S and 3-1=2$\in$S => vertices 0 and 2, 3 and 1 are adjacent.

From the following graph, we say that G(R) is **1- regular graph**

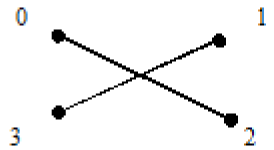


Fig. 3.1: 1- Regular graph

Example 3.2: Let r=3 => n= 2^3 then R = $Z_{2^3} = Z_8$.

Z_8={0,1,2,3,4,5,6,7} is a ring of integers modulo 8.

Therefore S = {2, 4, 6}. Here 2-0, 4-0, 6-0, 4-2, 6-2, 6-4, 7-1, 3-1, 5-1, 5-3, 7-3, 7-5 $\in$ S

Then we have the following graph.

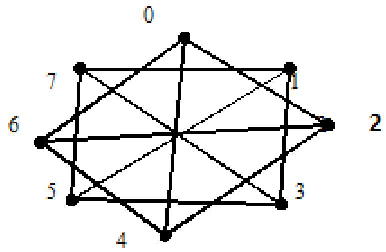


Fig. 3.2: 3- Regular graph

Since every vertices in G(R) has degree 3. Therefore it is a **3- Regular graph**. Continuing like this we get **5- Regular graph, 7- Regular graph** and so on which is shown in the following table

Number of vertices(r$\geq$1) n = 2^r	G(R) =(2^{r-1}-1)- Regular
2^1	Null graph
2^2	1 – Regular
2^3	3- Regular
2^4	7- Regular
2^5	15- Regular
2^6	31- Regular
.	.
.	.
.	.

Table-3.1

Next we consider R be a commutative ring with identity and let R= Z_n x Z_n where Z_n be a ring of integers modulo n. Let G(R) be the graph of R. Defined by G(R) = (V(R), E(R)) where V(R) be the vertex set of G(R) and E(R) be the edge set of G(R), where the set of all the elements of ring R are consider as the vertices of graph G(R). For any two elements x, y$\in$R be considered as vertices of G(R), if x and y are adjacent in G(R). the edge set E(R) ={x, y$\in$R /x and y are adjacent iff x.y = 0, x$\neq$y}

Theorem 3.2: Let R be a commutative ring with identity and let G(R)) = (V(R), E(R)) be denote the graph of R, where V(R) be the vertex set of G(R) and E(R) be the edge set of G(R). Defined E(R) ={x, y $\in$R / x is adjacent to y <=> x.y=0 and x, y $\neq$ 0}.Then the graph G(R) is consisting a complete bipartite graph of order n-1. where n is a prime number (n $\geq$3).

Proof: Let R be a commutative ring, where n is a prime number such that $n \geq 2$. Then R contains n^2 elements.

Let G(*R*) be the graph of R such that E(*R*) = {X, Y∈R / X is adjacent to Y <=> X.Y=0 and X, Y ≠ 0}.

Let X, Y ∈ R. Then $X=(x_1, x_2)$, $Y=(y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in R$.
X is adjacent to Y<=> X . Y = 0

<=> $(x_1, x_2). (y_1,y_2) = (0, 0)$

<=> $(x_1.y_1, x_2.y_2) = (0, 0)$

<=> $x_1.y_1= 0$ and $x_2.y_2 = 0$

<=> $x_1 = 0$ or $y_1 = 0$ and $x_2 = 0$ or $y_2 = 0$

<=> since n is prime, there are exactly (n-1) pair of elements adjacent in G(*R*) and the resultant graph is a complete bipartite graph of order n-1.

If n = 2(n-1), where n ≥3. Then the graph of G(*R*) = 4k for any positive integer k ≥ 1.

Example 3.3: Consider the ring R= Z_n x Z_n where **n =2**. Then Z_n x Z_n = Z_2 x Z_2 which consists of 2^2 elements,

i.e., Z_2 x Z_2 = {(0, 0) (1, 0) (0, 1) (1, 1)}.

Let G(*R*) = {X, Y ∈ R / X is adjacent Y <=>X.Y=0, X≠Y and X, Y ≠ 0}.
(0, 1). (1, 0)= (0, 0)

From the following **graph**

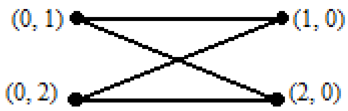
Example 2.5: Let **n = 3**.Then Z_n x Z_n = Z_3x Z_3 which consists of 3^2 elements,

i.e,.Z_3 x Z_3 ={(0,0)(0,1)(0,2)(1,0)(1,1) (1,2)(2,0)(2,1) (2,2).

Let G(*R*) ={X, Y ∈ R / X is adjacent Y <=>X.Y=0, X≠Y and X, Y ≠ 0}.

(0, 1) . (1, 0) = (0, 0)   (0, 1) . (2, 0) = (0, 0)
(0, 2) . (1, 0) = (0, 0)   (0, 2) . (2, 0) = (0, 0)

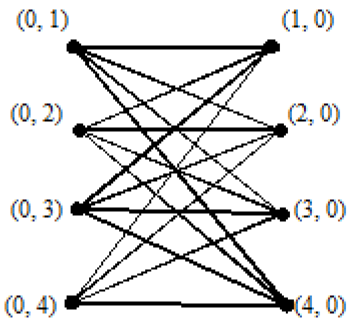
From the following **graph.**



Example 2.6: Let **n = 5**. Then Z_n x Z_n = Z_5 x Z_5 which consists of 3^2 elements,
i.e,.Z_5xZ_5={(0,0)(0,1)(0,2)(0,3)(0,4)(1,0)(1,1)(1,2)(1,3)(1,4)(2,0)(2,1)(2,2)(2,3)(2,4)(3,0)(3,1)(3,2)(3,3)(3,4)(4,0)(4,1)(4,2)(4,3)(4,4)}.

Let G(*R*) = {X,Y ∈ R / X is adjacent Y <=>X.Y=0, X≠Y and X,Y ≠ 0}.
(0,1) . (1,0) = (0,0)   (0,1) . (3,0) = (0,0)
(0,2) . (1,0) = (0,0)   (0,2) . (3,0) = (0,0)
(0,3) . (1,0) = (0,0)   (0,3) . (3,0) = (0,0)
(0,4) . (1,0) = (0,0)   (0,4) . (3,0) = (0,0)
(0,1) . (2,0) = (0,0)   (0,1) . (4,0) = (0,0)
(0,2) . (2,0) = (0,0)   (0,2) . (4,0) = (0,0)
(0,3) . (2,0) = (0,0)   (0,3) . (4,0) = (0,0)
(0,4) . (2,0) = (0,0)   (0,4) . (4,0) = (0,0)

From the following **figure.**



In the same way Z_7x Z_7 and Z_{11}x Z_{11} are also form complete bipartite graph i.e., all prime numbers form complete bipartite graphs.

The resultant graph are obtained in the following table

Number of vertices n = r (r>1) (n is prime)	$\lvert R \rvert = n^2$	$G(R)=K_{r-1\ r-1}$ –Complete bipartite graph
2	4	K_{11}-Completite bipartite graph
3	9	K_{22}-Completite bipartite graph
5	25	K_{44}-Completite bipartite graph
7	49	K_{66}-Completite bipartite graph
.	.	:
:	.	:

Table-3.2

i.e., $\lvert G(R) \rvert$ = 4, 8, 12, 20, ---

Therefore, $\lvert G(R) \rvert$ = Multiple of 4 = 4.k for any positive integer $k \geq 1$.

REFERENCES

1. Ashrafi N, Maiman H.R, Pournaki M.R, and Yassemi S "Unit graph Associated with rings" Communications in Algebra (R), 38 (2010) 2851-2871.
2. Anderson D.F and Livingston P.S, "The zero-divisor graph of a commutative ring", J. Algebra 217 (1999), 434-447.
3. Anderson D.F, Frazier A, Lauve A, and Livingston P.S, "The zero-divisor graph of a commutative ring 11". Lecture Notes in pure and Appl. Math. 220 (2001), 61-72.
4. Bhattacharya P.B, Jain.S.k, Nagpul "Basic Abstract Algebra", Plika Press Pvt. Lt Red, New Delhi-2007.
5. Grimaldi R.P. "Graphs from rings". Proceedings of the 20th South-eastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989). Congv. Numer. Vol. 71, pp. (1990) 95-103.
6. Chartrand G., Oellermann, O.R "Applied and Algorithmic Graph Theory". New York: McGraw-Hill, Inc. (1993).
7. Frank Harary "Graph theory", Narosa publishing house, New Delhi 2001, ISBN-978-81-85015-55-2.

Source of support: Nil, Conflict of interest: None Declared